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The control problem associated with heating process of a rod

Abstract. This paper studies the boundary control problem for a one-dimensional heat transfer equation under periodic boundary conditions, which arise in practical scenarios such as the thermal regulation of cylindrical rods or ring-shaped domains. The main objective is to determine the optimal control function, prescribed at the boundary, that ensures the rod reaches a specified average temperature over time. By applying the method of separation of variables, the control problem is reduced to a Volterra integral equation of the first kind. As is well known, such equations are classically ill-posed and notoriously difficult to analyze. To address this, we derive the necessary estimates for the kernel of the integral equation and employ the Laplace transform method to establish the existence and admissibility of the control function within appropriate Sobolev spaces. Furthermore, we demonstrate the regularity properties of the solution. In addition, a concrete example is provided to illustrate how the control function can be explicitly constructed for specific parameter values. This result contributes to the broader understanding of boundary control in parabolic partial differential equations and offers potential applications in the optimal control of thermal processes.
Keywords: heat conduction equation, average temperature, periodic boundary condition, control problem, Volterra integral equation, Laplace transform.

1. Introduction

It is known that due to the widespread use of partial differential equations in physics and engineering, there is always a great interest in the study of boundary control problems. Therefore, in recent years, the control issues for heat transfer equations have been widely studied by many researchers.

It is well known that publications [1, 2] have examined preliminary findings on optimal time control problems for parabolic type PDEs. The control problem for a linear parabolic type equation in a one-dimensional domain with Robin boundary condition was studied in [3]. The control problem of parabolic type PDE equations in an infinite dimensional domain was first examined in [4].

The time-varying bang-bang property of time optimal controls for heat equation and its applications is studied in [5].

Early work on the control problem considered in our work is studied in detail in [6]. In [7], the boundary control problem for the heat equation with the Robin boundary condition is studied and developed a mathematical model of the heating process of a cylindrical domain. Important work in this field can be seen in articles [8, 9]. These works addressed control problems in one, two, and three dimensions. They also demonstrated the admissibility of the control by demonstrating the existence of an integral equation solution by the application of the Laplace transform method. Many optimal control problems for parabolic equations can be seen in [10-13].

In [14], Guo and Littman considered the null-boundary control problem for a semilinear heat equation with Dirichlet boundary condition in a one-dimensional bounded domain.

In [15], a mathematical model of thermocontrol processes was studied. The optimal time problem in boundary control of the heat conduction equation and its relationship to the "bang-bang" principle were examined in [16]. It was proved that the optimal time control with respect to arbitrary target temperature distribution in boundary control is “Bang-Bang”. In [17], the minimal time control problem for a linear heat equation with memory was considered. The purpose of such a problem is to find a control, which steers the solution of the heat equation with memory from a given initial state to a given target as soon as possible.

Many details regarding optimal control problems can be found in the monographs [18, 19]. In [20], a specific time optimum control problem with a closed ball centered at zero as the target was studied. The problem was governed by the internal controlled heat equation. Some practical problems of the control problem with different boundary conditions for the linear heat transfer equation were studied in [21].

We can see that the control problems for the heat equation in our work come down to solving the Volterra integral equation of the first kind. As a result of the research, it was found that when such a control problem is considered in pseudo-parabolic equations, the problem of finding control comes to Volterra integral equation of the second kind. We can see the control problems for the pseudo-parabolic equation in works [22, 23].

The current work examines the heat transfer equation's control problem with a periodic boundary condition. In the past, control problems using heat equations' Dirichlet, Neumann, and Robin boundary conditions were examined. The primary objective of this work is to determine the control function that is required to heat the rod to a specified average temperature. By applying the separation of variables method, we are able to solve the heat equation's initial boundary value problem. This reduces the control problem to the Volterra integral equation of the first kind. It is well known that the Volterra integral equation of the first kind is typically ill-posed and that it is not always simple to solve. For this, the required estimates for the integral equation's kernel were found, and the Laplace transform method was used to demonstrate the existence of the equation's solution. In the final part, a sample of finding a control function at a given parameter value is provided.

