IRSTI 27.31.17 https://doi.org/10.26577/ijmph.2024v15i1a5

Lazhar Bougoffa
Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia
e-mail: lbbougoffa@imamu.edu.sa

A new transform for solving linear second-orders ODE with variable coefficients

Abstract: In this paper, we present a novel symmetry-enhanced transform

$$\mathcal{L}(f(x);a)=\int_{-\infty}^{+\infty}e^{-ax^2}f(x)dx=\mathcal{B}(a),a>0$$

to evaluate Gaussian integrals commonly used in mathematical and physical domains, particularly in quantum field theory. Additionally, we utilize this original transformation methodology to solve a wide range of second-order linear ordinary differential equations (ODEs) that have variable coefficients, which is a common occurrence in physics. Notable examples encompass Weber, Euler-Cauchy, and Bessel equations, highlighting the broad applicability of our proposed method. Diverging from established transforms like the widely used Laplace transform, our innovative approach introduces a symmetrical model, offering a distinct and founding perspective to the field.

Key words: New integral transform; Gaussian integral; Equation of free oscillations; Weber's equation; Euler-Cauchy Equation; Bessel's equation.

1 Introduction

Recently, Bougoffa and Rach proved the following formula [1]

Lemma 1

$$\int_{-\infty}^{\infty}e^{-ax^2}f(x)dx=\sqrt{\frac{\pi}{a}}\sum_{n=0}^{\infty}\frac{f^{(2n)}(0)}{(4a)^n n!},a>0 \quad (1.1)$$

provided that the function f is infinitely differentiable in $\mathbb{R}$ and, in a neighborhood of $x=0$, it equals its Maclaurin series expansion about x. Here we must assume that f is such that the integral in the left hand side of (1.1) exists (that is, has some finite value). This assumption is usually satisfied in applications. We shall discuss this in section 2.

This formula has obtained by a new combination with the Adomian decomposition method [2, 3] and the explicit solution

$$u(x,t)=\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty}f(\xi)\,e^{-\frac{(\xi-x)^2}{4kt}}d\xi,$$

$$-\infty<x<\infty,t>0$$

(1.2) of the initial – value problem for the heat equation

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2},k>0,t\geq 0,u(x,0)=f(x). \quad (1.3)$$

This is a new helpful tool in calculating the Gaussian integrals [4] as a convenient convergent series.

Indeed, if we substitute $f(x)=1$ into (1.1), then we obtain

$$\int_{-\infty}^{\infty}e^{-ax^2}dx=\sqrt{\frac{\pi}{a}},a>0. \quad (1.4)$$

The alternatives of the Gaussian integral can be derived from (1.1). For example, the evaluation of

$$\int_{-\infty}^{\infty}e^{-ax^2}x^{2n}dx=\frac{\sqrt{\pi}(2n)!}{4^n a^{n+\frac{1}{2}}n!} \quad (1.5)$$

is obtained by letting $f(x)=x^{2n},n\geq 0$ into (1.1). It can be checked easily that different definite integrals of the above Gaussian form can be derived from formula (1.1).

1- The evaluation of the well-known integral: $\int_0^{\infty}\frac{e^{-a^2(1+x^2)}}{1+x^2}dx=\frac{\pi}{2}\operatorname{erf}(ca)$ can be evaluated directly from (1.1) by letting $f(x)=\frac{1}{x^2+1}$ with

$f(0)=1, f''(0)=-2!, f^{(4)}(0)=4!$. A simple substitution leads to

2-

$$\int_{-\infty}^{\infty}\frac{e^{-a^2x^2}}{1+x^2}dx=\frac{\sqrt{\pi}}{a}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n(2n)!}{4^na^{2n}n!}\right]. \quad (1.6)$$

Since $\frac{(2n)!}{n!}=2^n\big(1\cdot 3\cdot 5\cdots(2n-1)\big)$. Hence

$$\int_{-\infty}^{\infty}\frac{e^{-a^2x^2}}{1+x^2}dx=\frac{\sqrt{\pi}}{a}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2a^2)^n}\right]. \quad (1.7)$$

Using the asymptotic expansion of the complementary error function

$$\operatorname{erfc}(x)=\frac{e^{-x^2}}{\sqrt{\pi}x}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2x^2)^n}\right], \quad (1.8)$$

we obtain

$$\int_{-\infty}^{\infty}\frac{e^{-a^2(1+x^2)}}{1+x^2}dx=\pi\operatorname{erfc}(a). \quad (1.9)$$

3- The Dawson's integral [5, 6]:

This integral $\int_{-\infty}^{\infty}\boldsymbol{D(x)\,dx}$ is called the Dawson's integral, where $\boldsymbol{D(x)=e^{-x^2}\int_0^x e^{t^2}\,dt}$ and it arises in computation of the Voigt lineshape in heat conduction and in the theory of electrical oscillations in certain special vacuum tubes. It can be derived from formula (1.1) that

$$\int_{-\infty}^{\infty}\boldsymbol{D(x)\,dx}=\int_{-\infty}^{\infty}\boldsymbol{e^{-x^2}\left[\int_0^x e^{t^2}\,dt\right]dx=0}. \quad (1.10)$$

4- The plasma dispersion function [7]:

The plasma dispersion function when the imaginary component of ξ is equal to zero and is defined as $Z(\xi)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-x^2}}{x-\xi}dx$.

