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Long-term analysis of solutions with initial jumps in singularly perturbed equations

Abstract. This paper addresses a dilemma settled by contour stipulations involving a minor fluctuation related to a third-order linear integro-differential equation (IDE) that features a small parameter modifying the two top slopes. The study specifically examines cases in which the intercepts of the corresponding attribute equation are detrimental. The objective of this report is to provide approaching assessments for the outcome of a problem settled by contour stipulations involving a minor fluctuation with preliminary discontinuities, as well as to analyze the approaching convergence of the outcome of a preliminary value issue subjected to prominent fluctuation toward the outcome of a steady preliminary value issue. This manuscript constructs the primary structure of outcomes and introductory operations for a distinct modified consistent differential statement, while also deriving their eventual assessments. The approaching response of the outcome to the distinctively modified a dilemma settled by contour stipulations at the points of preliminary shifts is demonstrated. A degenerate dilemma settled by contour stipulations is constructed. It is demonstrated that the outcome of the preliminary outstandingly ruffled problem settled by contour stipulations approaches the outcome of the corresponding degenerate dilemma.

Key words: degenerate, contour, BVP, a third-order linear IDE.

1. Introduction

Equations that have a minor specification in the coefficients of the peak-order slopes are referred to as distinctively unsettled relations. Distinctively modified relations serve as a mathematical framework for various implemented (field-based) challenges.

The investigation of the distinctively modified an issue settled by contour stipulations was initiated by the inquiry of M. I. Vishik, L. A. Lyusternik [1], and K. A. Kassymov [2]. In [3-16], Kassymov and his students further investigated the preliminary and dual-point division issues involving preliminary disruptions. One can apply the method for solving a sharply tweaked obstacle if in a part of its domain the condition of the well-known Tikhonov theorem is valid. Sharply tweaked differential equations with a piecewise constant argument were examined in [12]. In [15] further investigated common cases of the Cauchy Problem (CP) with initial vaults. Various asymptotic methods can be found in [17]. For basic differential and IDE, a triple-point obstacle settled by contour stipulations are examined solely in cases involving a top-order slope with preliminary spikes manifesting [18-20].

In contrast to the earlier papers on this topic, the converging transition implemented to approach the original dilemma (1)-(2) led to both a simplified guise of the unaltered challenge and a modified version of it. In the study of certain sharply tweaked dilemmas, it is observed that near the contour of the realm, the fast variable of the outcome can become infinitely large as the minor parameter approaches zero. K.A. Kassymov further investigated common cases of the CP with initial vaults. A distinctive feature of such obstacles is that, as the minor parameter tends to zero, the solution converges to that of a degenerate equation with adjusted pioneering stipulations. This phenomenon is referred to as the "initial vault" of the outcome. Boundary value problem (BVP) for sharply tweaked IDE with initial vaults were studied in [21-28]. These studies revealed the phenomenon of "boundary vaults," where certain derivatives of the outcome become infinitely large at both ends of the interval for sufficiently minor parameter values. At the interval boundaries, vaults of varying orders were observed.

This work focuses on integral BVPs for sharply tweaked third-order linear IDE, where the outcome exhibits vault of the same order at the ends of a specified interval. The primary objective is to determine the asymptotic behavior of the outcome as the minor parameter approaches zero and to construct the corresponding modified degenerate obstacles. We develop a new concept through this altered fluctuation dilemma. This concept is referred to as the preliminary spikes in the integral summand. Hence, this challenge also features a preliminary spike in the integral summand. Ergo, the converging transition has been entrenched.

2. Statement of the problem and preliminaries

A third-order linear IDE with a subtle adjustment examine at the two-peak slopes

$$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_0(t)y'' + A_1(t)y' + A_2(t)y = \\ = F(t) + \int_0^1 \sum_{i=0}^{2} H_i(t,x)y^{(i)}(x,\varepsilon)dx \tag{1}$$

with the boundary stipulations

$$h_1 y \equiv y(0,\varepsilon) = \alpha,\ h_2 y \equiv y'(0,\varepsilon) = \beta, \\ h_3 y \equiv y(1,\varepsilon) = \gamma, \tag{2}$$

Where $\varepsilon > 0 -$ small parameter, $0 < t_0 < 1,$ and $\alpha, \beta, \gamma -$ known invariants.