2. Statement of problem

In the present article, we consider the heat conduction equation

$$\frac{\partial u(x,t)}{\partial t} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x,t) \in \Omega := (-L, L) \times (0, \infty), \tag{1}$$

with boundary value conditions

$$u(-L,t) - u(L,t) = \nu(t), \quad t \geq 0, \tag{2}$$

and

$$u_x(-L,t) - u_x(L,t) = 0, \tag{3}$$

and initial condition

$$u(x,0) = 0, \quad -L \leq x \leq L, \tag{4}$$

where $L > 0$ denotes the length of the interval, $a > 0$ is the thermal conductivity coefficient, and $\nu(t)$ is the control function, which gives the flow amplitude.

In the next steps, we assume that the constant representing the coefficient of conductivity is $a = 1$.

Definition 2.1. *A control function $\nu : [0, \infty) \to \mathbb{R}$ is called an admissible if it belongs to the Sobolev space $W_2^1(\mathbb{R}_+)$, and satisfies the conditions $\nu(0) = 0$, $|\nu(t)| \leq 1$ for all $t \geq 0$.*

According to the properties of Sobolev spaces, such a function is continuous on the half-line $t \geq 0$.

We now consider the following control problem.

Control Problem. *Assume that the function $\phi(t)$ is given. Then, find the control function $\nu(t)$ from the following equation:*

$$\int_{-L}^{0} u(x,t,\nu(t))dx = \phi(t), \quad t \geq 0, \tag{5}$$

where $u(x,t,\nu(t))$ is the solution of the mixed problem (1)-(4) and it depends on the control function $\nu(t)$.

In the above control problem, the physical meaning of equation (5) refers to the average temperature in the $[-L, 0]$ section of the $2L$ long thin rod. Our main goal in this work is to show how the control function should be so that the average temperature in the rod is equal to $\phi(t)$.

For any $M > 0$, we denote by $W(M)$ the set of functions $\phi \in W_2^2(-\infty, +\infty)$, which satisfy the following conditions:

$$\|\phi\|_{W_2^2(\mathbb{R}_+)} \leq M, \quad \phi(t) = 0 \quad for \quad t \leq 0.$$

We now offer the primary theorem for demonstrating admissible control's existence.

Theorem 2.1. *There is a* $M>0$ *such that, for any function* $\phi\in W(M)$*, the solution* $\nu(t)$ *to equation (5) exists and satisfies the condition* $|\nu(t)|\le 1$.

We will consider the proof of Theorem 2.1 step by step in the next sections.

3. Integral equation for control function

This section contains the solutions to the initial boundary value problem and the spectral problem. Consequently, we possess the primary integral equation for determining the control function.

We now consider the spectral problem

$$X''(x)+\lambda X(x)=0,\quad x\in(-L,L),$$

with periodic boundary conditions

$$X(-L)=X(L),$$

and

$$X'(-L)=X'(L),\quad x\in[-L,L].$$

Then we have the following eigenvalues:

$$\lambda_k=\frac{\pi^2k^2}{L^2},\quad k\in\mathbb{N}_0=\mathbb{N}\cup\{0\},$$

and the eigenfunctions

$$X_{1k}=\sin\frac{k\pi}{L},\quad k\in\mathbb{N},$$

$$X_{2k}=\cos\frac{k\pi}{L},\quad k\in\mathbb{N}_0.$$

For an arbitrary Banch space B and for $T>0$ by the symbol $C\big([0,T]\to B\big)$ we denote the Banach space of all continuous maps $u:[0,T]\to B$ with the norm

$$\|u\|=\max_{0\le t\le T}|u(t)|.$$

By symbol $\tilde{W}_2^1(\Omega)$ we denote the subspace of the Sobolev space $W_2^1(\Omega)$ formed by functions, whose trace on $\partial\Omega$ is equal to zero. Note that due to the closure $\tilde{W}_2^1(\Omega)$ the sum of series of functions from $\tilde{W}_2^1(\Omega)$, converging in metric $W_2^1(\Omega)$ also belongs to $\tilde{W}_2^1(\Omega)$, where $\Omega:=\{x:-L<x<L\}$.

