IRSTI 41.03.02 https://doi.org/10.26577/ijmph.2024v15i2b12

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(*Received 15 October 2024; revised 20 November 2024; accepted 30 November 2024)*

# The problem of translational-rotational motion of a non-stationary axisymmetric small body in the gravitational field of two spherical bodies with variable mass

**Abstract.** In the new formulation, the translational-rotational motion of a non-stationary axisymmetric body under the gravitational influence of two spherical bodies with time-varying masses is considered. The non-stationarity of the axisymmetric body is characterized by changes in its mass, dimensions, and compression along its axis of symmetry. Additionally, the axisymmetric body is assumed to possess an equatorial plane of symmetry. This case is examined within a specialized version of the non-stationary restricted three-body problem. The problem becomes significantly more complex due to the time-dependent changes in the masses and dimensions of the bodies, which may also generate reactive forces. Equations of motion for the problem under consideration have been derived in both absolute and relative coordinate systems. For the first time, the equations of motion for the barycenter of two spherical bodies with variable masses, accounting for reactive forces, have been obtained. Based on these equations, the equations of motion for the problem in the barycentric coordinate system have also been derived.
**Key words:** restricted three-body problem, variable mass, non – stationary axisymmetric celestial bodies, translational-rotational motion.

## 1 Introduction

Mathematical models of gravitating point bodies are well studied Duboshin [1]. Nowadays in astronomy and in astrophysics the investigations of problems of dynamics of non-stationary gravitating non-point bodies are actual Eggleton [2], Omarov [3], Omarov [4], Minglibayev [5].

Let us note some works on the dynamics of gravitating systems with variable masses close to the subject of the present work. In the works Lukyanov [6], Bekov [7], Letelier and Silva [8], Abouelmagd et al. [9] interesting questions on the restricted three-body-point problem with variable masses are studied. The problem has primarily been studied when the masses of the primary bodies change according to the Meshchersky law at the same rate Bekov [10], Lukyanov [11], Rystygulova [12]. In Suraj et al. [13], the restricted three-body problem with variable mass is investigated, focusing on the system's dynamics when one of the bodies loses or gains mass according to Jeans' law. Article Abouelmagd and Guirao [14] is dedicated to studying the perturbed three-body problem where the bodies are considered as oblate spheroids. The main focus is on finding libration points and their linear stability under the influence of Coriolis and centrifugal forces. Periodic orbits around these points are also studied. In Mittal et al. [15], the three-body problem with variable mass and small thrust is considered. The authors introduce the concept of artificial equilibrium points, created by a constant small thrust to balance gravitational and centrifugal forces in the system. This study is based on the law of mass change according to Jeans, where the spacecraft loses mass over time. Equations of motion were derived using Meshchersky transformations, and the stability of these points was analyzed. In Abouelmagd and Mostafa [16], within the framework of the restricted three-body problem with variable masses, the motion of celestial bodies where the third body changes its mass according to Jeans' law is investigated. In Abouelmagd et al. [17], a new model arising from the restricted three-body problem with anisotropic mass changes is considered. This mass change occurs when mass leaves or enters the system from points on the infinitesimally small body. The model has interesting applications for studying small objects such as cosmic dust, as well as for future space colonization and parking of spacecraft. In Celletti and Vartolomei [18], classical perturbation methods are applied to the dynamics of space debris, which can be considered a small body within the restricted three-body problem framework. Mass variability occurs naturally in many celestial bodies, such as stars, asteroids, comets, or artificial satellites, which lose or gain mass during their motion [19]. Accounting for mass variability can significantly alter orbital trajectories [20], especially when reactive forces are involved. When considering a system with variable masses, especially in the context of the three-body problem, it is important to account for how mass changes influence the equilibrium points and orbits of the system. In Huda et al. [21], equilibrium points were studied in a modified restricted three-body problem where one of the primary masses is an elongated body. It was found that parameter changes, including variable mass, shift the positions of equilibrium points and affect their stability.