Postulate for a moment that:

I. $A_i(t) \in C^2[0,1], i = \overline{0,2}, F(t) \in C[0,1]$

and $H_0(t,x), H_1(t,x) \in C(D)$

where $D = \{0 \le t \le 1, 0 \le x \le 1\}$.

II. $A_1(t) \ne 0, \quad 0 \le t \le 1$.

II. The intercepts of the identity

$$\mu^2 + A_0(t)\cdot\mu + A_1(t) = 0$$

adhere to the provisions

$$\mu_1(t) \ne \mu_2(t),\ \mu_1(t) < 0, \quad \mu_2(t) < 0.$$

Deliberate the succeeding isotropic identity concomitant alongside (1):

$$L_\varepsilon y \equiv \varepsilon^2 \cdot y''' + \varepsilon \cdot A_0(t)y'' + \\ + A_1(t)y' + A_2(t)y = 0 \tag{3}$$

Lemma. If provisions I-III are adhered to, therefore the indispensable bundle of outcomes of the (3) comprises as outlined convergent trend depiction as $\varepsilon \to 0$ [6]:

$$y_1^{(q)}(t,\varepsilon) = \frac{1}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx\right)\cdot \\ \cdot\left(\mu_1^q(t)y_{10}(t) + O(\varepsilon)\right), q = \overline{0,2} \\ y_2^{(q)}(t,\varepsilon) = \frac{1}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_0^t \mu_2(x)dx\right)\cdot \\ \cdot\left(\mu_2^q(t)y_{20}(t) + O(\varepsilon)\right), q = \overline{0,2} \\ y_3^{(q)}(t,\varepsilon) = y_{30}^{(q)}(t) + O(\varepsilon), q = \overline{0,2} \tag{4}$$

provided that $y_{30}(t) = \exp\left(-\int_0^t \frac{A_2(x)}{A_1(x)}dx\right)$, and the mappings $y_{i0}(t), i = 1,2$ are outcomes of the conundrums

$$a_i(t)y_{i0}'(t) + b_i(t)y_{i0}(t) = 0,\ y_{i0}(0) = 1, \quad i = 1,2,$$

assuming

$$a_i(t) = (A_0(t) + 2\mu_i(t))\mu_i(t); \\ b_i(t) = A_2(t) + A_0(t)\mu_i'(t) + 3\mu_i(t)\mu_i'(t).$$

Assemble core mappings:

$$K(t,s,\varepsilon) = \frac{W(t,s,\varepsilon)}{W(s,\varepsilon)} \tag{5}$$

provided that $W(s,\varepsilon)$ is the wronskian of the constituent aggregate of outcomes of (3), and $W(t,s,\varepsilon)$ is a determinant yielded through the $W(s,\varepsilon)$ by swapping its third tuples with the system of fundamental solutions (SFS).

Owing to (4), for the $K(t,s,\varepsilon)$ the ensuing converging portrayal clutches as $\varepsilon \to 0$:

$$K^{(q)}(t,s,\varepsilon) = \varepsilon^2 \left(\frac{y_{30}^{(q)}(t)}{y_{30}(s)\mu_1(s)\mu_2(s)} - \frac{\mu_1^q(t) y_{10}(t)}{\varepsilon^q y_{10}(s)\mu_1(s)\left(\mu_2(s)-\mu_1(s)\right)} \exp\left(\frac{1}{\varepsilon} \int_s^t \mu_1(x)dx \right) + \right.$$
$$\left. + \frac{\mu_2^q(t) y_{20}(t)}{\varepsilon^q y_{20}(s)\mu_2(s)\left(\mu_2(s)-\mu_1(s)\right)} \exp\left(\frac{1}{\varepsilon} \int_s^t \mu_2(x)dx \right) \right) + O(\varepsilon), q = \overline{0,2}. \tag{6}$$

Through (6) for the $K(t,s,\varepsilon)$ we fetch converging forecasts as $\varepsilon \to 0$:

$$\left| K^{(q)}(t,s,\varepsilon) \right| \le C\varepsilon^2 + \frac{C}{\varepsilon^{q-2}} e^{-\gamma \frac{t-s}{\varepsilon}}, \; q = 0,1,2. \tag{7}$$

whereby $C > 0, \gamma > 0$ are invariants unconstrained of ε. A recurring theme in the treatise is that for each obstacle we first establish the form of a uniformly valid vanishing expansion for its outcome. Then, knowing this form, we are able to proceed to calculate the appropriate inner and outer expansions from the differential equation (DE) for the obstacle.