An immediate consequence of this is

$$Z(\xi)=-\sum_{n=0}^{\infty}\frac{(2n)!}{4^nn!}\frac{1}{\xi^{2n+1}}=$$
$$=-\frac{1}{\xi}\left(1+\frac{1}{2}\frac{1}{\xi^2}+\frac{3}{4}\frac{1}{\xi^4}+\frac{15}{8}\frac{1}{\xi^6}+\cdots\right). \quad (1.11)$$

This follows simply by letting the following in (1.1)

$$f(x)=\frac{1}{x-\xi},$$

$$f^{(2n)}(0)=-(2n)!\,\xi^{-2n-1}, n\geq 1. \quad (1.12)$$

The reader will find in the Table 1 different definite integrals of the above forms which can be easily calculated by (1.1).

No.	$f(x)$	$\int_{-\infty}^{\infty}e^{-ax^2}f(x)dx, a>0$
1	1	$\frac{\sqrt{\pi}}{\sqrt{a}}$
2	x^2	$\frac{\sqrt{\pi}}{2a\sqrt{a}}$
3	$x^n, n\ is\ odd$	0
4	x^4	$\frac{3\sqrt{\pi}}{4a^2\sqrt{a}}$
5	x^{2n}	$\frac{(2n)!\sqrt{\pi}}{4^na^{n+\frac{1}{2}}n!}$
6	$\frac{x^2}{x^2+b^2}$	$\frac{\sqrt{\pi}}{\sqrt{a}}-\pi be^{ab}(1-\operatorname{erf}(\sqrt{ab}))$
7	$\frac{1}{x^2+b^2}$	$\frac{\sqrt{\pi}}{b}e^{ab}(1-\operatorname{erf}(\sqrt{ab}))$
8	$\cos rx$	$\frac{\sqrt{\pi}}{\sqrt{a}}e^{-\frac{r^2}{4a}}$
9	$e^{\pm rx}$	$\frac{\sqrt{\pi}}{\sqrt{a}}e^{\frac{r^2}{4a}}$

Table continuation

No.	$f(x)$	$\int_{-\infty}^{\infty} e^{-ax^2} f(x)dx, a > 0$
10	xe^{-2rx}	$\frac{r}{a}\frac{\sqrt{\pi}}{\sqrt{a}}e^{\frac{r^2}{a}}$
11	x^2e^{-2rx}	$\frac{1}{2a}\frac{\sqrt{\pi}}{\sqrt{a}}(1+\frac{2r^2}{a})e^{\frac{r^2}{a}}$
12	$e^{\frac{-r}{x^2}}$	$\frac{\sqrt{\pi}}{\sqrt{a}}e^{-2\sqrt{ra}}$

Table 1: **Some functions and their transforms $\int_{-\infty}^{\infty} e^{-ax^2} f(x)dx, a > 0$.**

Another important observation on some elementary integrals is

Lemma 2 Define the function F_n by

$$F_n(x) = \int_x^{\infty} t^n e^{-at^2} dt, a > 0. \quad (1.13)$$

Then F_n satisfies

$$\begin{cases} F_{2n}(x) + F_{2n}(-x) = \frac{\sqrt{\pi}(2n)!}{4^n a^{n+\frac{1}{2}} n!} \\ F_{2n+1}(x) + F_{2n+1}(-x) = 0. \end{cases} \quad (1.14)$$

Proof. Replacing x with$-x$, we obtain

$$F_n(-x) = \int_{-x}^{\infty} t^n e^{-at^2} dt, a > 0. \quad (1.15)$$

If we make the transformation $t \to -t$, we get $F_n(-x) = \int_{-\infty}^{x} (-1)^n t^n e^{-at^2} dt, a > 0$.

Thus,

$$F_{2n}(x) + F_{2n}(-x) = \int_{-\infty}^{\infty} t^{2n} e^{-at^2} dt. \quad (1.16)$$

Hence, from Table 1, we have

$$F_{2n}(x) + F_{2n}(-x) = \frac{\sqrt{\pi}(2n)!}{4^n a^{n+\frac{1}{2}} n!}. \quad (1.17)$$

We note that if we choose $n = 0$ and $a = \frac{1}{2}$, then $Q(x) + Q(-x) = 1$, where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt$ is the Q- function corresponds to the tail probability of standard normal distribution [8].

Similarly, we have $F_{2n+1}(x) + F_{2n+1}(-x) = 0$.