There are many circumbinary planetary systems in the NASA database, of which more than 250 systems consist of two stars and one planet NASA [22]. The dynamical evolution of a system with two stars and one planet of variable masses can be studied in the framework of the restricted three-body problem with variable masses. In Minesaki [23], the restricted three-body problem is considered, where two massive points and a massless point attract each other according to Newton's law, and the Hill regions pulsate. In these regions, the massless point moves inside closed regions surrounding only one of the massive points. The work of Langford and Loren [24] is dedicated to studying the dynamics of planets orbiting close binary stars within the framework of the restricted three-body problem.

In this paper we study the problem of translational-rotational motion of a non-stationary axisymmetric small body in the gravitational field of two spherical bodies of variable mass in the framework of the restricted three-body problem.

## 2 The physical statement of the problem and assumptions

We considered the motion of three bodies $P_1$, $P_2$ and $P_3$ with variable masses $m_1 = m_1(t)$, $m_2 = m_2(t)$, $m_3 = m_3(t)$ and varying characteristic dimensions $l_1 = l_1(t)$, $l_2 = l_2(t)$, $l_3 = l_3(t)$, interacting according to Newton's law. Of these, the first two bodies are primary massive spherically symmetric bodies of variable mass, whose motion is determined by the problem of two bodies with variable masses. The third small mass body is an axisymmetric body of variable mass, variable size, and variable compression.

The physical problems of such a variable mass body problem, in particular, can be

a) The translational-rotational motion of a planet in the Newtonian gravitational field of two stars

b) The translational-rotational motion of an asteroid in the Newtonian gravitational field of a star and a planet

c) The translational-rotational of an artificial celestial body in the Newtonian gravitational field of a planet and its natural satellite.

To obtain a mathematical model of such physical problems, we adopt the following assumptions:

1. Moments of inertia of the second order of the considered bodies are variable and known functions of time

$$A_i = A_i(t),\ B_i = B_i(t),\ C_i = C_i(t),\quad i = 1,\ 2,\ 3 \tag{1}$$

2. The problem is set in a restricted formulation

$$m_3 << m_1,\ m_3 << m_2,\ m_1 \ge m_2. \tag{2}$$

In other words, a non – stationary axisymmetric body of small mass does not affect the motion of two primary spherical bodies with variable masses.

3. Masses and characteristic dimensions of bodies known functions of time change with different specific rates

$$\frac{\dot{m}_1(t)}{m_1(t)} \ne \frac{\dot{m}_2(t)}{m_2(t)} \ne \frac{\dot{m}_3(t)}{m_3(t)},\quad \frac{\dot{l}_1(t)}{l_1(t)} \ne \frac{\dot{l}_2(t)}{l_2(t)} \ne \frac{\dot{l}_3(t)}{l_3(t)}. \tag{3}$$

4. Bodies $P_1$, $P_2$ are spherically symmetric

$$A_1(t) = B_1(t) = C_1(t),\quad A_2(t) = B_2(t) = C_2(t) \tag{4}$$

5. The axisymmetric shape of a small non-stationary body $P_3$ remains unchanged during evolution

$$A_3(t) = B_3(t) \neq C_3(t). \tag{5}$$

6. The small body has an equatorial plane of symmetry. Three mutually perpendicular planes passing through the center of inertia of an axisymmetric non-stationary body define its principal axes of inertia.