Definition 3.1. *By the solution of the initial boundary value problems (1)–(4) we mean a function* $u(x,t)$*, represented in the form*

$$u(x,t)=\nu(t)\frac{L-x}{2L}-w(x,t),\qquad(6)$$

where the function $w(x,t)$ *is a generalized solution from the class* $C\big([0,T]\to\tilde{W}_2^1(\Omega)\big)$ *of the following problem:*

$$w_t(x,t)-w_{xx}(x,t)=\frac{L-x}{2L}\nu'(t),$$

with periodic boundary conditions

$$w(-L,t)=w(L,t),\quad w_x(-L,t)=w_x(L,t),$$

and initial condition

$$w(x,0)=0.$$

We solve the solution of the above mixed problem by the Fourier method.

Thus, we obtain (see [24])

$$w(x,t)=\frac{1}{2}\nu(t)+\sum_{k=1}^{\infty}\frac{(-1)^k}{\pi k}\left(\int_0^t e^{-\lambda_k(t-s)}\nu'(s)ds\right)\sin\frac{\pi k}{L}x.\qquad(7)$$

Note that the class $C\big([0,T]\to\tilde{W}_2^1(\Omega)\big)$ is a subset of the class $W_2^{1,0}(\Omega)$, which was considered in monograph [25] for defining a solution to the problem homogeneous boundary conditions (see the corresponding uniqueness theorem in Ch. III, Theorem 3.2, pp 173-176). Therefore, the above introduced generalized solution is also a generalized solution in the sense of [25]. However, unlike a solution from the class $W_2^{1,0}(\Omega)$, which is guaranteed to have a trace for almost everywhere

$t \in [0,T]$, a solution from a class $C\left([0,T] \to \tilde{W}_2^1(\Omega)\right)$ continuously depends on $t \in [0,T]$ in the metric $L_2(\Omega)$.

Lemma 3.1. *Let* $\nu \in W_2^1(\mathbb{R}_+)$ *and* $\nu(0) = 0$. *Then the function*

$$u(x,t,\nu(t)) =$$
$$\frac{\pi}{L^2}\sum_{k=1}^{\infty}(-1)^k k\left(\int_0^t e^{-\lambda_k(t-s)}\nu(s)ds\right)\sin\frac{\pi k}{L}x, \quad (8)$$

is the solution of the initial-boundary problem (1)-(4).

Proof. Using (6) and (7), we rewrite the solution of problem (1)-(4) in the form

$$u(x,t,\nu(t)) = \frac{L-x}{2L}\nu(t) - \frac{1}{2}\nu(t) -$$
$$-\sum_{k=1}^{\infty}\frac{(-1)^k}{\pi k}\left(\int_0^t e^{-\lambda_k(t-s)}\nu'(s)ds\right)\sin\frac{\pi k}{L}x.$$

We will prove that function $w(x,t)$ represented by the indicated Fourier series, belongs to the class $C\left([0,T] \to \tilde{W}_2^1(\Omega)\right)$. It suffices to prove that gradient of this function, taken with respect to $x \in \Omega$, continuously depends on $t \in [0,T]$ on the norm of the space $L_2(\Omega)$. According to Parseval's equality, the norm of this gradient is equal to

$$\|w_x(\cdot,t)\|^2_{L_2[-L,L]} = \frac{1}{L^2}\sum_{k=1}^{\infty}\left(\int_0^t e^{-\lambda_k(t-s)}\nu'(s)ds\right)^2 \le$$
$$\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\|\nu'\|^2_{L_2} = \frac{1}{6}\|\nu'\|^2_{L_2}.$$

The fact that the function $w(x,t)$ is a generalized solution in the sense of the integral identity (3.5) of monograph [25] immediately follows from Parseval's equality.

Lemma is proved.

Using the condition (5) and the solution of the mixed problem (8), we can write

$$\phi(t) = \frac{1}{L}\sum_{k=1}^{\infty}(1-(-1)^k)\int_0^t e^{-\lambda_k(t-s)}\nu(s)ds, \quad (9)$$

where $\lambda_k = \dfrac{\pi^2 k^2}{L^2}$.

Let us introduce the function

$$B(t) = \sum_{k=1}^{\infty}\beta_k e^{-\lambda_k t}, \quad t > 0, \quad (10)$$

where β_k is defined as

$$\beta_k = \frac{1-(-1)^k}{L}.$$

Then equality (9) takes the form

$$\int_0^t B(t-s)\nu(s)ds = \phi(t), \quad t > 0. \quad (11)$$

The resulting Volterra integral equation (11) is the main equation for admissible control $\nu(t)$.