3. Main results

Enable the $\Phi_i(t,\varepsilon), i = 1,2,3$ occur outcomes of the dilemma

$$L_\varepsilon \Phi_i(t,\varepsilon) = 0, \; h_k \Phi_i(t,\varepsilon) = \delta_{ki}, \; i,k = 1,2,3.$$

We convene these a boundary mappings and discern:

$$\Phi_i(t,\varepsilon) = \frac{I_i(t,\varepsilon)}{I(\varepsilon)}, \; i = 1,2,3, \tag{8}$$

where

$$I(\varepsilon) = \begin{vmatrix} h_1 y_1(t,\varepsilon) & h_1 y_2(t,\varepsilon) & h_1 y_3(t,\varepsilon) \\ h_2 y_1(t,\varepsilon) & h_2 y_2(t,\varepsilon) & h_2 y_3(t,\varepsilon) \\ h_3 y_1(t,\varepsilon) & h_3 y_2(t,\varepsilon) & h_3 y_3(t,\varepsilon) \end{vmatrix},$$

and $I_i(t,\varepsilon)$, ($i = 1,2,3$) are determinants fetched through the $I(\varepsilon)$ by overriding its i-th chains with the chain $y_1(t,\varepsilon), y_2(t,\varepsilon), y_3(t,\varepsilon)$.

With regard to (4), for the $\Phi_i(t,\varepsilon), i = 1,2,3$ we procure converging portrayals as $\varepsilon \to 0$:

$$\Phi_1^{(q)}(t,\varepsilon) = \frac{1}{\varepsilon^q} \frac{\mu_1^q(t)\mu_2(0) y_{10}(t)}{\mu_2(0)-\mu_1(0)} \exp\left(\frac{1}{\varepsilon} \int_0^t \mu_1(x)dx \right) - \frac{1}{\varepsilon^q} \frac{\mu_2^q(t)\mu_1(0) y_{20}(t)}{\mu_2(0)-\mu_1(0)} \exp\left(\frac{1}{\varepsilon} \int_0^t \mu_2(x)dx \right) +$$
$$+ O\left(\frac{1}{\varepsilon^{q-1}} \exp\left(\frac{1}{\varepsilon} \int_0^t \mu_1(x)dx \right) + \frac{1}{\varepsilon^{q-1}} \exp\left(\frac{1}{\varepsilon} \int_0^t \mu_2(x)dx \right) \right),$$

$$\Phi_2^{(q)}(t,\varepsilon) = \frac{1}{\varepsilon^{q-1}} \frac{\mu_2^q(t) y_{20}(t)}{\mu_2(0)-\mu_1(0)} \exp\left(\frac{1}{\varepsilon} \int_0^t \mu_2(x)dx \right) - \frac{\mu_1^q(t) y_{10}(t)}{\varepsilon^{q-1} \cdot (\mu_2(0)-\mu_1(0))} \exp\left(\frac{1}{\varepsilon} \int_0^t \mu_1(x)dx \right) +$$
$$+ O\left(\frac{1}{\varepsilon^{q-2}} \left(\exp\left(\frac{1}{\varepsilon} \int_0^t \mu_1(x)dx \right) + \exp\left(\int_0^t \mu_2(x)dx \right) \right) \right), \tag{9}$$

$$\Phi_3^{(q)}(t,\varepsilon)=\frac{1}{\varepsilon^q}\cdot\frac{\mu_2^q(t)\mu_1(0)y_{20}(t)}{(\mu_2(0)-\mu_1(0))y_{30}(1)}\cdot\exp\left(\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx\right)-$$
$$-\frac{1}{\varepsilon^q}\cdot\frac{\mu_1^q(t)\mu_2(0)y_{10}(t)}{(\mu_2(0)-\mu_1(0))y_{30}(1)}\cdot\exp\left(\frac{1}{\varepsilon}\int_0^t\mu_1(x)dx\right)+\frac{y_{30}^{(q)}(t)}{y_{30}(1)}+$$
$$+O\left(\frac{1}{\varepsilon^{q-1}}\left(\exp\left(\frac{1}{\varepsilon}\int_0^t\mu_1(x)dx\right)+\exp\left(-\frac{1}{\varepsilon}\int_t^1\mu_2(x)dx\right)\right)+\varepsilon\right)$$