7. The masses of the bodies vary both isotropically and non-isotropically. In the case of non-isotropic vary of masses, reactive forces arise. The total reactive forces in case of non-isotropic change of masses of bodies are not equal to zero and are applied to the center of inertia of the corresponding bodies. Then additional reactive moments are equal to zero:

$$\vec{F}_{reac} \neq 0, \qquad \vec{M}_{reac} = 0. \tag{6}$$

8. Let's limit to an approximate expression of the second zonal harmonic force function

$$U = U_{12} + U_{23} + U_{31} \tag{7}$$

$$U_{12} = f\left(\frac{m_1 m_2}{R_{12}}\right), \tag{8}$$

$$U_{23} = f\left(\frac{m_2 m_3}{R_{23}}\right) + f m_3 \left(C_3 - A_3\right)\frac{1-3\gamma_{23}^2}{2R_{23}^3},$$

$$U_{31} = f\left(\frac{m_1 m_3}{R_{31}}\right) + f m_3 \left(C_3 - A_3\right)\frac{1-3\gamma_{31}^2}{2R_{31}^3}. \tag{9}$$

Under such celestial-mechanical assumptions, the formulation of a new problem is formulated. At the same time, the laws of mass change and moments of inertia uniquely determine the evolution of these non-stationary bodies and additional internal degrees of freedom do not appear. The translational motion of the center of mass of two primary bodies with variable masses and the translational-rotational motion of a non – stationary axisymmetric body in joint consideration are studied.

Note that under such assumptions, the translational-rotational motion of a non-stationary axisymmetric body in a central non-stationary field of attraction was considered in Bizhanova et al. [25] within the framework of the two-body problem.

Similar assumptions for three non-stationary axisymmetric bodies were used in Minglibayev and Kushekbay [26] to study the secular evolution in the three-body problem in the unrestricted formulation.

However, in contrast to Bizhanova et al. [25], Minglibayev and Kushekbay [26] in this paper we have considered the translational-rotational motion of a non-stationary axisymmetric body in a non-stationary gravitational field of two spherical bodies in the framework of the restricted three-body problem.

### 3 Equations of translational-rotational motion of the center of mass in absolute coordinate system

In Ibraimova and Minglibayev [29], the equations of translational motion of two spherically symmetric non-stationary bodies in the framework of the two-body problem with variable masses were derived from the generalised Meshchersky equations Meshchersky [27], Markeev [28]

$$m_1 \ddot{\vec{R}}_1 = grad_{\vec{R}_1} U_{12} + \vec{F}_{1reac}, \; U_{12} = f\left(\frac{m_1 m_2}{R_{12}}\right), \tag{10}$$

$$\vec{F}_{1reac} = \dot{m}_{11}\vec{V}_{11} + \dot{m}_{12}\vec{V}_{12},$$
$$\vec{V}_{11} = \vec{u}_{11} - \dot{\vec{R}}_1, \quad \vec{V}_{12} = \vec{u}_{12} - \dot{\vec{R}}_1 \tag{11}$$

$$m_1 = m_1(t_1) - |m_{11}| + m_{12} = \\ = m_1(t_0) - \int_{t_0}^{t} \left(|\dot{m}_{11}|\right) dt + \int_{t_0}^{t} \left(\dot{m}_{12}\right) dt \tag{12}$$

Here, the notation corresponds to the conventions adopted in Ibraimova and Minglibayev [29].

Under the assumptions (1)-(8), the equations of rotational motion in Euler variables of body $P_1$ in absolute coordinate system in the considered formulation are as follows

$$\frac{d}{dt}\left(A_1 p_1\right) = 0, \; \frac{d}{dt}\left(B_1 q_1\right) = 0, \\ \frac{d}{dt}\left(C_1 r_1\right) = 0, \tag{13}$$

$$p_1 = \dot{\psi}_1 \sin\theta_1 \sin\varphi_1 + \dot{\theta}_1 \cos\varphi_1,$$
$$q_1 = \dot{\psi}_1 \sin\theta_1 \cos\varphi_1 - \dot{\theta}_1 \sin\varphi_1, \quad (14)$$
$$r_1 = \dot{\psi}_1 \cos\theta_1 + \dot{\varphi}_1$$