Lemma 3.2. *The following estimate holds for the kernel* $B(t)$*:*

$$0 < B(t) \le \frac{C}{\sqrt{t}}, \quad 0 < t \le 1,$$

where function $B(t)$ *is defined by (10).*

Proof. For any $q > 0$ consider the following relations:

$$\sum_{n=1}^{\infty}e^{-qn^2} = \sum_{n=1}^{\infty}\int_n^{n+1}e^{-q[s]^2}ds = \int_1^{\infty}e^{-q[s]^2}ds =$$
$$= \int_1^{\infty}e^{-qs^2}e^{q(s^2-[s]^2)}ds,$$

where $[s]$ is an integer part of s, and $s \ge 1$.

It is clear that

$$e^{q(s^2-[s]^2)} = e^{q(s-[s])(s+[s])} \le e^{2qs}.$$

Then we obtain the estimate

$$\int_1^\infty e^{-qs^2} e^{q(s^2-[s]^2)} ds \le \int_1^\infty e^{-qs^2+2qs} ds = e^q \int_1^\infty e^{-q(s-1)^2} ds.$$

Hence, for $0 < q \le \text{const}$ we get

$$\sum_{n=1}^\infty e^{-qn^2} \le \int_1^\infty e^{-qs^2} e^{q(s^2-[s]^2)} ds \le$$
$$e^q \int_0^\infty e^{-qs^2} ds \le \frac{C}{\sqrt{q}}. \qquad (12)$$

As is well known,

$$\beta_k = \frac{1-(-1)^k}{L} \le \frac{2}{L}. \qquad (13)$$

Thus, we obtain the following estimate by using estimates (12) and (13):

$$0 < B(t) = \sum_{k=1}^\infty \beta_k e^{-\lambda_k t} \le \frac{2}{L}\sum_{k=1}^\infty e^{-\lambda_k t} \le \frac{C}{\sqrt{t}}.$$

Lemma 3.2 is proved.

4. Proof of Theorem 2.1

It o establish the existence of a solution to equation (5), and thus prove Theorem 2.1, we must first show that the Volterra integral equation of the first kind (11) admits an admissible solution. In this section, we provide a detailed analysis of Volterra integral equation (11) and prove the existence of its solution.

For the function $\nu(t)$, we know that its Laplace transform is

$$\tilde{\nu}(p) = \int_0^\infty e^{-pt} \nu(t) dt,$$

where $p = \sigma + i\zeta$, $\sigma > 0$, $\zeta \in \mathbb{R}$.

After that, by applying the Laplace transform to the integral equation (11), we get

$$\tilde{\phi}(p) = \int_0^\infty e^{-pt} \int_0^t B(t-s)\nu(s) ds dt = \tilde{B}(p)\tilde{\nu}(p).$$

Then we can write

$$\tilde{\nu}(p) = \frac{\tilde{\phi}(p)}{\tilde{B}(p)},$$

and for $p = \sigma + i\zeta$, we get

$$\nu(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\tilde{\phi}(p)}{\tilde{B}(p)} e^{pt} dp =$$
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{\phi}(\sigma+i\zeta)}{\tilde{B}(\sigma+i\zeta)} e^{(\sigma+i\zeta)t} d\zeta. \qquad (14)$$

Lemma 4.1. *For the Laplace transform of the function $B(t)$, the following estimate holds:*

$$|\tilde{B}(\sigma+i\zeta)| \ge \frac{C_\sigma}{\sqrt{1+\zeta^2}}, \quad \sigma > 0, \quad \zeta \in \mathbb{R},$$

where $C_\sigma > 0$ is a constant only depending on σ.