The following recurrence follows by integration by parts

Lemma 3 The function F_n satisfies the following recurrence

$$F_n(x) = \frac{n-1}{2a} F_{n-2}(x) + \frac{x^{n-1}}{2a} e^{-ax^2} \quad (1.18)$$

with

$$F_0(x) = \frac{\sqrt{\pi}}{2\sqrt{a}}\left(1 - \text{erfc}\left(\sqrt{a}x\right)\right) \text{ and}$$
$$F_1(x) = \frac{1}{2a} e^{-ax^2}. \quad (1.19)$$

Also, an alternative expression for F_{2n+1}can be obtained from the recurrence and mathematical induction

Lemma 4

$$\boldsymbol{F_{2n+1}(x) = \frac{n!}{2a^{n+1}} e^{-ax^2} \sum_{k=0}^{n} \frac{(\sqrt{a}x)^{2k}}{k!}}. \quad (1.20)$$

Many integral transforms of the familiar Laplace transform $\mathcal{L}(f;s) = \int_0^{\infty} e^{-sx} f(x)dx$ with kernel $k(s,x) = e^{-sx}$ are introduced such as the Laplace-Carson transform, which is defined by [9] $\mathcal{L}(f(x);p) = p\int_0^{+\infty} e^{-px} f(x)dx, p > 0$ and the so-called Sumudu transform: $\mathcal{F}(f(x);q) = \frac{1}{q}\int_0^{+\infty} e^{-\frac{1}{q}x} f(x)dx, q > 0$, which has been the subject of several papers. This integral transform is obtained from the Laplace-Carson transform by means of the trivial change of variable $\boldsymbol{p = \frac{1}{q}}$. All the properties demonstrated for the Sumudu transform may very readily be deduced from the corresponding properties for the standard Laplace transform. These integral transforms have been demonstrated to provide accurate and computable solutions for a wide class of linear differential equations. It is imperative to acknowledge that while the Laplace transform is a powerful tool for solving certain differential equations, it is not universally applicable. Indeed, not all functions possess a Laplace transform, rendering it inadequate for certain classes of linear ordinary differential equations (ODEs). For in- stance, the function $\boldsymbol{f(x) = e^{x^2}}$ does not have a Laplace transform, as the integral diverges for all values of $\boldsymbol{s}$.

The transformation $\boldsymbol{\mathcal{L}(f(x); a)}$ with its Properties 1-5 (Theorem 2) offers a powerful solution for tackling different linear ordinary differential equations. Among the equations are the Euler-Cauchy equation, Weber's equation [10], and associated Bessel's equation [11], etc. By converting them into more manageable forms, we can compute one solution with ease, and then the second independent solution can be deduced by the method of reduction of order to such equations. This approach is de- tailed after Section 3. This integral transform is also particularly significant in physics boundary value problems, as it can solve linear equations with initial and boundary conditions that the Laplace transform cannot handle. Therefore, our newly proposed integral transform presents a promising alter- native for addressing such equations.

The symmetry transformation technique involves simplifying the differential equation into a first-order linear differential equation, which can then be solved with ease through an inverse transform. Although this method is effective, it may sometimes yield a second-order ODE with variable coefficients. Hence, this approach is only successful under suitable conditions on the coefficients of the ODE.

To provide a comprehensive understanding of the meaning of this transformation in mathematical terms, we will present a detailed discussion of its integral transform definition, including its formula, properties, and practical applications. This will allow for a thorough exploration of its conceptual underpinnings and how it can be effectively utilized.

2 A new integral transform

We begin by stating the following definition:

Definition 1 *Let f be a function defined for all* $x \geq 0$. *Our new integral transform is the integral of* f *times* $e^{-ax^2}, a > 0$ *from* $-\infty$ *to* $+\infty$, *and is denoted by* $\mathcal{L}(f(x); a)$:

$$\mathcal{L}(f(x); a) = \int_{-\infty}^{+\infty} e^{-ax^2} f(x)dx, a > 0. \quad (2.1)$$

Here we must assume that f satisfies

$$|f(x)| \leq Me^{bx^2} \text{for all} -\infty < x < +\infty, a > b, \quad (2.2)$$

where b *and* M *are some constants, such that the integral exists. then*
$\mathcal{L}(f(x); a)$ *is a function of a, say,* $B(a)$, *and*

$$\mathcal{L}(f(x); a) = \int_{-\infty}^{+\infty} e^{-ax^2} f(x)dx = \mathcal{B}(a), \quad (2.3)$$

$$a > 0.$$

Furthermore, if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is infinitely differentiable and, in a neighborhood of $x = 0$, it equals its Maclaurin series expansion about x, *then*

$$\mathcal{B}(a) = \sqrt{\frac{\pi}{a}} \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(4a)^n n!}. \quad (2.4)$$

The given function f in (2.3) is called the inverse transform of $B(a), a > 0$ *and is denoted by* $\mathcal{L}^{-1}(B)$; *that is,* $f(x) = \mathcal{L}^{-1}(\mathcal{B}(a)), a > 0$ provided that $\mathcal{L}(f(x); a) \neq 0$.