Similarly, we obtained the translational-rotational motion of a non-stationary spherically symmetric body $P_3$

$$m_2 \ddot{\vec{R}}_2 = grad_{\vec{R}_2} U_{21} + \vec{F}_{2reac},$$
$$U_{21} = f\left(\frac{m_2 m_1}{R_{21}}\right), \quad (15)$$

$$\vec{F}_{2reac} = \dot{m}_{21}\vec{V}_{21} + \dot{m}_{22}\vec{V}_{22},$$
$$\vec{V}_{21} = \vec{u}_{21} - \dot{\vec{R}}_2, \qquad \vec{V}_{22} = \vec{u}_{22} - \dot{\vec{R}}_2 \quad (16)$$

$$m_2 = m_2(t_0) - |m_{21}| + m_{22} =$$
$$= m_2(t_0) - \int_{t_0}^{t} (|\dot{m}_{21}|) dt + \int_{t_0}^{t} (\dot{m}_{22}) dt \quad (17)$$

$$\frac{d}{dt}(A_2 p_2) = 0, \ \frac{d}{dt}(B_2 q_2) = 0,$$
$$\frac{d}{dt}(C_2 r_2) = 0, \quad (18)$$

$$p_2 = \dot{\psi}_2 \sin\theta_2 \sin\varphi_2 + \dot{\theta}_2 \cos\varphi_2,$$
$$q_2 = \dot{\psi}_2 \sin\theta_2 \cos\varphi_2 - \dot{\theta}_2 \sin\varphi_2, \quad (19)$$
$$r_2 = \dot{\psi}_2 \cos\theta_2 + \dot{\varphi}_2$$

The translational-rotational motion of a non-stationary axisymmetric small body $P_3$, accordingly, had the form

$$m_3 \ddot{\vec{R}}_3 = grad_{\vec{R}_3} \tilde{U}_3 + \vec{F}_{3reac} \quad (20)$$

$$\tilde{U}_3 = f m_3 \left(\frac{m_1}{R_{31}} + \frac{m_2}{R_{32}}\right) +$$
$$+ f m_3 (C_3 - A_3) \frac{1}{2} \left(\frac{1 - 3\gamma_{23}^2}{R_{23}^3} + \frac{1 - 3\gamma_{31}^2}{R_{31}^3}\right), \quad (21)$$

$$\vec{F}_{3reac} = \dot{m}_{31}\vec{V}_{31} + \dot{m}_{32}\vec{V}_{32},$$
$$\vec{V}_{31} = \vec{u}_{31} - \dot{\vec{R}}_3, \qquad \vec{V}_{32} = \vec{u}_{32} - \dot{\vec{R}}_3 \quad (22)$$

$$m_3 = m_3(t_0) - |m_{31}| + m_{32} =$$
$$m_3(t_0) - \int_{t_0}^{t} (|\dot{m}_{31}|) dt + \int_{t_0}^{t} (\dot{m}_{32}) dt \quad (23)$$

$$\frac{d}{dt}(A_3 p_3) - (A_3 - C_3) q_3 r_3 =$$
$$= \left[\frac{\partial U}{\partial \psi_3} - \cos\theta_3 \frac{\partial U}{\partial \varphi_3}\right] \frac{\sin\varphi_3}{\sin\theta_3} + \cos\varphi_3 \frac{\partial U}{\partial \theta_3},$$
$$\frac{d}{dt}(A_3 q_3) - (C_3 - A_3) r_3 p_3 = \quad (24)$$
$$= \left[\frac{\partial U}{\partial \psi_3} - \cos\theta_3 \frac{\partial U}{\partial \varphi_3}\right] \frac{\cos\varphi_3}{\sin\theta_3} - \sin\varphi_3 \frac{\partial U}{\partial \theta_3},$$
$$\frac{d}{dt}(C_3 r_3) = 0.$$

$$p_3 = \dot{\psi}_3 \sin\theta_3 \sin\varphi_3 + \dot{\theta}_3 \cos\varphi_3,$$
$$q_3 = \dot{\psi}_3 \sin\theta_3 \cos\varphi_3 - \dot{\theta}_3 \sin\varphi_3, \quad (25)$$
$$r_3 = \dot{\psi}_3 \cos\theta_3 + \dot{\varphi}_3$$

$\vec{R}_j$ is the radius vector of the center of mass of the bodies, $\vec{R}_{ij}$ are the mutual distances between the centers of mass of the bodies, $f$ is the gravitational constant.

