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An eighth-order two-step with-memory adaptive method base on Hansen-Patrick's method and its dynamic

Abstract. A tri-parametric family of two-point iterative methods with six-order convergence for solving nonlinear equations has been proposed. Each derivative-free method member of the family requires only three evaluations of the given function per iteration. It is optimal in the sense of the Kung and Traub conjecture. Based on these methods, with-memory methods with convergence orders of 6, 7, 7.53, and 8 are constructed. The parameters of the self-accelerator are calculated using Newton interpolation and information from the current and previous iterations. The proposed family has an efficiency index of 1.96 and 2. Numerical comparisons have been made to reveal the high efficiency of the developed method. The dynamical study of iterative schemes reflects a good overview of their stability, convergence properties, and graphical aspects by drawing attraction basins in the complex plane. Also, we have examined the dynamic behavior of new methods to select the best weight function that has the largest attraction basins for different polynomials.

Key words: Adaptive method with memory, Accelerator parameter, Basin of attraction, Nonlinear equations.

1 Introduction

Definition 1 In 1960, Ostrowski presented a measure called the efficiency index for comparing iterative methods. The efficiency index is still used to compare iterative methods. Its definition is as follows [1]

$$IE(IM) = \sqrt[\theta_f]{r}, \tag{1}$$

where r and θ_f are the R-order of convergence and the number of function evaluations per iteration, respectively.

Literature

Many problems from Engineering, Chemistry, Physics, and other fields can be obtained in the form of equations using Mathematical Modelling. The field of computational sciences offers a lot of possibilities to researchers to solve these problems and has seen notable growth in mathematics. We remark that in computational sciences, the practice of numerical study for finding such solutions is essentially connected to variants of the iterative method. Iterative methods for finding the solutions of nonlinear equations are the first purpose of the work. Fariborzi Araghi et al. [2] and Ullah et al. [3] have solved nonlinear equations using adaptive methods.

Existing iterative methods

Can attribute the first two-step without-memory method to Ostrowski. In 1960, he presented the family of optimal-methods as follows [1].

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, k = 0,1,2,\ldots, \\ x_{k+1} = y_k - \dfrac{f(y_k)}{f'(x_k)} . \dfrac{f(x_k)}{f(x_k) - 2f(y_k)}. \end{cases} \tag{2}$$

In 1977, Hansen and Patrick [4] built an optimal third-order method as follows

$$y_k = x_k - \frac{(\alpha+1) f(x_k)}{\alpha f'(x_k) \pm ((f'(x_k))^2 - (\alpha+1) f(x_k) f''(x_k))^{\frac{1}{2}}} \tag{3}$$

In 2003, Petkovic et al. [5] regained Hansen-Patrick's method by Laguerre's method. Sharma et al. [6] suggested a two-point fourth-order family of iterative methods for solving nonlinear equations (p = 1/3, α = 1)

$$\begin{cases} y_k = x_k - \rho \dfrac{f(x_k)}{f'(x_k)}, k = 0,1,2,\ldots, \\ x_{k+1} = x_k \dfrac{(\alpha+1) f(x_k)}{\alpha f'(x_k) \pm ((f'(x_k))^2 - (\alpha+1) f(x_k) f''(x_k))^{\frac{1}{2}}} \dfrac{f(x_k)}{f'(x_k)}. \end{cases} \tag{4}$$

In 2015, Kansal et al. [12] proposed a fourth-order two-step method as follows:

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, k = 0,1,2,\ldots, \\ x_{k+1} = x_k \dfrac{(\alpha+1)}{\alpha \pm \left(\dfrac{(f(x_k))^2 - (\beta - 2\alpha - 2) f(x_k) f(y_k) - \beta(\alpha+1)(f(y_k))^2}{(f(x_k))^2 + f(x_k) f(y_k)}\right)^{\frac{1}{2}}} \dfrac{f(x_k)}{f'(x_k)}. \end{cases} \tag{5}$$

Kansal et al. [7] presented a new family of fourth-order methods based on the method (3) that they have obtained by Hansen-Patrick's method. This method, which we will refer to as KKBM, is given by

$$\begin{cases} H(0) = 1, H'(0) = -\dfrac{\alpha+1}{2}, H''(0) < \infty, t_k = \dfrac{f(y_k)}{f(x_k)}, k = 0,1,2,\ldots, \\ y_k = x_k - \dfrac{f(x_k)}{f[x_k, z_k] + \lambda f(z_k)}, z_k = x_k + \gamma f(x_k), \\ x_{k+1} = y_k - \dfrac{f(x_k)}{f[y_k, z_k] + \lambda f(z_k)} \left(-1 + \dfrac{(\alpha+1)}{\alpha + \left(1 - \dfrac{2(\alpha-1) f(y_k)}{f(x_k)}\right)^{\frac{1}{2}}}\right) H(t_k). \end{cases} \tag{6}$$

Some weight functions satisfy the condition of (6) are as follows:

$$\begin{cases} H_1(t)=1-\frac{\alpha+1}{2}, H_2(t)=\frac{1}{1+\frac{\alpha+1}{2}t}, H_3(t)=\frac{1}{1+\frac{\alpha+1}{2}t+\theta t^2}, H_4(t)=\frac{1+t}{1+\frac{\alpha+3}{2}t}, \alpha,\theta\in R, \\ H_5(t)=\frac{1+t+\theta t^2}{1+\frac{\alpha+3}{2}t}, H_6(t)=1+\text{Arctan}(-\frac{\alpha+1}{2}t), H_7(t)=\cos(t)-\sin(\frac{\alpha+1}{2}t), \\ H_8(t)=1+\text{Arcsin}(-\frac{\alpha+1}{2}t), H_9(t)=\frac{1+t}{1+\frac{\alpha+3}{2}t+\theta t^2}, H_{10}(t)=\cos(t)-\frac{\alpha+1}{2}\sin(t), \\ \quad H_{11}(t)=1-\frac{\alpha+1}{2}\text{Arcsin}(t), H_{12}(t)=1-\tan(\frac{\alpha+1}{2}t), H_{13}(t)=1+\frac{\alpha+1}{2}t+\theta t^2. \end{cases} \tag{7}$$

The two-step iterative method (6) has fourth-order satisfies (considering the weight functions mentioned in equation (7)) the following error equation:

$$\begin{aligned} e_{k+1} = & \left(1+\gamma f'(\zeta)\right)^2(\lambda+c_2)((3+2H''(0)+4\alpha+\alpha^2)(1+\gamma f'(\zeta)\lambda^2+2(-1+\gamma f'(\zeta)+2H''(0) \\ & (1+\gamma f'(\zeta))+\alpha(4+\alpha)(1+\gamma f'(\zeta))(\lambda+c_2+(-5+3\gamma f'(\zeta)+2H''(0)(1+\gamma f'(\zeta)) \\ & +\alpha(4+\alpha)(1+\gamma f'(\zeta))c_2^2+4c_3))\frac{1}{-4}e_k^4+O\left(e_k^5\right). \end{aligned} \tag{8}$$