In order to guarantee the convergence of the series $\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(4a)^n n!}$ for $a > 0$, we must assume that the higher order derivatives of $f(x)$ at $x = 0$ are bounded. We first note that, it may happen in certain cases that $\mathcal{L}(f(x); a)$ exists for a given function f, but $\mathcal{L}^{-1}(\mathcal{B}(a))$ is not uniquely determined. This can be seen from the example given in Table 1-No. 9: $\mathcal{L}(e^{\pm rx}; a) = \frac{\sqrt{\pi}}{\sqrt{a}} e^{\frac{r^2}{4a}}$, which shows that the inverse of this transform is not essentially unique. This is true in many situations in the application of this transform for solving second-order ODEs. It is a very convenient one, since it allows us to find two solutions. This can be seen from a simple example in Section 3.

Another observation that we need is that it can be easily checked that $\mathcal{L}(f(x); a)$ for any odd function, for example $f(x) = x^n, n = 1,3, ...$ In this case, a useful modified of this integral transform is given by
$\mathcal{L}(f(x); a) = \int_{0}^{+\infty} e^{-ax^2} f(x)dx, a > 0. (2.5)$ In general, for any even or odd function, we may say that the inverse of a given transform is essentially unique. In particular, if two continuous functions have the same transform, they are completely identical, and in this case we could determine the function $f(x)$ from its transform $\mathcal{B}(a)$ provided that

it was possible to express $\mathcal{B}(a)$in terms of simpler functions with known inverse transforms. Indeed, any odd or even function f can be determined from its transform $B(a)$ using the Mellin's Inversion Formula

$$g(t) = \mathcal{L}^{-1}(\mathcal{B}(a)) =$$

$$= \frac{1}{2\pi i} \lim_{T\to+\infty} \int_{\gamma-iT}^{\gamma+iT} e^{at}\, \mathcal{B}(a)da, t > 0, \quad (2.6)$$

where γ is a positive constant and is greater than the real part of all singularities of $\mathcal{B}(a)$, and the function $g(t) = \frac{f(\sqrt{t})}{\sqrt{t}}$; can be done with the substitution $t = x^2$. We give an example to illustrate the application of this inversion. Let $\mathcal{B}(a) = \frac{1}{\sqrt{a}}$. Substituting into the Mellin's inversion formula, we find that

$$g(t) = \frac{1}{2\pi i} \lim_{T\to+\infty} \int_{\gamma-iT}^{\gamma+iT} e^{at} \frac{1}{\sqrt{a}} da =$$

$$=\frac{1}{\sqrt{\pi t}} \lim_{T\to+\infty} [\text{erf}(\sqrt{at})]_{\gamma-iT}^{\gamma+iT} = \frac{1}{\sqrt{\pi t}} \quad (2.7)$$

Consequently, $f(x) = \frac{1}{\sqrt{\pi}}$.

We would now like to examine some examples of functions that do not have a Maclaurin expansion. The first example in this section reproduces a related integral to $\mathcal{L}(f(x); a)$.

Examples [12, 13]

1- $\int_{-\infty}^{+\infty} \left(e^{-a_1x^2} - e^{-a_2x^2}\right) \frac{1}{x^2} dx, a_1, a_2 > 0$.

This integral can be obtained by writing $\left(e^{-a_1x^2} - e^{-a_1x^2}\right)\frac{1}{x^2} = \int_{a_1}^{a_1} e^{-tx^2}\, dt$. Thus

2-

$$\int_{-\infty}^{+\infty} (e^{-a_1x^2} - e^{-a_2x^2}) \frac{1}{x^2} dx = 2\sqrt{\pi}(\sqrt{a_2} - \sqrt{a_1}).$$

3- $\mathcal{L}(e^{\frac{-b}{x^2}}; a) = \int_{-\infty}^{+\infty} e^{-ax^2}\, e^{\frac{-b}{x^2}} dx = \frac{\sqrt{\pi}}{\sqrt{a}} e^{-2\sqrt{ab}}, b > 0.$

4- $\mathcal{L}\left(\ln x^2 ; a = \frac{1}{2\sigma^2}\right) = \sqrt{2\pi}\sigma (\ln \sigma^2 - \sigma - \ln 2)$.

5- $\int_{-\infty}^{+\infty} e^{-ax^2}\, [\int_{-x}^{+\infty} e^{-by^2}] dx = \frac{\pi}{2\sqrt{ab}}, a, b > 0.$

We have

Theorem 1 *If f is defined and piecewise continuous on every finite inter-val on the x −axis and satisfies the condition (2.2). Then the new integral transform $\mathcal{L}(f(x); a)$ exists for $a > b$.*

Proof. Since f is piecewise continuous and $e^{-ax^2} f(x)$ is integrable over any finite interval on the x −axis

$$|\mathcal{L}(f(x); a)| =$$
$$=\left|\int_{-\infty}^{\infty} e^{-ax^2}\, f(x)dx\right| \le \int_{-\infty}^{\infty} e^{-ax^2}\, |f(x)|dx \le M \int_{-\infty}^{\infty} e^{-(a-b)x^2}\, dx.$$

From (No. (1)-Table 1), we have $\int_{-\infty}^{\infty} e^{-(a-b)x^2}\, dx = \frac{\pi}{\sqrt{a-b}}$.