For further convenience, we have rewritten the equations of translational motion of the three bodies under consideration in the form

$$\ddot{\vec{R}}_1 = grad_{\vec{R}_1} U_{12} + \frac{1}{m_1}\vec{F}_{1reac}, \ U_{12} = f\left(\frac{m_2}{R_{12}}\right), \quad (26)$$

$$\ddot{\vec{R}}_2 = grad_{\vec{R}_2} U_{21} + \frac{1}{m_2}\vec{F}_{2reac}, \ U_{21} = f\left(\frac{m_1}{R_{21}}\right), \quad (27)$$

$$\ddot{\vec{R}}_3 = grad_{\vec{R}_3} \tilde{U}^* + \frac{1}{m_3}\vec{F}_{3reac},$$
$$\tilde{U}^* = f\left(\frac{m_1}{R_{31}} + \frac{m_2}{R_{32}}\right) +$$
$$+ f (C_3 - A_3) \frac{1}{2} \left(\frac{1 - 3\gamma_{23}^2}{R_{23}^3} + \frac{1 - 3\gamma_{31}^2}{R_{31}^3}\right), \quad (28)$$

Equations (26)-(28) describe the problem under consideration in the absolute coordinate system.

We will assume that the reactive forces are applied at the centers of the respective non-stationary bodies, as noted by Lukyanov [30].

## 4 Equations of translational-rotational motion in relative coordinate system

Proceeding from the equations of motion in the absolute coordinate system (26)-(28), we obtain the equation of motion in the relative coordinate system with the origin at the center of the more massive body $P_1$. The equation of motion of the two primary bodies problem has the form

$$\ddot{\vec{R}}_{12} = grad_{\vec{R}_{12}} \tilde{U}_{12} + \frac{1}{m_2}\vec{F}_{2reac} - \frac{1}{m_1}\vec{F}_{1reac} \quad ,$$

$$\tilde{U}_{12} = f\frac{m_1+m_2}{R_{12}} \tag{29}$$

The rotational motion of the two primary spherical bodies is determined from simple equations (13)-(14) and (18)-(19), therefore, in the future, the rotational motion of the two primary spherical bodies will be considered known.

In relative coordinates, the equations of motion of a body with a small mass, in the field of attraction of two primary bodies, can be written in the following form

$$\ddot{\vec{R}}_{13} = grad_{\vec{R}_{13}} \tilde{U}^* + \frac{1}{m_3}\vec{F}_{3reac} - \frac{1}{m_1}\vec{F}_{1reac}, \tag{30}$$

$$\vec{R}_{12} = \vec{R}_2 - \vec{R}_1, \qquad \vec{R}_{13} = \vec{R}_3 - \vec{R}_1 . \tag{31}$$

The rotational motion of a non-stationary axisymmetric body $P_3$ is determined by equations (24)-(25), but they now refer to a relative coordinate system.

## 5 Equations of motion in the barycentric coordinate system

### *5.1 Determination of the barycenter motion in the absolute coordinate system*

Let us denote the radius of the vector of the three bodies under consideration in the barycentric coordinate system by $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ , respectively. In order to determine the motion of the barycenter, we need to know the relative motion of the two primary spherical bodies with variable masses in the relative coordinate system.