Through (9) for the contour mappings $\Phi_i(t,\varepsilon), i=1,2,3$, we procure the ensuing converging forecasts as $\varepsilon\to 0$:

$$\left|\Phi_1^{(q)}(t,\varepsilon)\right|\le\frac{C}{\varepsilon^q}e^{-\gamma\frac{t}{\varepsilon}},\quad q=0,1,2,$$

$$\left|\Phi_2^{(q)}(t,\varepsilon)\right|\le\frac{C}{\varepsilon^{q-1}}e^{-\gamma\frac{t}{\varepsilon}},\quad q=0,1,2, \tag{10}$$
$$\left|\Phi_3^{(q)}(t,\varepsilon)\right|\le C+\frac{C}{\varepsilon^q}e^{-\gamma\frac{t}{\varepsilon}},\quad q=0,1,2.$$

$C>0,\gamma>0$ are invariants unconstrained of ε.

Enable us introduce the ensuing notation:

$$z(t,\varepsilon)=F(t)+\int_0^1\left[H_0(t,x)y(x,\varepsilon)+H_1(t,x)y'(x,\varepsilon)+H_2(t,x)y''(x,\varepsilon)\right]dx. \tag{11}$$

Henceforth the (1) takes the guise:

$$\begin{gathered}L_\varepsilon y\equiv\varepsilon^2\cdot y'''+\varepsilon\cdot A_1(t)y''+\\+A_2(t)y'+A_3(t)y=z(t,\varepsilon)\end{gathered} \tag{12}$$

We pursue an outcome of the (12) in the guise

$$\begin{gathered}y(t,\varepsilon)=C_1\Phi_1(t,\varepsilon)+C_2\Phi_2(t,\varepsilon)+\\+C_3\Phi_3(t,\varepsilon)+\frac{1}{\varepsilon^2}\int_0^t K(t,s,\varepsilon)z(s,\varepsilon)ds,\end{gathered} \tag{13}$$

$\Phi_i(t,\varepsilon), i=1,2,3-$ boundary mappings, and $K(t,s,\varepsilon)-$ initial mapping stipulated by the (5). $C_i, i=1,2,3-$ inconnu invariants, $z(t,\varepsilon)-$ inconnu.

Supplanting (13) into the (12) we fetch the ensuing 2nd type Fredholm integral equation corresponding to the $z(t,\varepsilon)$:

$$z(t,\varepsilon)=f(t,\varepsilon)+\int_0^1 H(t,s,\varepsilon)z(s,\varepsilon)ds, \tag{14}$$

here

$$\begin{gathered}f(t,\varepsilon)=F(t)+C_1\int_0^1\sum_{i=0}^2 H_i(t,x)\Phi_1^{(i)}(x,\varepsilon)dx+\\C_2\int_0^1\sum_{i=0}^2 H_i(t,x)\Phi_2^{(i)}(x,\varepsilon)dx+C_3\int_0^1\sum_{i=0}^2 H_i(t,x)\Phi_3^{(i)}(x,\varepsilon)dx,\end{gathered} \tag{15}$$
$$H(t,s,\varepsilon)=\frac{1}{\varepsilon^2}\int_s^1\sum_{i=0}^2 H_i(t,x)K^{(i)}(x,s,\varepsilon)dx.$$

IV. Infer that 1 is not an latent of the hub $H(t,s,\varepsilon)$.

Henceforth the outcome of the (14) is inimitably discerned in the ensuing guise:

$$z(t,\varepsilon)=f(t,\varepsilon)+\int_0^1 R(t,s,\varepsilon)f(s,\varepsilon)ds \;, \quad (16)$$

whereby $R(t,s,\varepsilon)$ is the remedy of the $H(t,s,\varepsilon)$. Supplanting (16) into (13) we procure:

$$y(t,\varepsilon)=\sum_{i=1}^{3}C_i\left[\Phi_i(t,\varepsilon)+\frac{1}{\varepsilon^2}\int_0^t K(t,s,\varepsilon)\overline{\phi_i}(s,\varepsilon)ds+\right.$$
$$\left.+\frac{1}{\varepsilon^2}\int_1^t K_1(t,s,\varepsilon)\overline{\phi_i}(s,\varepsilon)ds\right]+$$
$$+\frac{1}{\varepsilon^2}\int_0^t K_0(t,s,\varepsilon)\overline{F}(s,\varepsilon)ds+\frac{1}{\varepsilon^2}\int_1^t K_1(t,s,\varepsilon)\overline{F}(s,\varepsilon)ds. \quad (17)$$