Proof. We can express the Laplace transform of the function $B(t)$ as follows

$$\tilde{B}(p) = \int_0^\infty B(t) e^{-pt} dt = \sum_{k=1}^\infty \beta_k \int_0^\infty e^{-(p+\lambda_k)t} dt =$$
$$\sum_{k=1}^\infty \frac{\beta_k}{p+\lambda_k}.$$

Then, when $p = \sigma + i\zeta$ in the above equality, we can write

$$\tilde{B}(\sigma+i\zeta) = \sum_{k=1}^\infty \frac{\beta_k}{\sigma+\lambda_k+i\zeta} =$$
$$= \sum_{k=1}^\infty \frac{\beta_k(\sigma+\lambda_k)}{(\sigma+\lambda_k)^2+\zeta^2} - i\zeta \sum_{k=1}^\infty \frac{\beta_k}{(\sigma+\lambda_k)^2+\zeta^2} =$$
$$Re\tilde{B}(\sigma+i\zeta) + i Im\tilde{B}(\sigma+i\zeta),$$

where $Re\tilde{B}(\sigma+i\zeta)$ and $\operatorname{Im}\tilde{B}(\sigma+i\zeta)$ are as follows

$$Re\tilde{B}(\sigma+i\zeta) = \sum_{k=1}^\infty \frac{\beta_k(\sigma+\lambda_k)}{(\sigma+\lambda_k)^2+\zeta^2},$$

and

$$Im\widetilde{B}(\sigma+i\zeta)=-\zeta\sum_{k=1}^{\infty}\frac{\beta_k}{(\sigma+\lambda_k)^2+\zeta^2}.$$

We can see that the following inequality holds

$$(\sigma+\lambda_k)^2+\zeta^2\le((\sigma+\lambda_k)^2+1)(1+\zeta^2).$$

Therefore, we obtain

$$\frac{1}{(\sigma+\lambda_k)^2+\zeta^2}\ge\frac{1}{1+\zeta^2}\frac{1}{(\sigma+\lambda_k)^2+1}. \quad (15)$$

Then, using the inequality (15), we obtain the following estimates:

$$|Re\widetilde{B}(\sigma+i\zeta)|=\sum_{k=1}^{\infty}\frac{\beta_k(\sigma+\lambda_k)}{(\sigma+\lambda_k)^2+\zeta^2}\ge$$
$$\ge\frac{1}{1+\zeta^2}\sum_{k=1}^{\infty}\frac{\beta_k(\sigma+\lambda_k)}{(\sigma+\lambda_k)^2+1}=\frac{C_{1,\sigma}}{1+\zeta^2}, \quad (16)$$

and

$$|Im\widetilde{B}(\sigma+i\zeta)|=|\zeta|\sum_{k=1}^{\infty}\frac{\beta_k}{(\sigma+\lambda_k)^2+\zeta^2}\ge$$
$$\ge\frac{|\zeta|}{1+\zeta^2}\sum_{k=1}^{\infty}\frac{\beta_k}{(\sigma+\lambda_k)^2+1}=\frac{C_{2,\sigma}|\zeta|}{1+\zeta^2}, \quad (17)$$

where $C_{1,\sigma}$ and $C_{2,\sigma}$ are defined as follows

$$C_{1,\sigma}=\sum_{k=1}^{\infty}\frac{\beta_k(\sigma+\lambda_k)}{(\sigma+\lambda_k)^2+1},\ C_{2,\sigma}=\sum_{k=1}^{\infty}\frac{\beta_k}{(\sigma+\lambda_k)^2+1}.$$

Using estimates (16) and (17), we obtain the required estimate

$$|\widetilde{B}(\sigma+i\zeta)|\ge\frac{C_\sigma}{\sqrt{1+\zeta^2}}, \quad (18)$$

where $C_\sigma=min(C_{1,\sigma},C_{2,\sigma})$ is bounded for all $\sigma>0$.

Lemma 4.1 is proved.

If we proceed to the limit as $\sigma\to 0$ in the equality (14), we have

$$\nu(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\phi}(i\zeta)}{\widetilde{B}(i\zeta)}e^{i\zeta t}d\zeta. \quad (19)$$

Also, to prove Theorem 2.1, we need the following lemma.

Lemma 4.2. [8] *Let $\phi(t)\in W(M)$. Then, the following inequality holds for the Laplace transform of the function $\phi(t)$:*

$$\int_{-\infty}^{+\infty}|\widetilde{\phi}(i\tau)|\sqrt{1+\zeta^2}\,d\zeta\le C_1\|\phi\|_{W_2^2(R_+)},$$

where the constant C_1 is positive.

We now present the proof of Theorem 2.1.