The relative equation of motion of the two-body problem is defined by equations (31), which we will rewrite in the form of

$$\ddot{\vec{r}}_{12} = -f\left(m_1+m_2\right)\frac{\vec{r}_{12}}{r_{12}^3} + \vec{F}_{12reac},$$

$$\vec{r}_{12} = \vec{R}_2 - \vec{R}_1 = \vec{R}_{12} . \tag{32}$$

The motion of the barycenter is determined by the known formula [1], [5]

$$\vec{R} = \frac{m_1\vec{R}_1 + m_2\vec{R}_2}{m_1+m_2} = \frac{m_1}{m_1+m_2}\vec{R}_1 + \\ + \frac{m_2}{m_1+m_2}\vec{R}_2 = \nu_1\vec{R}_1 + \nu_2\vec{R}_2, \tag{33}$$

$$\nu_2 = \frac{m_2}{m_1+m_2} = \frac{m_2}{m},\ \nu_1 = \frac{m_1}{m_1+m_2} = \frac{m_1}{m},$$

$$m = m_1 + m_2 \tag{34}$$

Differentiating twice the expressions (33), we obtain

$$\ddot{\vec{R}} = \ddot{\nu}_1\vec{R}_1 + 2\dot{\nu}_1\dot{\vec{R}}_1 + \nu_1\ddot{\vec{R}}_1 + \\ + \ddot{\nu}_2\vec{R}_2 + 2\dot{\nu}_2\dot{\vec{R}}_2 + \nu_2\ddot{\vec{R}}_2. \tag{35}$$

In equation (35) we substitute formulas (26), (27) and considering $\vec{R}_1 = \vec{R} - \nu_2\vec{r}_{12}$, $\vec{R}_2 = \vec{R} + \nu_1\vec{r}_{12}$, we obtain

$$\ddot{\vec{R}} = \ddot{\nu}_1\left[\vec{R} - \nu_2\vec{r}_{12}\right] + 2\dot{\nu}_1\left[\dot{\vec{R}} - \dot{\nu}_2\vec{r}_{12} - \nu_2\dot{\vec{r}}_{12}\right] + \\ + \nu_1\left[fm_2\frac{\vec{r}_{12}}{r_{12}^3} + \frac{1}{m_1}\vec{F}_{1reac}\right] + \ddot{\nu}_2\left[\vec{R} + \nu_1\vec{r}_{12}\right] + \\ + 2\dot{\nu}_2\left[\dot{\vec{R}} + \dot{\nu}_1\vec{r}_{12} + \nu_1\dot{\vec{r}}_{12}\right] + \nu_2\left[-fm_1\frac{\vec{r}_{12}}{r_{12}^3} + \frac{1}{m_2}\vec{F}_{2reac}\right]. \tag{36}$$

Grouping the coefficients with the same values, we obtain

$$\begin{aligned}\ddot{\vec{R}} &= \left(\ddot{\nu}_1+\ddot{\nu}_2\right)\vec{R}+\left(\ddot{\nu}_2\nu_1-\ddot{\nu}_1\nu_2\right)\vec{r}_{12}+\\ &+2\left(\dot{\nu}_1+\dot{\nu}_2\right)\dot{\vec{R}}+2\left(\dot{\nu}_2\dot{\nu}_1-\dot{\nu}_1\dot{\nu}_2\right)\vec{r}_{12}+\\ &+2\left(\dot{\nu}_2\nu_1-\dot{\nu}_1\nu_2\right)\dot{\vec{r}}_{12}+f\left(\nu_1 m_2-\nu_2 m_1\right)\frac{\vec{r}_{12}}{r_{12}^3}+\\ &+\frac{\nu_1}{m_1}\vec{F}_{1реак}+\frac{\nu_2}{m_2}\vec{F}_{2реак},\end{aligned} \tag{37}$$