We redraft (17) in the ensuing guise:

$$y(t,\varepsilon)=\sum_{i=1}^{3}C_iQ_i(t,\varepsilon)+P(t,\varepsilon). \quad (18)$$

where

$$Q_i(t,\varepsilon)=\Phi_i(t,\varepsilon)+\frac{1}{\varepsilon^2}\int_0^t K(t,s,\varepsilon)\overline{\phi_i}(s,\varepsilon)ds$$

$$P(t,\varepsilon)=\frac{1}{\varepsilon^2}\int_0^t K(t,s,\varepsilon)\overline{F}(s,\varepsilon)ds,$$

$$\overline{\varphi_i}(s,\varepsilon)=\int_0^1\sum_{j=0}^{2}\overline{H_j}(s,x,\varepsilon)\Phi_i^{(j)}(x,\varepsilon)dx,$$

$$\overline{H_j}(s,x,\varepsilon)\equiv H_j(s,x)+ \quad (19)$$

$$+\int_0^1 R(s,p,\varepsilon)H_i(p,x)dp=\overline{H}_j^0(s,x)+O(\varepsilon),$$

$$\overline{F}(s,\varepsilon)=F(s)+\int_0^1 R(s,p,\varepsilon)F(p)dp.$$

To spot the invariants C_i, $i=1,2,3$ we use the contour stipulation (2) and after that we procure the ensuing system of algebraic equation (AE):

$$\begin{cases} C_1Q_1(0,\varepsilon)+C_2Q_2(0,\varepsilon)+C_3Q_3(0,\varepsilon)=\alpha-P(0,\varepsilon),\\ C_1Q_1'(0,\varepsilon)+C_2Q_2'(0,\varepsilon)+C_3Q_3'(0,\varepsilon)=\beta-P'(0,\varepsilon),\\ C_1Q_1(1,\varepsilon)+C_2Q_2(1,\varepsilon)+C_3Q_3(1,\varepsilon)=\gamma-P(1,\varepsilon).\end{cases} \quad (20)$$

outcome of the system (20) can occur written in the guise:

$$C_1=\alpha, C_2=\beta, C_3=$$
$$=\frac{\gamma-\displaystyle\int_0^1\frac{y_{30}(1)\left(\overline{F}(\mathrm{s})+\alpha\overline{\varphi}_1(\mathrm{s})+\beta\overline{\varphi}_2(\mathrm{s})\right)}{y_{30}(\mathrm{s})\mu_1(\mathrm{s})\mu_2(\mathrm{s})}ds}{\overline{\Delta}}.$$

Enable $\Delta(\varepsilon)$ occur the main determinant of the (20) and the converging of this determinant is settled by:

$$\Delta(\varepsilon)=\overline{\Delta}+O(\varepsilon),$$

V. $\overline{\Delta}\equiv 1+\displaystyle\int_0^1\frac{y_{30}(1)\overline{\varphi_3}(s)}{y_{30}(s)A_1(s)}ds\neq 0$

When mathematically describing real processes with short-term fluctuations, it is possible to disregard these fluctuations and consider them as having an "instantaneous" nature. Such idealization necessitates the study of dynamical systems represented by discontinuous trajectories. Henceforth ensuing theorems clutch legitimate.

Theorem 1. Enable the assumpsions I-V occur legitimate, henceforth the solution of the problem settled by boundary stipulations (1)-(2) prevails, is nonpareil on the $[0,1]$, and can occur penning in the guise

$$y^{(q)}(t,\varepsilon)=\sum_{i=1}^{3}C_i(\varepsilon)Q_i^{(q)}(t,\varepsilon)+P^{(q)}(t,\varepsilon),$$
$$q=0,1,2, \tag{21}$$

where $Q_i^{(q)}(t,\varepsilon), i=1,2,3$, $P^{(q)}(t,\varepsilon)$ are the slope by t of the $Q_i(t,\varepsilon), i=1,2,3$, $P(t,\varepsilon)$ stipulated in (19) and $C_i(\varepsilon)$, $i=1,2,3$ are outcomes of the (20).