Proof of Theorem 2.1. First, we prove $\nu\in W_2^1(\mathbb{R}_+)$. Using estimate (18) and equality (19), we obtain

$$\int_{-\infty}^{+\infty}|\widetilde{\nu}(\zeta)|^2\,(1+|\zeta|^2)d\zeta=$$
$$\int_{-\infty}^{+\infty}\left|\frac{\widetilde{\phi}(i\zeta)}{\widetilde{B}(i\zeta)}\right|^2(1+|\zeta|^2)d\zeta\le$$
$$\le\frac{1}{C}\int_{-\infty}^{+\infty}|\widetilde{\phi}(i\zeta)|^2\,(1+|\zeta|^2)^2d\zeta$$
$$=const\|\phi\|^2_{W_2^2(\mathbb{R})}.$$

Now, we show that the function $\nu(t)$ satisfies the Lipschitz condition. Actually,

$$|\nu(t)-\nu(s)|=\left|\int_s^t\nu'(\xi)d\xi\right|\le\|\nu'\|_{L_2}\sqrt{t-s}.$$

Using (18), (19) and Lemma 4.2, we have the following estimate:

$$|\nu(t)|\le\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|\widetilde{\phi}(i\zeta)|}{|\widetilde{B}(i\zeta)|}d\zeta\le$$
$$\le\frac{1}{2\pi C}\int_{-\infty}^{+\infty}|\widetilde{\phi}(i\zeta)|\sqrt{1+\zeta^2}\,d\zeta\le$$
$$\le\frac{C_1}{2\pi C}\|\phi\|_{W_2^2(\mathbb{R}_+)}\le\frac{C_1M}{2\pi C}=1,$$

whre

$$M=\frac{2\pi C}{C_1}.$$

Theorem 2.1 is proved.

5. An example

Let us now examine the function that follows:

$$\phi(t)=\begin{cases} 0, & \text{for} \quad t\le 0; \\ Ht^2e^{-t}, & \text{for} \quad t>0, \end{cases} \tag{20}$$

where $H>0$ is a constant number. The average temperature in the thin rod is the function $\phi(t)$ in its physical sense.

Assume that $a=1$ and $L=1$ in equation (1). Then the function $B(t)$ determined by equality (10) is as follows:

$$B(t)=\sum_{k=1}^{\infty}(1-(-1)^k)e^{-\lambda_k t}, \quad t>0,$$

where $\lambda_k=\pi^2k^2$.

We can represent the kernel $B(t)$ in the form

$$B(t)=2e^{-\lambda_1 t}+\sum_{k=3}^{\infty}(1-(-1)^k)e^{-\lambda_k t}$$
$$=e^{-\lambda_1 t}(2+O(1)e^{-t(\lambda_3-\lambda_1)}), \quad \lambda_1=\pi^2.$$

Consequently, we can write $B(t)\simeq 2e^{-\pi^2 t}$ for $t>0$.

In this instance, the approximation can be used in place of the primary integral equation (11)

$$2\int_0^t e^{-\pi^2(t-s)}\nu(s)ds=Ht^2e^{-t}, \quad t>0.$$

Using the Laplace transform, we may solve the aforementioned integral equation as follows

$$\tilde{\nu}(p)=H\frac{p+\pi^2}{(p+1)^3}.$$

Thus, we define the original function $\nu(t)$, which is the solution of the integral equation, as

$$\nu(t)=He^{-t}(t+(\pi^2-1)t^2). \tag{21}$$

It is known that $\nu(0)=0$. We set

$$f(t)=e^{-t}(t+(\pi^2-1)t^2), \quad t>0.$$

Note that $\lim_{t\to\infty}f(t)=0$. Let the function $f(t)$ reach its maximum value at the point T^*. Therefore, we can write

$$\max f(t)=f(T^*)=A,$$

where $A=const>0$.

If we take as $H\le\frac{1}{A}$, then we have the following estimate:

$$|\nu(t)|\le H|f(t)|\le HA\le 1.$$

Thus, when the average temperature in the rod is given by equation (20), we found the control function $\nu(t)$ in the form (21) and verified that it is admissible.

Conclusion

The control of the heat conduction equation with periodic boundary condition was examined in this article. The provided mixed issue was reduced to the Volterra integral equation of the first type using the separation of variables approach. The Laplace transform method was used to demonstrate the presence of admissible control, which is the solution to the integral problem.