Thus $|\mathcal{L}(f(x); a)| \le M \frac{\sqrt{\pi}}{\sqrt{a-b}}, a > b$.

It can be checked easily that the condition (2.2) holds. Indeed, if we choose, for example, $f(x) = x^{2n}$, then from Maclaurin series that is, $x^{2n} < n!\, e^{x^2}$ and so on.

Theorem 2 *Let* $\mathcal{L}(f(x); a) = \int_{-\infty}^{\infty} e^{-ax^2}\, f(x)dx = \mathcal{B}(a), a > 0$, then

1- $\mathcal{L}(x^2 f(x); a) = -\frac{d\mathcal{B}}{da}(a)$.

2- $\mathcal{L}(xf'(x); a) = -[\mathcal{B}(a) + 2a \frac{d\mathcal{B}}{da}(a)]$.

3- $\mathcal{L}(f''(x); a) = -2a \left[\mathcal{B}(a) + 2a \frac{d\mathcal{B}}{da}(a)\right]$.

4- $\mathcal{L}(x^2 f''(x); a) = \frac{d}{da}\left[2a\left(\mathcal{B}(a) + 2a \frac{d\mathcal{B}}{da}(a)\right)\right]$.

5- $\mathcal{L}(e^{bx^2} f(x); a) = \mathcal{B}(a - b), a > b$ (Shifting Property).

6- $e^{bx^2} f(x) = \mathcal{L}^{-1}(\mathcal{B}(a - b)), a > b$.

Property 2 holds if f is continuous on (−∞, ∞) and satisfies the condition (2.2) and f' is piecewise continuous on every finite interval on (−∞, ∞). Similarly, Property 3 holds if f and f' are continuous on (−∞, ∞) and satisfy (2.2) and f'' is piecewise continuous on every finite interval of (−∞, ∞).

Proof.

1- $\mathcal{B}(a) = \int_{-\infty}^{\infty} e^{-ax^2}\, f(x)dx, a > 0$.

Differentiating $\mathcal{B}(a)$ under the integral sign with respect to the parameter a, we obtain

$$\frac{d\mathcal{B}}{da} = \mathcal{B}'(a) = -\int_{-\infty}^{\infty} e^{-ax^2}\, x^2 f(x)dx = -\mathcal{L}(x^2 f(x); a).$$

2- Assume that $f'(x)$ is continuous on $(-\infty, \infty)$. Thus, $\mathcal{L}(xf'(x); a) = \int_{-\infty}^{\infty} e^{-ax^2}\, xf'(x)dx$.

Since f satisfies$|\, f\,(x)\,| \leq\, Me^{bx^2}$, integration by parts yields

$$\mathcal{L}(xf'(x); a) = -\int_{-\infty}^{\infty} e^{-ax^2} f(x)dx + 2a\int_{-\infty}^{\infty} e^{-ax^2}\, x^2 f(x)dx$$

The integrals $\mathcal{L}(f(x); a)$ and $\mathcal{L}(x^2 f(x); a)$; a) exist for $a\, >\, b$. So that $\mathcal{L}(xf'(x); a)$ exists for $a\, >\, b$. Using now $\mathcal{B}(a) = \mathcal{L}(f(x); a)$ and $\mathcal{B}'(a) = -\mathcal{L}(x^2 f(x); a)$, we obtain the desired property.

If f' is piecewise continuous on$(-\infty, \infty)$. Then, the proof is similar.

3- Assume that f'' is continues on $(-\infty, \infty)$. Thus, by Definition 1, we have $\mathcal{L}(f''(x); a) = \int_{-\infty}^{\infty} e^{-ax^2} f''(x)dx$.

Integration by parts yields

$$\mathcal{L}(f''(x); a) = -2a\int_{-\infty}^{\infty} e^{-ax^2} f(x)dx + (2a)^2\int_{-\infty}^{\infty} e^{-ax^2}\, x^2 f(x)dx.$$

Since f and f' satisfy (2.2), the two integrals $\mathcal{L}(f(x); a)$and $\mathcal{L}(x^2 f(x); a)$exist for $a\, >\, b$. So that $\mathcal{L}(f''(x); a)$ exists for $a > b$, and the proof is complete.

The proof of this for f'' is piecewise continuous on $(-\infty, \infty)$ is similar.

4- Properties 4 follows simply from Definition 1 and Property 3.

5- $\mathcal{L}\big(e^{bx^2} f(x); a\big) = \int_{-\infty}^{\infty} e^{-(a-b)x^2} f(x)dx = \mathcal{B}(a-b), a > b.$

In order to verify the accuracy of our present method, we present some elementary examples.