In the right part of the equation (37) the expressions in brackets after calculations are considerably simplified

$$\begin{aligned}\dot{\nu}_1+\dot{\nu}_2 &= \frac{d}{dt}\left(\nu_1+\nu_2\right)=\frac{d}{dt}\left(\frac{m_1}{m}+\frac{m_2}{m}\right)=\\ &=\frac{d}{dt}\left(\frac{m}{m}\right)=\frac{d}{dt}(1)=0,\\ \ddot{\nu}_1+\ddot{\nu}_2 &= \frac{d^2}{dt^2}\left(\nu_1+\nu_2\right)=\frac{d^2}{dt^2}(1)=0,\\ \nu_1 m_2-\nu_2 m_1 &= \frac{m_1}{m}m_2-\frac{m_2}{m}m_1=0,\end{aligned} \tag{38}$$

$$\dot{\nu}_2\nu_1-\dot{\nu}_1\nu_2=\frac{m_1\dot{m}_2-m_2\dot{m}_1}{m^2}=A, \tag{39}$$

$$\begin{aligned}\ddot{\nu}_2\nu_1-\ddot{\nu}_1\nu_2 &= \frac{1}{m^2}\left[m_1\ddot{m}_2-m_2\ddot{m}_1\right]+\\ &+\frac{2}{m^3}\left[m_2\dot{m}_1-m_1\dot{m}_2\right]\dot{m}=B.\end{aligned} \tag{40}$$

Then finally we obtain the equation of motion of the barycenter of two primary spherical bodies with variable masses in the form

$$\ddot{\vec{R}}=B\vec{r}_{12}+2A\dot{\vec{r}}_{12}+\frac{1}{m}\left[\vec{F}_{1reac}+\vec{F}_{2reac}\right], \tag{41}$$

***5.2 Equation of translational-rotational motion of two primary bodies and small body with variable masses in barycentric coordinate system***

Considering that in the barycentric coordinate system

$$m_1\vec{r}_1+m_2\vec{r}_2=0,\ \vec{r}_{12}=\vec{r}_2-\vec{r}_1 \tag{42}$$

we obtain

$$\vec{r}_1=-\nu_2\vec{r}_{12},\ \vec{r}_2=\nu_1\vec{r}_{12}, \tag{43}$$

Differentiating formulas (43), we obtain equations of motion of two primary bodies $P_1$ and $P_2$ in barycentric coordinate system

$$\ddot{\vec{r}}_1=-fm{\nu_2}^3\frac{\vec{r}_1}{r_1^3}+A_1\dot{\vec{r}}_1+B_1\vec{r}_1+\vec{F}_{12reac}. \tag{44}$$

$$A_1=2\frac{\dot{\nu}_2}{\nu_2},\ B_1=\left(\frac{\ddot{\nu}_2}{\nu_2}-2\frac{\dot{\nu}_2^{\,2}}{\nu_2^{\,2}}\right),$$

$$\vec{F}_{12reac}=\frac{\nu_2}{m_1}\vec{F}_{1reac}-\frac{1}{m}\vec{F}_{2reac}$$

$$\ddot{\vec{r}}_2=-fm{\nu_1}^3\frac{\vec{r}_2}{r_2^3}+A_2\dot{\vec{r}}_2+B_2\vec{r}_2+\vec{F}_{21reac}. \tag{45}$$

$$A_2=2\frac{\dot{\nu}_1}{\nu_1},\ B_2=\left(\frac{\ddot{\nu}_1}{\nu_1}-2\frac{\dot{\nu}_1^{\,2}}{\nu_1^{\,2}}\right),$$

$$\vec{F}_{21reac}=\frac{\nu_1}{m_2}\vec{F}_{2reac}-\frac{1}{m}\vec{F}_{1reac}$$

The equations of a small non-stationary axisymmetric body in the barycentric coordinate system are obtained from the following relations

$$\vec{r}_3=\vec{R}_3-\vec{R},\ \ddot{\vec{r}}_3=\ddot{\vec{R}}_3-\ddot{\vec{R}}\ . \tag{46}$$

Considering the differential equation of motion of a small body in absolute coordinate system (28) and the derived equation of motion of the barycenter of two primary bodies (41) we obtain

$$\begin{aligned}\ddot{\vec{r}}_3 &= grad_{\vec{R}_3}\tilde{U}^*+\frac{1}{m_3}\vec{F}_{3reac}-\\ &-\left(B\vec{r}_{12}+2A\dot{\vec{r}}_{12}+\frac{1}{m}\left[\vec{F}_{1reac}+\vec{F}_{2reac}\right]\right)\end{aligned} \tag{47}$$

Equations (47) describe the motion of a non – stationary axisymmetric small body in a barycentric coordinate system in the Newtonian gravitational field of two primary spherically symmetric bodies with variable masses.