Motivation and organization

In this study, we convert Hansen-Patrick's method into the optimal-method fourth-order. They have three free parameters. Also, we increase the convergence order from three to eight. The proposed methods use the information from all steps. The efficiency index of them is 2. The based Hansen-Patrick method of the third-order has been presented in Section 2. In Section 3, we have converted the without-memory schemes into with-memory methods. Methods with 100% convergence order improvement in Section 4 have been developed for the first time using Hansen-Patrick's method. This section presents the purpose of this paper by proving the original theorem. The presented multipoint methods are tested and compared with existing methods of the same order in Section 5. The dynamical behavior of the proposed methods is analyzed in Section 6. We finish the work with some remarks and conclusions

2 Derivation of the family of without memory methods

Based on Kansal et al.'s method [7], we have derived a general optimal family four order as follows:

$$\begin{cases} H(0)=1, H'(0)=-\frac{\alpha+1}{2}, |H''(0)|<\infty, t_k=\frac{f(y_k)}{f(x_k)}, k=0,1,2,\ldots, \\ y_k=x_k-\frac{f(x_k)}{f[x_k,z_k]+\lambda f(z_k)}, z_k=x_k+\gamma f(x_k), \\ x_{k+1}=y_k-\frac{f(x_k)}{f[y_k,z_k]+\lambda f(z_k)+\beta(y_k-x_k)(y_k-z_k)}(-1+\frac{(\alpha+1)}{\alpha\pm(1-\frac{2(\alpha-1)f(y_k)}{f(x_k)})^{\frac{1}{2}}})H(t_k). \end{cases} \tag{9}$$

We have shown this method with TM4.

Theorem 2.1 Let I ⊆ R be an open interval, f : I → R be a scalar function which has a simple root ζ in the open-interval I, and also the initial approximation x_0 is sufficiently close to the simple zero, then, the two-step iterative method (9) has four-order satisfies the following error equation:

$$e_{k+1} = \left(1+\gamma f'(\zeta)\right)^2 (\lambda + c_2)(-4 + f'(\zeta)(2H''(0) + (1+\alpha)(3+\alpha))(1+\gamma f'(\zeta))\lambda^2 + \\ + f'(\zeta)c_2(2(-1+\gamma f'(\zeta)+2H''(0))(1+\gamma f'(\zeta)) + \alpha(4+\alpha)(1+\gamma f'(\zeta))\lambda + \\ +(-5+3\gamma f'(\zeta)+2H''(0)(1+\gamma f'(\zeta)) + \alpha(4+\alpha)(1+\gamma f'(\zeta)))c_2 + 4f'(\zeta)c_3))\frac{1}{-4}e_k^4 + \mathrm{O}\left(e_k^5\right). \quad (10)$$

Proof. Now, to check the convergence order of (9), we avoid retyping the widely practiced approach in the literature and put forward the following self-explained Mathematica code:

$$f(x_k) = \\ = f'(\zeta)\left[e_k + c_2 e_k^2 + c_3 e_k^3 + c_4 e_k^4 + O\left(e_k^5\right)\right], \quad (11)$$

$$f'(x_k) = f'(\zeta)\left[1 + 2c_2 e_k + 3c_3 e_k^2 + 4c_4 e_k^3 + 5c_5 e_k^4 + O\left(e_k^5\right)\right], \quad (12)$$

where e = x − ζ and $c_k = \frac{f^{(k)}(\zeta)}{k!f'(\zeta)}$, $k = 2,3,,...$.

In[1] : $f[e\_] = fla(e + c_2e_2 + c_3e_3 + c_4e_4)$;

(*fla denotes $f'(\zeta)$*)

In[2] : $f[x\_,y\_] = \frac{f[x]-f[y]}{x-y}$;

In[3] : $ez = e+\gamma f[e]$;

In[4] : $ey = e - Series[\frac{f[e]}{f[e,ez]+\lambda f[ez]},$

$\{e,0,4\}]$;

In[5] : $t = \frac{f[ey]}{f[e]}$;

In[6] : $H[t\_] = H0 + H1*t + H2*\frac{t^2}{2}$;

(*$H0 = H(0), H1 = H'(0), H2 = H''(0)$*)

In[7] : $e_{k+1} = ey - Series \times$

$\times[\frac{f[e]}{f[ey,ez]+\lambda f[ez]+\beta(ey-e)(ey-ez)}$

$*(-1+\frac{(\alpha+1)}{\alpha+(1-\frac{2(\alpha-1)f[ey]}{f[e]})^{\frac{1}{2}}}),\{e,0,4\}]$

The expression of the asymptotic error of $e_{k+1} = x_{k+1} - \zeta$ can be presented as

$$e_{k+1} = A1*e + A2*e^2 + \\ +A3*e^3 + A4*e^4 + O(e^5). \quad (13)$$

The family (9) will have the order of convergence equal to four if the coefficients A1, A2, and A3 in (13) all vanish. First, for A1, we have

A1 = Coefficient[e_{k+1}, e]//Simplify
Out[A1] = 0

$A2 = Coefficient[e_{k+1}, e^2]//Simplify$

$Out[A2] = -(H0-1)(1+\gamma fla)(\lambda + c_2)$

H(0)=1; Comment : This condition vanishes the coefficient of e^2,

A3 = Coefficient[e_{k+1}, e^3]//Simplify

Out[A3] =

$=-\frac{1}{2}(1+\alpha+2H1)(1+\gamma fla)^2(\lambda + c_2)$

$H(1) = -\frac{1+\alpha}{2}$; Comment: This condition vanishes the coefficient of e^3,

$$A4 = \text{Coefficient}[e_{k+1}, e^4]//\text{Simplify}$$

$$\text{Out}[A4] = (1+\gamma fla)^2 (\lambda + c_2)(-4 + fla(2H''(0) + (1+\alpha)(3+\alpha))(1+\gamma fla)\lambda^2 + flac_2(2(-1+\gamma fla + 2H''(0))(1+\gamma fla) + \alpha(4+\alpha)(1+\gamma fla)\lambda + (-5 + 3\gamma fla + 2H''(0)(1+\gamma fla) + \alpha(4+\alpha)(1+\gamma fla))c_2 + 4flac_3))\frac{1}{-4}e_k^4 + O(e_k^5)$$

This proof reveals that the two-step class of Hansen-Patrick's method (9) reaches the order of convergence four by using only three functional evaluations per full iteration. Which completes the proof of the Theorem 2.1.