Theorem 2. Beneath stipulations I-V, for the outcome of the (1)-(2) the ensuing converging forecasts clutch as $\varepsilon\to 0$:

$$|y^{(q)}(t,\varepsilon)|\le C\left|\overline{F}(t)+\alpha\overline{\phi}_1(s)+\beta\overline{\phi}_2(s)\right|+\frac{C}{\varepsilon^{q-1}}\left|\frac{\mu_2^{q-2}-\mu_1^{q-2}}{\mu_2(0)-\mu_1(0)}\right|+$$
$$+\frac{C}{\varepsilon^{q}}e^{-\delta\frac{t}{\varepsilon}}\left|\frac{\mu_1^q(t)\mu_2(0)y_{10}(t)-\mu_2^q(t)\mu_1(0)y_{20}(t)}{\mu_2(0)-\mu_1(0)}\right| \tag{22}$$

$q=1,2$ and $C>0$ is invariant unconstrainedof ε.

Proof: Alongside using the converging forecasts (7), (10) of the $K(t,s,\varepsilon)$, $\Phi_i(t,\varepsilon), i=1,2,3$ We fetch the ensuing converging forecasts for the $Q_i(t,\varepsilon), i=1,2,3$ and $P(t,\varepsilon)$:

$$\left|Q_1^{(q)}(t,\varepsilon)\right|\le C+\frac{C}{\varepsilon^{q}}e^{-\gamma\frac{t}{\varepsilon}},\quad q=0,1,2,$$
$$\left|Q_2^{(q)}(t,\varepsilon)\right|\le C+\frac{C}{\varepsilon^{q-1}}e^{-\gamma\frac{t}{\varepsilon}},\quad q=0,1,2,$$
$$\left|Q_3^{(q)}(t,\varepsilon)\right|\le C+\frac{C}{\varepsilon^{q}}e^{-\gamma\frac{t}{\varepsilon}},\quad q=0,1,2. \tag{23}$$
$$\left|P^{(q)}(t,\varepsilon)\right|\le \max_{0\le t\le 1}|F(t)|\frac{C}{\varepsilon^{q-1}}e^{-\gamma\frac{t}{\varepsilon}},\quad q=0,1,2,$$

through (23) we procure (22). Alongside the Theorem 2, one can fetch

$$y(0,\varepsilon)=C, y'(0,\varepsilon)=C,$$
$$y''(0,\varepsilon)=O\left(\frac{1}{\varepsilon^2}\right),\varepsilon\to 0.$$

Henceforth at point $t=0$ there is a zero-order primary vault of the stipulated outcome. It conveys that, the outcome of the dilemma settled by contour stipulations (1), (2) has the manifestations of primary vaults of zero order at the point $t=0$.

We will ponder the ensuing degenerate problem (DP):

$$L_0\overline{y}=A_2(t)\overline{y}'+A_3(t)\overline{y}=F(t)+$$
$$+\int_0^1\sum_{i=0}^{2}H_i(t,x)y^{(i)}(x,\varepsilon)dx+\Delta(t), \tag{24}$$

$$\overline{y}(0)=\alpha,\quad \overline{y}(1)=\gamma. \tag{25}$$

Where $\Delta(t)$ and Δ_0 the primary vaults of the inconnu integral term and outcome, respectively.

Another significant aspect of this field is DE with impulsive effects. However, studying such obstacles from all perspectives is not considered novel, as these chalenges initially arose in nonlinear mechanics and attracted attention due to their ability to adequately describe processes in nonlinear undulating systems. One well-known specimen of such concerns is the clock model.

Theorem 3. Enable the stipulations I-V are satiated, henceforth for the inconsistency between the outcomes $y(t,\varepsilon)$ and $\overline{y}(t)$ of the singularly perturbed (SP) dilemma settled by contour stipulations (1)-(2) and the DP (24)-(25) ensuing converging estimate clutches as $\varepsilon\to 0$:

$$\left|y^{(q)}(t,\varepsilon)-\overline{y}^{(q)}(t)\right| \le \frac{Ce^{-\gamma_1\frac{t}{\varepsilon}}}{\varepsilon^q}\Big(\max\left|\Delta_0 H_1(t,0)-\Delta(t)\right|+\varepsilon(\left|\Delta_0\right|+\left|\gamma\right|+\max\left|\Delta_0 H_1(t,0)-\Delta(t)\right|)\Big)+$$
$$+\frac{Ce^{-\gamma_2\frac{1-t}{\varepsilon}}}{\varepsilon^{q-1}}(\left|\Delta_0\right|+\left|\gamma\right|+\max\left|\Delta_0 H_1(t,0)-\Delta(t)\right|)+$$
$$+C\varepsilon\Big(\max\left|\Delta_0 H_1(t,0)-\Delta(t)\right|(\varepsilon^{1-q}+1)+(\left|\Delta_0\right|+\left|\gamma\right|)\Big)$$

$q=1,2$ and $C>0$ is invariant unconstrained of ε.