The equations of rotational motion of the body $P_3$ around the center of mass in the absolute coordinate system retain the form (24)-(25), but these equations already refer to the barycentric coordinate system.

## 6 Two different systems of differential equations of the considered problem in barycentric coordinate system

Note that the obtained two primary body equations (44)-(45) and the equations of motion of an axisymmetric non – stationary small body (47) are not independent. By the center-of-mass invariant (42), it is sufficient to consider one of the two equations (44)-(45) together with equation (47). Below we will write them in the form of a system of joint differential equations.

### *6.1 System of differential equations of motion of bodies $P_1$ and $P_3$ in barycentric coordinate system*

From equations (44) and (47), considering (43), we obtain the following system of independent equations

$$\ddot{\vec{r}}_1 = -fm\nu_2 \frac{\vec{r}_1}{r_1^3} + A_1\dot{\vec{r}}_1 + B_1\vec{r}_1 + \vec{F}_{12reac}. \quad (48)$$

$$\ddot{\vec{r}}_3 = grad_{\vec{r}_3}\tilde{U}^* + A_{13}\dot{\vec{r}}_1 + B_{13}\vec{r}_1 + \vec{F}_{31reac} \quad (49)$$

$$A_{13} = \frac{2A}{\nu_2}, \quad B_{13} = \frac{B}{\nu_2} - 4A\frac{\dot{\nu}_2}{\nu_2^2},$$
$$\vec{F}_{31reac} = \frac{1}{m_3}\vec{F}_{3reac} - \frac{1}{m}\left[\vec{F}_{1reac} + \vec{F}_{2reac}\right], \quad (50)$$

Note that in the case of constant masses the obtained differential equations of motion transform into the known equations obtained by other authors Krasilnikov [31]. If all three bodies are spherically symmetric and with constant masses, we come to the classical restricted three-body problem with constant masses Markeev [32], Szebehely [33].

### *6.2 System of differential equations of motion of bodies $P_2$ and $P_3$ in barycentric coordinate system*

To study the problem under consideration, instead of the system (45)-(47), we can study the following system of equations

$$\ddot{\vec{r}}_2 = -fm\nu_1 \frac{\vec{r}_2}{r_2^3} + A_2\dot{\vec{r}}_2 + B_2\vec{r}_2 + \vec{F}_{21reac}. \quad (51)$$

$$\ddot{\vec{r}}_3 = grad_{\vec{r}_3}\tilde{U}^* + A_{23}\dot{\vec{r}}_2 + B_{23}\vec{r}_2 + \vec{F}_{32reac} \quad (52)$$

$$A_{23} = -\frac{2A}{\nu_1}, \quad B_{23} = -\frac{B}{\nu_1} + 4A\frac{\dot{\nu}_1}{\nu_1^2},$$
$$\vec{F}_{32reac} = \frac{1}{m_3}\vec{F}_{3reac} - \frac{1}{m}\left[\vec{F}_{1reac} + \vec{F}_{2reac}\right], \quad (53)$$

## 7. Results and Discussion

In the present paper, we formulate a new celestial-mechanical problem when condition (1) – (8) are satisfied. Differential equations of the formulated problem in absolute, relative and barycentric coordinate systems are obtained for the first time. The equations of motion in the relative coordinate system are convenient for studying the considered problem in the case of an inner restricted problem. The equations of motion in the barycentric coordinate system can be effectively used in the study of the outer restricted problem of the considered problem.

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