3 With Memory Methods

In this section, we first have proposed with-memory methods which, we have based on the without-memory-method mentioned in Equation (9). We have advanced the convergence rate of the methods (9) by varying the parameters per full iteration. With the choice $\gamma = \frac{-1}{f'(\zeta)}, \lambda = -c_2 = -\frac{f''(\zeta)}{2f'(\zeta)}, \beta = \frac{f'''(\zeta)}{6}$ it can confirm that the order of the presented method would be 7.5. However, the exact values of $f'(\zeta)$, $f''(\zeta)$ and $f'''(\zeta)$ are not available in practice. But we could approximate the parameters γ, λ, and β by γ_k, λ_k, and β_k. The exact value of a simple root is not known, and consequently, the derivatives of the function cannot compute. We have used information available from the current and prior iteration and have obtained the parameters γ_k, λ_k, and β_k

$$\gamma_k = \frac{-1}{f'(\zeta)} \approx \frac{-1}{N_3'(x_k)},$$
$$\lambda_k = -\frac{f''(\zeta)}{2f'(\zeta)} \approx \frac{N_4''(w_k)}{N_4'(w_k)}, \quad (14)$$
$$\beta_k = \frac{f'''(\zeta)}{6} \approx \frac{N_5'''(y_k)}{6}.$$

Hence, we use Newton's interpolation method to approximate the derivatives of f, where $N_3(x_k); N_4(w_k)$ *and* $N_5(y_k)$ are Newton's interpolation polynomials of degrees three, four, and five respectively. In this work, we have proposed the families of with-memory methods as follows.

(i) If we only interpolate parameter γ_k using Newton's method, a procedure by six order with-memory method will obtain.

$$\begin{cases} \gamma_k = \frac{-1}{N_3'(x_k)}, k = 1,2,3,..., \\ H(0) = 1, H'(0) = -\frac{\alpha+1}{2}, |H''(0)| < \infty, t_k = \frac{f(y_k)}{f(x_k)}, k = 0,1,2,\ldots, \\ y_k = x_k - \frac{f(x_k)}{f[x_k, z_k] + \lambda f(z_k)}, z_k = x_k + \gamma_k f(x_k), \\ x_{k+1} = y_k - \frac{f(x_k)}{f[y_k, z_k] + \lambda f(z_k) + \beta(y_k - x_k)(y_k - z_k)}(-1 + \frac{(\alpha+1)}{\alpha \pm (1 - \frac{2(\alpha-1)f(y_k)}{f(x_k)})^{\frac{1}{2}}})H(t_k). \end{cases} \quad (15)$$

(ii) We attempt to prove that the method with memory (9) has convergence seven-order provided that we use accelerators γ_k, λ_k.

$$\begin{cases} \gamma_k = \dfrac{-1}{N_3'(x_k)}, \lambda_k = \dfrac{N_4''(w_k)}{N_4'(w_k)}, k = 1,2,3,..., \\ H(0)=1, H'(0) = -\dfrac{\alpha+1}{2}, |H''(0)| < \infty, t_k = \dfrac{f(y_k)}{f(x_k)}, k = 0,1,2,\ldots, \\ y_k = x_k - \dfrac{f(x_k)}{f[x_k,z_k] + \lambda_k f(z_k)}, z_k = x_k + \gamma_k f(x_k), \\ x_{k+1} = y_k - \dfrac{f(x_k)}{f[y_k,z_k] + \lambda_k f(z_k) + \beta(y_k - x_k)(y_k - z_k)}(-1 + \dfrac{(\alpha+1)}{\alpha + (1 - \dfrac{2(\alpha-1)f(y_k)}{f(x_k)})^{\frac{1}{2}}})H(t_k)\,. \end{cases} \tag{16}$$

(iii) Replacing the fixed parameters γ, λ, and β in the iterative formula (9) by the varying γ_k, λ_k, and β_k calculated by (14), the following derivative-free two-points scheme with memory has achieved:

$$\begin{cases} \gamma_k = \dfrac{-1}{N_3'(x_k)}, \lambda_k = \dfrac{N_4''(w_k)}{N_4'(w_k)}, \beta_k = \dfrac{N_5'(y_k)}{6}, k = 1,2,3,..., \\ H(0)=1, H'(0) = -\dfrac{\alpha+1}{2}, |H''(0)| < \infty, t_k = \dfrac{f(y_k)}{f(x_k)}, k = 0,1,2,\ldots, \\ y_k = x_k - \dfrac{f(x_k)}{f[x_k,z_k] + \lambda_k f(z_k)}, z_k = x_k + \gamma_k f(x_k), \\ x_{k+1} = y_k - \dfrac{f(x_k)}{f[y_k,z_k] + \lambda_k f(z_k) + \beta_k(y_k - x_k)(y_k - z_k)}(-1 + \dfrac{(\alpha+1)}{\alpha + (1 - \dfrac{2(\alpha-1)f(y_k)}{f(x_k)})^{\frac{1}{2}}})H(t_k)\,. \end{cases} \tag{17}$$

Lemma 3.1 If $\gamma_k = \dfrac{-1}{N_3'(x_k)}$, then:

$$(1 + \gamma_k f'(\zeta)) \sim e_k e_{k,w} e_{k,y} \tag{18}$$

Theorem 3.1 If an initial approximation x_0 is sufficiently close to the zero ζ of f(x) = 0 and the parameter γ_k in the iterative scheme (15) is recursively calculated by lemma in (3.1), then the R-order of convergence is at least 6.

Lemma 3.2 If $\gamma_k = \dfrac{-1}{N_3'(x_k)}, \lambda_k = -\dfrac{N_4''(w_k)}{N_4'(w_k)}$, then:

$$(1 + \gamma_k f'(\zeta)) \sim e_k e_{k,w} e_{k,y}, \lambda_k + c_2 \sim e_k e_{k,w} e_{k,y} \tag{19}$$

Theorem 3.2 If an initial approximation x_0 is sufficiently close to the zero ζ of f(x) = 0 and the parameter γ_k, λ_k in the iterative scheme (16) is recursively calculated by lemma in (3.2), then the R-order of convergence is at least 7.

Proof The proof of this Theorem has been fully described in reference [7].

Lemma 3.3 If $\gamma_k = \dfrac{-1}{N_3'(x_k)}$, $\lambda_k = -\dfrac{N_4''(w_k)}{N_4'(w_k)}, \ and \ \beta_k = \dfrac{N_5'(y_k)}{6}$, then

$$\begin{cases} (1+\gamma_k f'(\zeta)) \sim e_k e_{k,w} e_{k,y}, \\ \lambda_k + c_2 \sim e_k e_{k,w} e_{k,y}\ , \\ (-4+f'(\zeta)(2H''(0)+(1+\alpha)(3+\alpha))(1+\gamma f'(\zeta))\lambda_k{}^2 + f'(\zeta)c_2(2(-1+\gamma_k f'(\zeta)+2H''(0)) \\ (1+\gamma_k f'(\zeta))+\alpha(4+\alpha)(1+\gamma_k f'(\zeta))\lambda_k + (-5+3\gamma_k f'(\zeta)+2H''(0)(1+\gamma_k f'(\zeta))+\alpha(4+\alpha) \\ (1+\gamma_k f'(\zeta)))c_2 + 4f'(\zeta)c_3)) \sim e_k e_{k,w} e_{k,y}. \end{cases} \quad (20)$$

where

$e_k = x_k - \zeta, e_{k,w} = w_k - \zeta,\ and\ \ e_{k,y} = y_k - \zeta$.