Conflicts of Interest

The author declare no conflicts of interest.

References

1. Fattorini, H. O. “Time-optimal control of solutions of operational differential equations.” *SIAM J. Control*. 2 (1965): 49-65.
2. Friedman, A. “Optimal control for parabolic equation.” *J. Math. Anal. Appl*. 18 (1967): 479-491.
3. Fattorini, H. O., D. L. Russell. “Exact controllability theorems for linear parabolic equations in one space dimension.” *Arch. Rational Mech. Anal*., 43(1971): 272-292.
4. Egorov, Yu. V. “Optimal control in Banach spaces.” *Dokl. Akad. Nauk SSSR* 150 (1963): 241-244. (In Russian)

5. Chen, N., Y. Wang and D. Yang. “Time-varying bang-bang property of time optimal controls for heat equation and its applications.” *Systems and Control Letters* 112 (2018): 18-23.

6. Albeverio, S., Sh.A. Alimov. “On one time-optimal control problem associated with the heat exchange process.” *Appl. Math. Opt*. 57 (2008): 58-68.

7. Alimov, Sh., N.M. Komilov. “Determining the thermal mode setting parameters based on output data.” *Differential Equations* 58(2022): 21-35.

8. Dekhkonov, F.N. “Boundary control associated with a parabolic equation.” *Journal of Mathematics and Computer Science* 33 (2024): 146-154.

9. Fayazova, Z.K. “Boundary control of the heat transfer process in the space.” *Russian Mathematics* 12(2019): 82-90.

10. Arada, N., J.P. Raymond. “Time optimal problems with Dirichlet boundary conditions.” *Discrete and continuous dynamic systems* 9 (2003): 1540-1570.

11. Li, X., Y. Yao. “Time optimal control for distributed parameter systems.” *Scientia Sinica* 24 (1981): 455-465.

12. Lebeau, G., E. Zuazua. “Null-controllability of a system of linear thermoelasticity.” *Arch. Rational Mech. Anal.* 141 (1998): 297-329.

13. Wang, G. “The existence of time optimal control of semilinear parabolic equations.” *System and Control Letters* 53(2004): 171-175.

14. Guo, Y. J. L., W. Littman. “Null boundary controllability for semilinear heat equations.” Appl. Math. Opt. 32 (1995): 281-316.

15. Alimov, Sh.A., G. Ibragimov. “Time optimal control problem with integral constraint for the heat transfer process.” Eurasian Math. J. 15 (2024): 8-22.

16. Schmidt, G. “The “Bang-Bang” principle for the time-optimal problem in boundary control of the heat equation.” SIAM Journal on Control and Optimization 18 (1980): 101-107.

17. Wang, L.J., X.X. Zhou. “Minimal time control problem of a linear heat equation with memory.” *Systems & Control Letters* 157(2021): 105052.

18. Lions, J.L. *Controle optimal de systemes gouvernes par des equations aux derivees partielles*. Paris: Dunod Gauthier-Villars, 1968.

19. Fursikov, A.V. *Optimal Control of Distributed Systems*, Theory and Applications, Translations of Math. Monographs, Amer. Math. Soc., Providence, Rhode Island, 2000.

20. Wang, G., L. Wang. “The Bang-Bang principle of time optimal controls for the heat equation with internal controls.” *Systems Control Letters* 56 (2007): 709-713.

21. Laroche, B., P. Martin and P. Rouchon. “Motion planning for the heat equation.” *International Journal of Robust and Nonlinear Control* 10 (2000): 629-643.

22. Dekhkonov, F.N. “On the control problem associated with a pseudo-parabolic type equation in an one-dimensional domain.” *International Journal of Applied Mathematics* 37(2024): 109-118.

23. Dekhkonov, F.N. “Boundary control problem associated with a pseudo-parabolic equation.” *Stochastic Modelling and Computational Sciences* 3 (2023): 117-128.

24. Tikhonov, A.N., A. Samarsky. *Equations of Mathematical Physics*. Moscow: Nauka, 1966.

25. Ladyzhenskaya, O.A., V.A. Solonnikov and N.N. Uraltseva. *Linear and quasi-linear equations of parabolic type*. Moscow: Nauka, 1967.

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