Proof. We denote by $u(t,\varepsilon)=y(t,\varepsilon)-\overline{y}(t)$. Henceforth with regard to (24)-(25), we procure the ensuing problem correspondingly for the mapping $u(t,\varepsilon)$:

$$L_\varepsilon u \equiv \varepsilon^2 u''' + \varepsilon A_1(t)u'' + A_2(t)u' + A_3(t)u =$$
$$= -\varepsilon^2 \overline{y}''' - \varepsilon A_1(t)\overline{y}'' - \Delta(t) + \tag{26}$$
$$+\int_0^1 \sum_{i=0}^{2} H_i(t,x)u^{(i)}(x,\varepsilon)dx$$
$$u(0,\varepsilon)=0, \quad u(1,\varepsilon)=0,$$
$$u'(0,\varepsilon)=\beta-\overline{y}'(0)=-\Delta_0 \tag{27}$$

Since the type of problem (26)-(27) is the same type of problem as (1)-(2), we see the validity of theorem 3 by applying the converging estimation of (22) to the outcome $u(t,\varepsilon)$.

Now we designate the inconnu mapping $\Delta(t)$ to ensure that upon $\varepsilon\to 0$, henceforth outcome $u(t,\varepsilon)$ of the challenge (26)-(27) nurtures to zero, i.e.

$$\Delta(t)=\Delta_0 H_2(t,0), \tag{28}$$

we fetch that the outcome of the (1)-(2) nurtures to the outcome of the (26)-(27) without deviation when the impartiality (28) is executed.

If the stipulations I-V clutch legitimate, henceforth for the $y(t,\varepsilon)$ outcome of the (1)-(2) the ensuing contour impartialities clutch legitimate:

$$\lim_{\varepsilon\to 0} y'(t,\varepsilon)=\overline{y}'(t), \quad \lim_{\varepsilon\to 0} y''(t,\varepsilon)=\overline{y}''(t),$$

for $0\le t\le 1$.

4. Conclusion

This treatise investigates the BVP for sharply tweaked IDE characterized by boundary vaults, where the fast outcome variable becomes unbounded at both boundaries. It highlights the unique qualitative influence of integral terms on the asymptotic behavior of outcomes in such equations. The presence of integral terms substantially alters the degenerate equation: the outcome of the sharply tweaked IDE does not converge to the outcome of the standard degenerate equation, derived by setting the minor parameter to zero. Instead, it converges to a specially modified degenerate IDE with an additional term, referred to as the vault of the integral term. Notably, the calculation of this integral term vault differs from the approach in [27] due to the occurrence of outcome vaults of the same order. The AE of the outcome to (1) and (2) was obtained. The distinction of this problem from previously considered problems is clearly demonstrated. This distinction is immediately noticeable, as the transition to the preliminary dilemmas (1) and (2) involves not a simple unperturbed problem (UP) but a modified unperturbed issue. Through this modified UP, a new concept is introduced. This concept is referred to as the initial vault of the integral term. Therefore, in this problem, the integral term also has an initial vault. Thus, the limiting transition is shown in this way. The concept emerging from this modified unperturbed problem is described as the initial vault of the integral term. This terminology reflects the unique characteristics of the integral term within the problem. Specifically, the integral term in these equations exhibits an initial vault, a feature that is not commonly found in standard unperturbed problems. This innovation underscores the distinctive nature of the limiting transition employed in this analysis. Consequently, the limiting transition is effectively demonstrated through this unique approach.

Summarizing this work, we have unveiled that the outcome of (1)-(2) at point $t=0$ has an initial vault of zero order, i.e.

$$y(0,\varepsilon)=O(1),\quad y'(0,\varepsilon)=O(1),\quad y''(0,\varepsilon)=O\left(\frac{1}{\varepsilon^2}\right),\quad \varepsilon\to 0,$$
$$y(1,\varepsilon)=O(1),\quad y'(1,\varepsilon)=O\left(1\right),\quad y''(1,\varepsilon)=O\left(1\right),\quad \varepsilon\to 0$$

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