The proof is very similar to the ones given previously in Torkashvand-Kazemi. [8].

Theorem 3.3 If an initial approximation x_0 is sufficiently close to the zero ζ of f(x) = 0 and the parameter $\gamma_k, \lambda_k, \beta_k$ in the iterative scheme (17) is recursively calculated by lemma in (3.3), then the R-order of convergence is at least 7.53.

Proof First, we assume that the R-order of convergence of sequence x_k, w_k, y_k is at least m, m_1, and m_2, respectively. Hence,

$$\begin{cases} e_{k+1} \sim e_k^m \sim e_{k-1}^{m^2}, \\ e_{k,w} \sim e_k^{m_1} \sim e_{k-1}^{mm_1}, \\ e_{k,y} \sim e_k^{m_2} \sim e_{k-1}^{mm_2}. \end{cases} \quad (21)$$

By (21), and lemma (3.3), we obtain

$$\begin{cases} (1+\gamma_k f'(\zeta)) \sim e_{k-1}^{1+m_1+m_2}, \\ \lambda_k + c_2 \sim e_{k-1}^{1+m_1+m_2}\ , \\ (-4+f'(\zeta)(2H''(0)+(1+\alpha)(3+\alpha))(1+\gamma f'(\zeta))\lambda_k{}^2 + f'(\zeta)c_2(2(-1+\gamma_k f'(\zeta)+2H''(0)) \\ (1+\gamma_k f'(\zeta))+\alpha(4+\alpha)(1+\gamma_k f'(\zeta))\lambda_k + (-5+3\gamma_k f'(\zeta)+2H''(0)(1+\gamma_k f'(\zeta))+\alpha(4+\alpha) \\ (1+\gamma_k f'(\zeta)))c_2 + 4f'(\zeta)c_3)) \sim e_{k-1}^{1+m_1+m_2}. \end{cases} \quad (22)$$

On the other hand, we get

$$e_{k,w} \sim (1+\gamma_k f'(\zeta))e_k \quad (23)$$

$$e_{k,y} \sim (1+\gamma_k f'(\zeta))(\lambda_k + c_2)e_k \quad (24)$$

$$\begin{aligned} e_{k+1} \sim \left(1+\gamma f'(\zeta)\right)^2 (\lambda + c_2)((3+2H''(0)+4\alpha+\alpha^2)(1+\gamma f'(\zeta)\lambda^2 + 2(-1+\gamma f'(\zeta)+2H''(0) \\ (1+\gamma f'(\zeta))+\alpha(4+\alpha)(1+\gamma f'(\zeta))(\lambda + c_2 + (-5+3\gamma f'(\zeta)+2H''(0)(1+\gamma f'(\zeta))+\alpha \\ (4+\alpha)(1+\gamma f'(\zeta))c_2^2 + 4c_3))e_k^4 \end{aligned} \quad (25)$$

Combining (22)-(23), (22)-(24), and (22)-(25), we conclude

$$\begin{cases} e_{k+1} = e_{k-1}^{4(1+m_1+m_2)+4m}, \\ e_{k,w} = e_{k-1}^{(1+m_1+m_2)+m}, \\ e_{k,y} = e_{k-1}^{2(1+m_1+m_2)+2m}. \end{cases} \quad (26)$$

Equating the powers of error exponents of e_{k-1} in pairs of relations (21), and (26), we have

$$\begin{cases} m^2 - 4(1+m_1+m_2) - 4m = 0, \\ mm_1 - (1+m_1+m_2) - m = 0, \\ mm_2 - 2(1+m_1+m_2) - 2m = 0. \end{cases} \quad (27)$$

This system has the solution

$m_1 = \frac{1}{8}(7+\sqrt{65}) \approx 1.88, m_2 = \frac{1}{4}(7+\sqrt{65}) \approx 3.77,$

$and \quad m = \frac{1}{2}(7+\sqrt{65}) \approx 7.53,$

which specifies the R−order of convergence of the derivative-free scheme with memory (15). So, the proof of Theorem 3.2 ends. □

3.1 Maximum improvement in convergence order

This part deals with the main contribution of this manuscript. In this work, we will use the idea of adaptive methods. This technique uses all previous and current information and increases the degree of convergence and efficiency index by one hundred percent. Thus, as iterations proceed, the degree of interpolation polynomials increases, and the best-updated approximations for computing the self-accelerator $\gamma_k, \lambda_k, \beta_k$ are obtained. We have extended the following recursive adaptive method with memory. Then

$$
\begin{cases}
x_0, \gamma_0, \lambda_0, \beta_0, \\
\gamma_k = \dfrac{-1}{N'_{3k}(x_k)}, \lambda_k = \dfrac{N''_{3k+1}(w_k)}{N'_{3k+1}(w_k)}, \beta_k = \dfrac{N'_{3k+2}(y_k)}{6}, k = 1,2,3,..., \\
H(0) = 1, H'(0) = -\dfrac{\alpha+1}{2}, |H''(0)| < \infty, t_k = \dfrac{f(y_k)}{f(x_k)}, k = 0,1,2,\ldots, \\
y_k = x_k - \dfrac{f(x_k)}{f[x_k, z_k] + \lambda_k f(z_k)}, z_k = x_k + \gamma_k f(x_k), \\
x_{k+1} = y_k - \dfrac{f(x_k)}{f[y_k, z_k] + \lambda_k f(z_k) + \beta_k (y_k - x_k)(y_k - z_k)} \left(-1 + \dfrac{(\alpha+1)}{\alpha + (1 - \dfrac{2(\alpha-1) f(y_k)}{f(x_k)})^{\frac{1}{2}}}\right) H(t_k).
\end{cases}
\tag{28}
$$

In what follows, we discuss the general convergence analysis of the recursive adaptive method with memory (28). It notes that the convergence order varies as the iteration goes ahead. First, we need the following lemma.