3 Applications: Solutions of linear second-orders ODE

It is important to note a key difference in our approach before using this transformation. Unlike the Laplace, this new integral transform transform focuses On expressing $\mathcal{B}(a)$ and its derivatives $\mathcal{B}'$(a), $\mathcal{B}''$(a),… with respect to a while dealing with the derivatives of functions. This is explained in Properties 1-6. Our approach works in general for finding one solution for the second-order differential equations by using the properties that govern this transform. Therefore, considering initial and boundary conditions at the first step is not necessary for any obtained solution.

It happens quite often that one solution can be found by this integral trans- form. Then a second linearly independent solution can be deduced by the method of reduction of order, which works easily in general. It is important to note that when using the new transform, the general solutions obtained should contain undetermined constants and these constants can be deter- mined by including the initial or boundary conditions that are given. To help understand this process, we provide some examples below:

3.1 Equation of free oscillations

We begin by considering the linear equation for free oscillations

$$y'' - w^2 y = 0, -\infty < x < \infty, \qquad (3.1)$$

Applying (2.3) to both sides of Eq.(3.1), we have

$$\mathcal{L}(y''; a) - w^2\mathcal{L}(y; a) = 0 \qquad (3.2)$$

Using Definition 1 and Property 3, we obtain

$$-2a[\mathcal{B}(a) + 2a\tfrac{d\mathcal{B}}{da}(a)] - w^2\mathcal{B}(a) = 0. \quad (3.3)$$

Thus,

$$(2a)^2\tfrac{d\mathcal{B}}{da}(a) + (w^2 + 2a)\mathcal{B}(a) = 0. \qquad (3.4)$$

which is a linear first-order equation. Solving for $\mathcal{B}$, we obtain $\mathcal{B}(a) = \frac{C}{\sqrt{a}} e^{\frac{w^2}{4a}}$, where C is a constant of integration. Hence from Table 1, we have $e^{\pm wx} = \mathcal{L}^{-1}\left(\frac{\sqrt{\pi}}{\sqrt{a}} e^{\frac{w^2}{4a}}\right)$. Thus $y_1 = C_1 e^{wx}$ and $y_2 = C_1 e^{-wx}$, where $C_1 = \frac{C}{\sqrt{\pi}}$. Consequently the general solution is then given by $y =\, C_1^* e^{wx} + C_2^* e^{-wx}$. The constants C_1^* and C_2^* follow easily from the initial or boundary conditions of this equation.

3.2 Weber Equation

Consider the linear Weber equation

$$y'' - (b^2x^2 + b)y = 0, -\infty < x < \infty, \quad (3.5)$$

which often arises in various applications. Applying (2.3) to both sides of Eq.(3.5), we have

$$\mathcal{L}(y''; a) - b^2\mathcal{L}(x^2y; a) - b\mathcal{L}(y; a) = 0. \quad (3.6)$$

Using Definition 1 and Properties 1 and 3, we obtain

$$-2a[\mathcal{B}(a) + 2a\frac{d\mathcal{B}}{da}(a)] +. \\ +b^2\frac{d\mathcal{B}}{da}(a) - b\mathcal{B}(a) = 0 \quad (3.7)$$

Thus,

$$(b^2 - (2a)^2)\frac{d\mathcal{B}}{da}(a) - (b + 2a)\mathcal{B}(a) = 0, \quad (3.8)$$

which is a first-order linear equation. Solving for $\mathcal{B}$, we obtain, $\mathcal{B}(a) = \frac{C}{\sqrt{a-\frac{b}{2}}}, a > \frac{b}{2}$, where C is a constant of integration. Hence from shifting property, we obtain $y_1 = C_1 e^{\frac{b}{2}x^2}, C_1 = \frac{C}{\sqrt{\pi}}$.The second independent solution y_2 can be obtained by the method of reduction of order. Indeed, if one solution y_1 is known to the homogeneous linear ODE: $y'' + p(x)y' + q(x)y = 0$, then, $y_2 = \int U(x)dx$ where $U(x) = \frac{1}{y_1^2}e^{-\int p(x)dx}$.

So that the desired second solution of Weber equation $y_2 = \frac{1}{C_1}e^{\frac{b}{2}x^2}\int e^{-bx^2}dx = \frac{\sqrt{\pi}}{2C_1\sqrt{b}}e^{\frac{b}{2}x^2}\operatorname{erf}(\sqrt{b}x)$. It follows that y_1 and y_2 form a basis of solutions. Hence, the general solution is obtained $y = C_1^* y_1 + C_2^* y_2$ and the particular solution can be deduced from the general solution and the initial conditions or boundary conditions.