Lemma 3.4 If $\gamma_k = \dfrac{-1}{N'_{3k}(x_k)}$, $\lambda_k = -\dfrac{N''_{3k+1}(w_k)}{N'_{3k+1}(w_k)}$, *and* $\beta_k = \dfrac{N'_{3k+2}(y_k)}{6}$, then:

$$
\begin{cases}
(1 + \gamma_k f'(\zeta)) \sim \prod_{s=0}^{k-1} e_s e_{s,w} e_{s,y}, \\
\lambda_k + c_2 \sim \prod_{s=0}^{k-1} e_s e_{s,w} e_{s,y}, \\
(-4 + f'(\zeta)(2H''(0) + (1+\alpha)(3+\alpha))(1+\gamma f'(\zeta))\lambda_k^2 + f'(\zeta) c_2 (2(-1+\gamma_k f'(\zeta)) + 2H''(0)) \\
(1+\gamma_k f'(\zeta)) + \alpha(4+\alpha)(1+\gamma_k f'(\zeta))\lambda_k + (-5 + 3\gamma_k f'(\zeta) + 2H''(0)(1+\gamma_k f'(\zeta)) + \alpha(4+\alpha) \\
(1+\gamma_k f'(\zeta)))c_2 + 4 f'(\zeta) c_3)) \sim \prod_{s=0}^{k-1} e_s e_{s,w} e_{s,y}.
\end{cases}
\tag{29}
$$

Main Theorem Let x_0 be a suitable-initial guess to the simple root ζ of f(x) =0. Also, suppose the initial values $\gamma_0, \lambda_0, \beta_0$ are chosen appropriately. Then, the R-order of the recursive adaptive method with memory (28) can obtain from the following system of nonlinear equations:

$$\begin{cases} m^k m_1 - (1+m_1+m_2)(1+m+m^2+m^3+...+m^{k-1}) - m^k = 0, \\ m^k m_2 - 2(1+m_1+m_2)(1+m+m^2+m^3+...+m^{k-1}) - 2m^k = 0, \\ m^{k+1} - 4(1+m_1+m_2)(1+m+m^2+m^3+...+m^{k-1}) - 4m^k = 0. \end{cases} \quad (30)$$

where m, m_1 *and* m_2 are the order of convergence of the sequences $\{x_k\},\{w_k\}$ *and* $\{y_k\}$ respectively.

Proof Let $\{x_k\},\{w_k\}$ *and* $\{y_k\}$ be convergent with orders m, m_1 *and* m_2, respectively. Then

$$\begin{cases} e_{k+1} \sim e_k^m \sim e_{k-1}^{m^2} \sim ... \sim e_0^{m^{k+1}}, \\ e_{k,w} \sim e_k^{m_1} \sim e_{k-1}^{mm_1} \sim ... \sim e_0^{m_1 m^k}, \\ e_{k,y} \sim e_k^{m_2} \sim e_{k-1}^{mm_2} \sim ... \sim e_0^{m_2 m^k}. \end{cases} \quad (31)$$

Now, by Lemma (3.4) and Eq (31), we obtain

$$\begin{aligned} (1+\gamma_k f'(\zeta)) &\sim \prod_{s=0}^{k-1} e_s e_{s,w} e_{s,y} = (e_0 e_{0,w} e_{0,y})(e_1 e_{1,w} e_{1,y})(e_s e_{s,w} e_{s,y})...(e_{k-1} e_{k-1,w} e_{k-1,y}) \\ &= (e_0 e_0^{m_1} e_0^{m_2})(e_0^m e_0^{mm_1} e_0^{mm_2})...(e_0^{m^{k-1}} e_0^{m^{k-1}m_1} e_0^{m^{k-1}m_2}) = e_0^{(1+m_1+m_2)+m(1+m_1+m_2)+...+m^{k-1}(1+m_1+m_2)} \\ &= e_0^{(1+m_1+m_2)(1+m+m^2+...+m^{k-1})}. \end{aligned} \quad (32)$$

Similarly, we get

$$(\lambda_k + c_2) = e_0^{(1+m_1+m_2)(1+m+m^2+...+m^{k-1})}. \quad (33)$$

and

$$\begin{aligned} &(-4+f'(\zeta)(2H''(0)+(1+\alpha)(3+\alpha))(1+\gamma f'(\zeta))\lambda_k^2 + f'(\zeta)c_2(2(-1+\gamma_k f'(\zeta)+2H''(0)) \\ &(1+\gamma_k f'(\zeta))+\alpha(4+\alpha)(1+\gamma_k f'(\zeta))\lambda_k + (-5+3\gamma_k f'(\zeta)+2H''(0)(1+\gamma_k f'(\zeta))+\alpha(4+\alpha) \\ &(1+\gamma_k f'(\zeta)))c_2 + 4f'(\zeta)c_3)) = e_0^{(1+m_1+m_2)(1+m+m^2+...+m^{k-1})}. \end{aligned} \quad (34)$$

By considering the errors of w_k, y_k, and x_{k+1} in Eq. (31), and Eqs. (32)-(34), we conclude:

$$e_{k,w} \sim (1+\gamma_k f'(\zeta))e_k \sim e_0^{(1+m_1+m_2)(1+m+m^2+...+m^{k-1})} e_0^{m^k} \quad (35)$$

$$e_{k,y} \sim (1+\gamma_k f'(\zeta))(\lambda_k + c_2)e_2^k \sim e_0^{((1+m_1+m_2)(1+m+m^2+...+m^{k-1}))^2} e_0^{2m^k} \quad (36)$$

$$\begin{aligned} e_{k+1} &\sim (1+\gamma f'(\zeta))^2 (\lambda + c_2)((3+2H''(0)+4\alpha+\alpha^2)(1+\gamma f'(\zeta)\lambda^2 + 2(-1+\gamma f'(\zeta)+2H''(0) \\ &(1+\gamma f'(\zeta))+\alpha(4+\alpha)(1+\gamma f'(\zeta))(\lambda + c_2 + (-5+3\gamma f'(\zeta)+2H''(0)(1+\gamma f'(\zeta))+\alpha \\ &(4+\alpha)(1+\gamma f'(\zeta))c_2^2 + 4c_3))e_k^4 \sim e_0^{((1+m_1+m_2)(1+m+m^2+...+m^{k-1}))^4} e_0^{4m^k} \end{aligned} \quad (37)$$

To obtain the desired result, it is enough to match the right-hand-side of the Eqs. (31), (35), (36), and (37). Then

$$\begin{cases} m^k m_1 - (1+m_1+m_2)(1+m+m^2+m^3+...+m^{k-1}) - m^k = 0, \\ m^k m_2 - 2(1+m_1+m_2)(1+m+m^2+m^3+...+m^{k-1}) - 2m^k = 0, \\ m^{k+1} - 4(1+m_1+m_2)(1+m+m^2+m^3+...+m^{k-1}) - 4m^k = 0. \end{cases} \quad (38)$$

Therefore, this completes the proof of the Theorem. □

Remark 3.1 For k = 4, we get the convergence order $m_1 \simeq 2$ $m_2 \simeq 4$ and $m \simeq 8$ (been shown by TM8). In this instance, the efficiency index is $8^{\frac{1}{3}} = 2$ which confirms that our proposed method competes for the whole of the present methods with-memory.

4 Numerical results and comparisons

In this section, we will check the effectiveness of the new optimal methods. We employ the present methods 2, 3, 4, 5, 6, 15, 16, 17, 36 (k=1) and 36 (k=4) denoted by OM, HPM, SGSM, KKBM4, KKBM7,TM6, TM7, TM7.5, and TM8, respectively, also to solve nonlinear equations given in Table 1.

The values of real parameters used for numerical calculations are suggested by authors of the original papers ($\alpha = 1$ for HPM; $\alpha = 1,\ \rho = \frac{1}{3}$ for SGSM; $\alpha = 1,\ \beta = 0$ for KKBM4, $\alpha = 1,\ \gamma_0 = \lambda_0 = 0.1$ for KKBM7; $\alpha = 1, \gamma_0 = \lambda_0 = \beta_0 = 0.1$ for TM6, TM7, TM7.5, and TM8). All computations are performed using the programming package Mathematica with multiple-precision arithmetic. Tables 2 − 5 also include, for each test function, the initial estimation values and the last value of the computational order of convergence r_c computed by the expression

$$r_c \approx \frac{log\ \left|f\left(x_n\right)/f\left(x_{n-1}\right)\right|}{log\ \left|f\left(x_{n-1}\right)/f\left(x_{n-2}\right)\right|}$$

Table 1 lists the exact-roots ζ and initial approximations x_0.Tables 2−5 show that the proposed methods compete with the previous methods. TM7.5 and TM8 have efficiency indices of $\sqrt[3]{7.53} = 1.96,\ \sqrt[3]{8} = 2$.

Table 1 – Test functions

Nonlinear function	Zero	Initial guess
$f_1\left(x\right) = x\log\left(1 + x\sin x\right) + e^{-1+x^2+x\cos x}\ \sin\pi x$	$\zeta = 0$	$x_0 = 0.5$
$f_2\left(x\right) = e^{x-x^2+2}$ -1	$\zeta = -1$	$x_0 = -0.8$
$f_3\left(x\right) = e^{-x+x^3}$ -cos(x^2-1)+x^{x^3} +1	$\zeta = -1$	$x_0 = -1.5$

Table 2 – Numerical results

functions		*OM*	*HPM, α = 1*	*SGSM, β = 1/3,α = 1*	*KKBM7, H₁(t),α = 1*	*TM7.5, H₁(t)*
$f_1, x_0 = 0.5$	$\|x_{n+1} - x_n\|$	5.53e-146	5.71e-33	1.12e-22	3.13e-849	3.05e-1089
	$\|f(x_{n+1})\|$	3.03e-582	1.55e-97	1.93e-88	2.21e-5938	4.04e-8199
	Iter	4	4	3	3	3
	r_c	4.00	3.00	4.00	7.00	7.53
$f_2, x_0 = -0.8$	$\|x_{n+1} - x_n\|$	2.79e-48	1.86e-25	3.04e-52	4.47e-747	6.56e-973
	$\|f(x_{n+1})\|$	1.84e-190	9.72e-75	1.67e-206	2.03e-5225	5.45e-7324
	Iter	3	4	3	3	3
	r_c	4.00	3.00	4.00	7.00	7.53
$f_3, x_0 = -1.5$	$\|x_{n+1} - x_n\|$	1.25e-44	9.11e-13	3.09e-29	5.93e-1058	2.52e-1459
	$\|f(x_{n+1})\|$	5.45e-176	3.53e-36	1.40e-114	8.86e-7402	3.92e-10987
	Iter	3	3	3	3	3
	r_c	4.00	3.00	4.00	7.00	7.53

Table 3 – Numerical results

functions		***TM*8, *H*8(*t*)**	***KKBM*4**	***TM*7, *H*1(*t*)**	***TM*6, *H*1(*t*)**	***TM*8, *H*12(*t*)**
$f_1, x_0 = 0.5$	$\lvert x_{n+1} - x_n\rvert$	7.02e-1038	4.63e-35	3.13e-849	9.66e-1218	7.02e-1038
	$\lvert f(x_{n+1})\rvert$	1.21e-8297	3.82e-135	2.21e-5939	4.54e-720	1.21e-8297
	Iter	4	3	3	3	4
	r_c	8.00	4.00	7.00	6.00	8.00
$f_2, x_0 = -0.8$	$\lvert x_{n+1} - x_n\rvert$	2.47e-940	1.91e-54	4.47e-747	1.37e-1484	4.21e-943
	$\lvert f(x_{n+1})\rvert$	3.91e-7517	2.34e-215	2.03e-5225	3.94e-8906	2.77e-7539
	Iter	4	3	3	3	4
	r_c	8.00	4.00	7.00	6.00	8.00
$f_3, x_0 = -1.5$	$\lvert x_{n+1} - x_n\rvert$	4.00e-1385	3.26e-45	5.93e-1058	1.48e-157	4.79e-1385
	$\lvert f(x_{n+1})\rvert$	1.41e-11075	2.10e-178	8.86e-7402	1.65e-961	4.98e-11075
	Iter	4	3	3	3	4
	r_c	7.99	4.00	7.00	6.00	7.99

Table 4 – Numerical results

functions		***TM*7.5, *H*7(*t*)**	***TM*7.5, *H*8(*t*)**	***TM*8, *H*4(*t*)**	***TM*8, *H*6(*t*)**	***TM*8, *H*7(*t*)**
$f_1, x_0 = 0.5$	$\lvert x_{n+1} - x_n\rvert$	2.92e-1089	3.05e-1089	8.18e-1038	7.02e-1038	6.75e-1038
	$\lvert f(x_{n+1})\rvert$	2.90e-8199	4.03e-8199	4.12e-8297	1.21e-8297	8.85e-8298
	Iter	3	3	4	4	4
	r_c	7.53	7.53	8.00	8.00	8.00
$f_2, x_0 = -0.8$	$\lvert x_{n+1} - x_n\rvert$	3.76e-1014	9.08e-976	5.36e-852	1.82e-932	3.74e-977
	$\lvert f(x_{n+1})\rvert$	1.50e-7634	1.61e-7345	1.93e-6810	3.52e-7454	1.09e-7811
	Iter	3	4	4	4	4
	r_c	7.53	7.53	8.00	8.00	8.00
$f_3, x_0 = -1.5$	$\lvert x_{n+1} - x_n\rvert$	3.05e-1459	1.66e-1320	2.33e-1385	2.33e-1385	2.83e-1393
	$\lvert f(x_{n+1})\rvert$	1.68e-10986	7.51e-10560	1.37e-10851	1.87e-11077	9.35e-11141
	Iter	3	3	4	4	4
	r_c	7.53	7.53	7.99	7.99	7.99

Table 5 – Comparison improvement of convergence order the proposed method with other schemes

with-memory methods	number of steps	optimal order	COC	percentage increase
CPJM[5]	2	4	5	25%
CLBTM[6]	2	4	7	75%
KKBM7[7]	2	4	7	75%
TKM[8]	2	4	7	75%
LSNKKM[9]	2	4	6	50%
LSMKKM[9]	2	8	12	50%
MWBM[10]	2	2	2.73	36.6%
MLAM[11]	2	4	5.95	48.75%
WM[12]	2	4	4.44	11.24%
TM6(15)	2	4	6	50%
TM7(16)	2	4	7	75%
TM7.5(17)	2	4	7.53	88.34%
TM8(28)	2	4	8	100%

Remark 4.1 As can be understood that the advancement in the convergence order from (100% of a development). It denotes the extremely high computational efficiency of our proposed methods. Hence, the efficiency index of the proposed method (28) is $8^{\frac{1}{3}} = 2$.