3.3 Euler-Cauchy Equation

Consider the linear Euler-Cauchy equation

$$x^2y'' + bxy' + cy = 0, -\infty < x < \infty. \quad (3.9)$$

A simple application of (2.3) with Properties 2 and 4 to both sides of Eq.(3.9) leads to

$$(2a)^2\frac{d^2\mathcal{B}}{da^2}(a) + (2a)(5 - b)\frac{d\mathcal{B}}{da}(a) + \\ +(2 - b + c)\mathcal{B}(a) = 0. \quad (3.10)$$

Note that the Laplace and Sumudu transforms convert Eq. (3.9) into $s^2Y''(s) + s(4 - b)Y'(s) + (2 - b + c)Y(s) = 0$ and $s^2Y''(s) + bsY'(s) + cY(s) = 0$, respectively.

If we assume that $2 - b + c = 0$, that is $b = c + 2$, then this integral transform takes the Euler-Cauchy differential equation and turns it under a suitable condition on its coefficients into a first-order linear differential equation. Thus, Eq.(3.10) becomes $(2a)^2\frac{d^2\mathcal{B}}{da^2}(a) + (2a)(5 - b)\frac{d\mathcal{B}}{da}(a) = 0$. So the Sumudu transform is not a best choice for Euler-Cauchy Equation. The solution is then simple $\mathcal{B}(a) = \frac{C}{b-3}a^{\frac{b-3}{2}} + D$, where C and D are constants of integration. If we assume that $3 - b = 2n + 1, n = 1,2,...,$ that is, $b = 2 - 2n$, then $\mathcal{B}(a) = -\frac{C}{2n+1}\frac{1}{a^n\sqrt{a}} + D$. Using $\mathcal{B}(a) \to 0$ as $a \to \infty$ to get $D = 0$. Hence, from Table 1, we obtain that $y_1 = C_n x^{2n}, C_n = -\frac{C\,4^n n!}{\sqrt{2}\,(2n)!(2n+1)}, n = 1,2,...,$ which is indeed a solution of the Cauchy-Euler equation:

$$x^2y'' + 2(1 - n)xy' - 2ny = 0,. \\ -\infty < x < \infty \quad (3.11)$$

The second independent solution y_2 can be also obtained by the method of reduction of order.

3.4 The associated Bessel Equation

Consider the linear associated Bessel equation

$$x^2y'' + 2(m + 1)xy' +, \\ [x^2 - l(l + 2m + 1)]y = 0, -\infty < x < \infty \quad (3.12)$$

where l and m are parameters. Applying (2.3) and using Properties 1, 2 and 4, we obtain

$$(2a)^2\frac{d^2\mathcal{B}}{da^2}(a) + (6a - 4am - 1)\frac{d\mathcal{B}}{da}(a) -. \\ [2m + l(l + 2m + 1)]\mathcal{B}(a) = 0 \quad (3.13)$$

Proceeding as in Example 3, if we assume that $2m + l(l + 2m + 1) = 0$, then, Eq.(3.21) can be converted into

$$(2a)^2\frac{d^2\mathcal{B}}{da^2}(a) + (6a - 4am - 1)\frac{d\mathcal{B}}{da}(a) = 0. \quad (3.14)$$

Hence $\frac{d\mathcal{B}}{da}(a) = \frac{C}{a^{\frac{3-2m}{2}}}e^{-\frac{1}{4a}}$. As a specific example, let $m = 1$. Thus $\frac{d\mathcal{B}}{da}(a) = \frac{C}{\sqrt{a}}e^{-\frac{1}{4a}}$. Using

Property 1: $\mathcal{L}(x^2y''; a) = -\frac{dB}{da}$ and from Table 1, we immediately obtain $y_1 = C_1\frac{\cos x}{x^2}$, which is a solution to Eq. (3.12) when

$$m = 1 \; and \; l = 1 \; or \; l = 2.$$

3.5 Other Equations

We would now like to examine other equations. Let

$$xy' + y = (x^2+1)e^{\frac{x^2}{2}}, -\infty < x < \infty. \quad (3.15)$$

Applying (2.3) to both sides of Eq.(3.15), we obtain, $\mathcal{L}(xy'; a) + \mathcal{L}(y; a) = \mathcal{L}\left((x^2+1)e^{\frac{x^2}{2}}; a\right)$. (3.16)

Using Property 2 and Definition 1, we obtain

$$-\left[\mathcal{B}(a) + 2a\frac{d\mathcal{B}}{da}(a)\right] + \mathcal{B}(a) = = \mathcal{L}\left((x^2+1)e^{\frac{x^2}{2}}; a\right). \quad (3.17)$$

Since $\mathcal{L}\left((x^2+1)e^{\frac{x^2}{2}}; a\right) = \mathcal{L}\left(x^2e^{\frac{x^2}{2}}; a\right) + \mathcal{L}\left(e^{\frac{x^2}{2}}; a\right)$, $\mathcal{L}\left(e^{\frac{x^2}{2}}; a\right) = \frac{\sqrt{\pi}}{\sqrt{a-\frac{1}{2}}}$ and $\mathcal{L}\left(x^2e^{\frac{x^2}{2}}; a\right) = \frac{\sqrt{\pi}}{2}\frac{1}{\left(a-\frac{1}{2}\right)\sqrt{a-\frac{1}{2}}}$, where $a > \frac{1}{2}$. We have

$$\frac{d\mathcal{B}}{da}(a) = -\frac{\sqrt{\pi}}{2}\frac{1}{2a\left(a-\frac{1}{2}\right)\sqrt{a-\frac{1}{2}}} - \frac{\sqrt{\pi}}{2a\sqrt{a-\frac{1}{2}}}, a > \frac{1}{2}. \quad (3.18)$$