As observed, the efficiency index of the proposed method is much higher than the method similar methods of Hansen-Patrick [13, 14,15].

5 Complex dynamics

In this section, we have examined the dynamic behaviors of the new with-memory method. For this purpose, we have assigned different values to the self-accelerating parameters and have used the weighted function $H(t_k) = 1\text{-}\frac{1}{2}t_k$ in the TM4 to select the most efficient one. Some significant results concerning the dynamic performances of the iterative methods have been obtained in [16,17,18]. We have compared with-memory methods (TM4) by using the basins of attraction for three complex polynomials $p_1(z) = z^2 - 1,\ p_2(z) = z^3 - 1,\ p_3(z) = z^3 - z,\ p_4(z) = z^4 - 1$. We have used similar material as in [18] to generate the basins of attraction. To produce the basins of attraction for the zeros of a polynomial and an iterative method, we use a framework of 500 × 500 points in a rectangle D = [−5, 5] × [−5, 5] ⊂ C, and we use these points as z_0. Whenever the sequence produced by the iterative method achieves a zero z∗ of polynomial $p_i(x)$, then we take with a tolerance $|z - z*| < 10^{-6}$ and a maximum of 25 iterations. Therefore, we determine that z_0 is in the basin of attraction of the zero and we paint this point in a color previously selected for this root. Figures 1, 2, 3, 4 and 5 show the basins of the attraction proposed method (TM4). Figures show that the accelerator parameter plays a decisive role in increasing the absorption domain of a repetitive method. As can be seen from Figures the smaller the size of the self-accelerator parameters, the greater the stability savings. In Figures (a)1, and (a)3, we have considered the number of points to be 50. As can be observed, the percentage of transparency with other shapes, which is 500 points, is much lower.

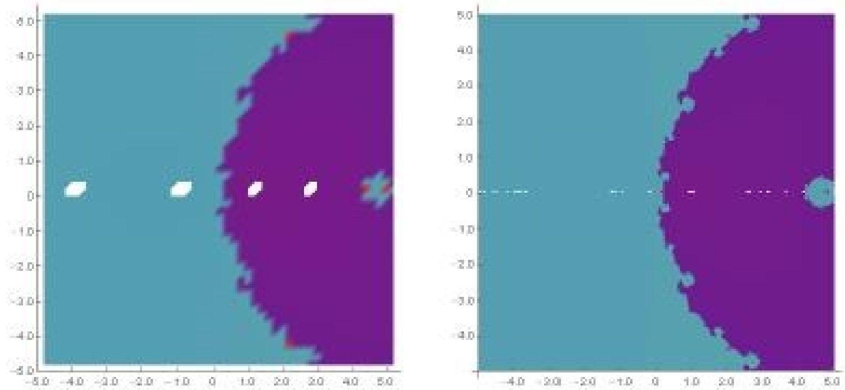
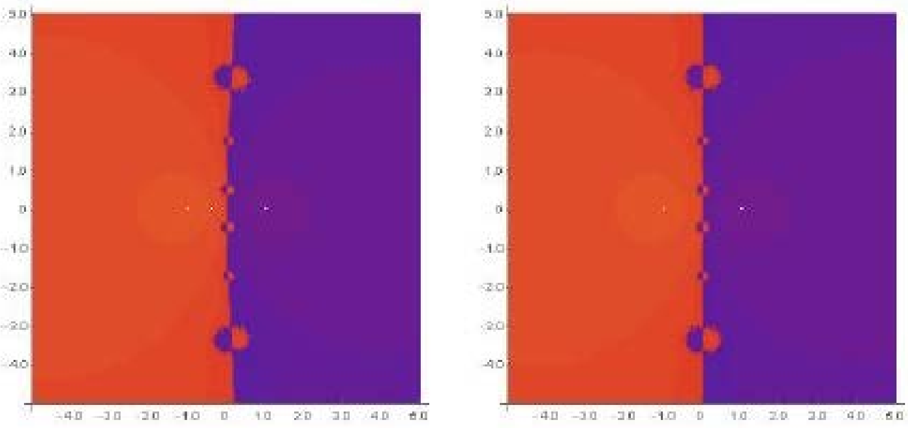
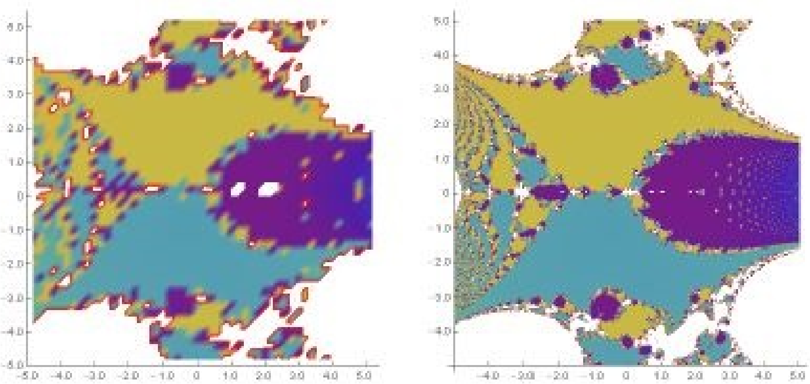

(a) TM4, $\gamma = \lambda = \beta = 0.1$ (b) TM4, $\gamma = \lambda = \beta = 0.1$