Solving for $\mathcal{B}$, we obtain

$$\mathcal{B}(a) = -\frac{\sqrt{\pi}}{2}\int\frac{1}{2a\left(a-\frac{1}{2}\right)\sqrt{a-\frac{1}{2}}}da - -\sqrt{\pi}\int\frac{1}{2a\sqrt{a-\frac{1}{2}}}da. \quad (3.19)$$

Since

$$\int\frac{1}{2a\sqrt{a-\frac{1}{2}}}da = \sqrt{2}\arctan(\sqrt{2a-1}) \quad (3.20)$$

And

$$\int\frac{1}{2a(a-\frac{1}{2})\sqrt{a-\frac{1}{2}}}da = -2^{\frac{3}{2}}\arctan(\sqrt{2a-1}). \quad (3.21)$$

We have $\mathcal{B}(a) = \frac{\sqrt{2\pi}}{\sqrt{2a-1}} + C$. Using $\mathcal{B}(a) \to 0 \; as \; a \to \infty$, to get $\mathcal{B}(a) = \frac{\sqrt{\pi}}{\sqrt{a-\frac{1}{2}}}$.

It follows that, from Table 1, $y_1(x) = e^{\frac{x^2}{2}}$. Consequently, the general solution to this equation is construct as the sum of y_1 and z, where z is the general solution to the corresponding homogeneous $xy' + y = 0$. Hence $y(x) = e^{\frac{x^2}{2}} + \frac{C}{x}$ and the constant C can be determined from the IC $y(x_0) = \alpha, x_0 \neq 0$.

4 Conclusion

This new integral transform has been demonstrated to provide accurate and computable solutions for a wide class of linear second-order differential equa- tions with variable coefficients such as Euler-Cauchy Equation, Weber's equa- tion and Bessel's equation. This integral transform takes a differential equation and turns it into a first-order linear differential equation, which is simpler than the given second-order and can be easily solved, applying the inverse transform gives us our desired solution. But sometimes the application of this integral transform gives a second-order ODE with variable coefficients, and this will show that the present method works well only under suitable conditions on the coefficients of this ODE.

Conflicts of Interest

The authors declare no conflicts of interest.

References

1. L. Bougoffa and R. Rach, Evaluation of Gaussian integrals by Ado- mian decomposition, *Scientia Series A: Mathematical Sciences*, Vol. 30 (2020), 1–5.
2. G. Adomian, Nonlinear Stochastic Systems Theory and Applications to Physics, Kluwer Aca- demic Publishers, Dordrecht, 1989.
3. G. Adomian and R. Rach, Evaluation of integrals by decomposition, *Journal of Computational and Applied Mathematics,* 23(1986), 99–101.
4. Victor H. Moll, Special Integrals of Gradshteyn and Ryzhik: The Proofs- Volume II, CRC Press, Taylor and Francis Group, 2016.
5. F. Dawson,On the Numerical Value of $\int_0^h e^{x^2} dx$, *London Math. Soc. Proc.,* 29, 519-522, 1898.
6. D.A. Dijkstra,A Continued Fraction Expansion for a Generalization of Dawson's Integral, *Math. Comp.,* 31, 503-510, 1977.
7. B. D. Fried and S. D. Conte, The Plasma Dispersion Function (Aca- demic, New York, 1961).
8. S. M. Stewart, Some alternative derivations of Craig's formula, *The Mathematical Gazette,* Vol. 101, No. 551, pp. 268-279, 2017.
9. V.A. Ditkin and A.P. Prudnikov, Integral Transforms and Operational Calculus (Oxford: Pergamon), 1965.
10. Andrei D. Polyanin, Valentin F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations, Second Edition, Chapman and Hall/CRC, 2003.
11. Isaac I.H. Chen, The associated Bessel functions and recurrence formulas, *INT. J. MATH. EDUC. SCI. TECHNOL.,* 1984, VOL. 15, NO. 3, 345–348.
12. Victor H. Moll, Special Integrals of Gradshteyn and Ryzhik: The Proofs- Volume II, CRC Press, Taylor and Francis Group, 2016.
13. Owen, D., A table of normal integrals, *Communications in Statistics: Simulation and Computation,* B9 (4): 389–419. 1980.

Information about authors:

Lazhar Bougoffa – Professor of Department of Mathematics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), (Riyadh, Saudi Arabia. e-mail address: lbbougoffa@imamu.edu.sa).

Received on 12 April, 2024
Accepted on 15 May, 2024