Figure 1 – Finding the roots of the polynomial $p_1(z) = z^2 - 1$



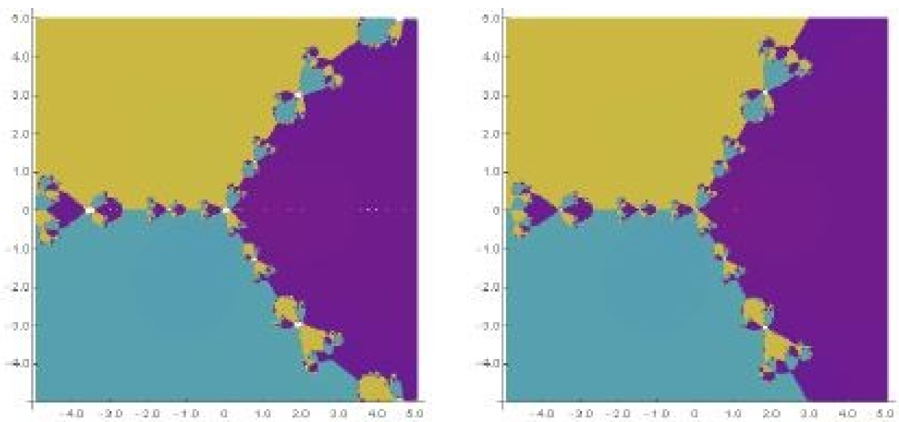
(a) TM4, $\gamma = \lambda = \beta = 0.01$ (b) TM4, $\gamma = \lambda = \beta = 0.001$

Figure 2 – Finding the roots of the polynomial $p_1(z) = z^2 - 1$



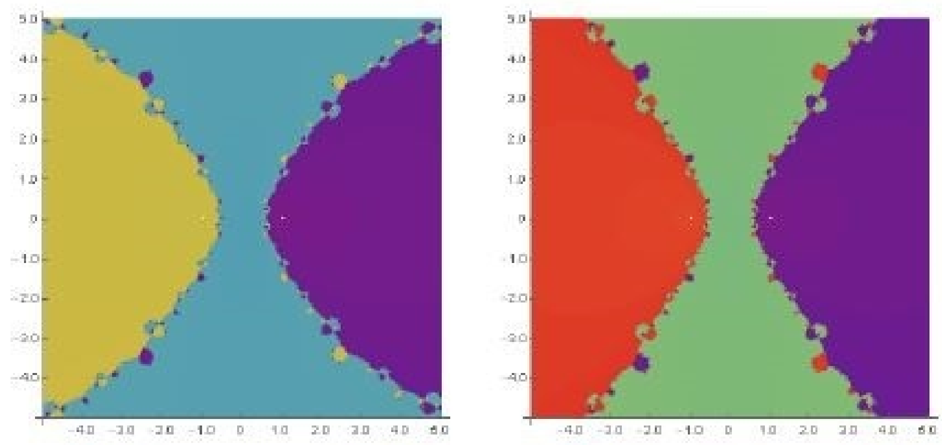
(a) TM4, $\gamma = \lambda = \beta = 0.1$ (b) TM4, $\gamma = \lambda = \beta = 0.1$

Figure 3 – Finding the roots of the polynomial $p_2(z) = z^3 - 1$



(a) TM4, $\gamma = \lambda = \beta = 0.01$ (b) TM4, $\gamma = \lambda = \beta = 0.001$

Figure 4 – Finding the roots of the polynomial $p_2(z) = z^3 - 1$



(a) TM4, $\gamma = \lambda = \beta = 0.01$ (b) TM4, $\gamma = \lambda = \beta = 0.001$

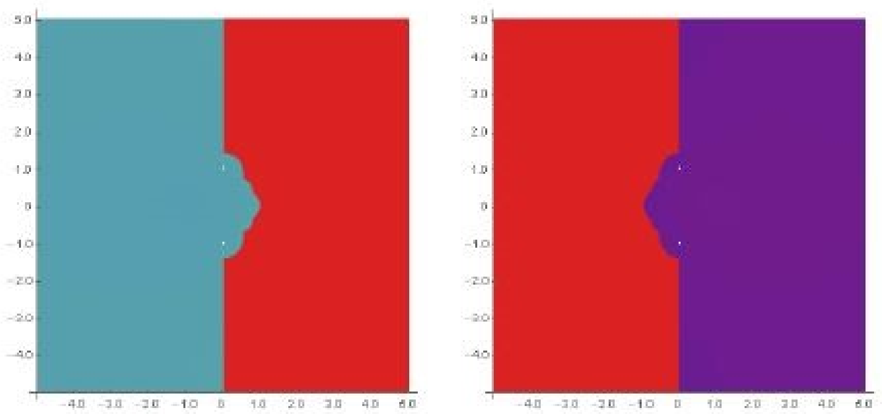
Figure 5 – Finding the roots of the polynomial $p_3(z) = z^3 - z$

In the following, we have drawn the absorption area of the Hansen-Patrick's method in two separate parts. The first part (a) is related to the HPM1 method,

$$x_{k+1} = x_k - \frac{(\alpha+1) f(x_k)}{\alpha f'(x_k) - (f'^2(x_k) - (\alpha+1) f(x_k) f''(x_k))^{\frac{1}{2}}} .$$

and we have considered the second part as the HPM2 method.

$$x_{k+1} = x_k - \frac{(\alpha+1) f(x_k)}{\alpha f'(x_k) + (f'^2(x_k) - (\alpha+1) f(x_k) f''(x_k))^{\frac{1}{2}}} .$$



(a) HPM1, $\alpha = 0$ (b) HPM2, $\alpha = 0$

Figure 6 – Finding the roots of the polynomial $p_1(z) = z^2 - 1$

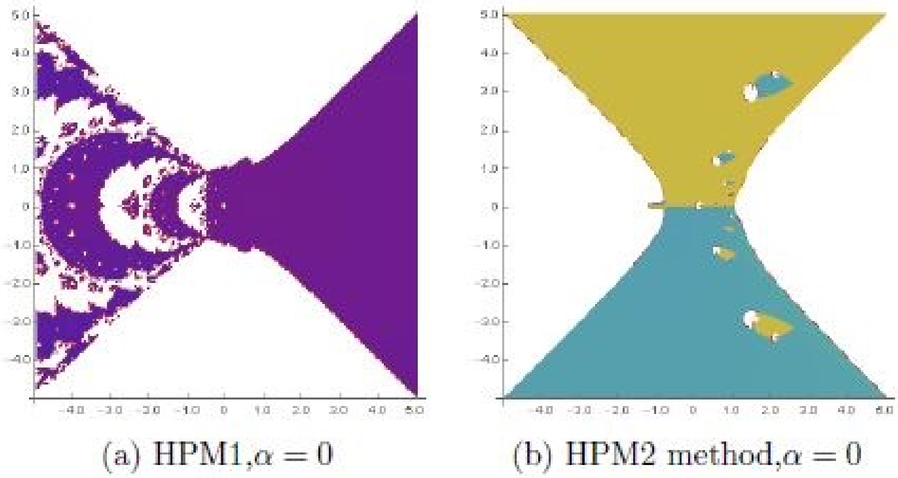


Figure 7 – Finding the roots of the polynomial $p_1(z) = z^2 - 1$

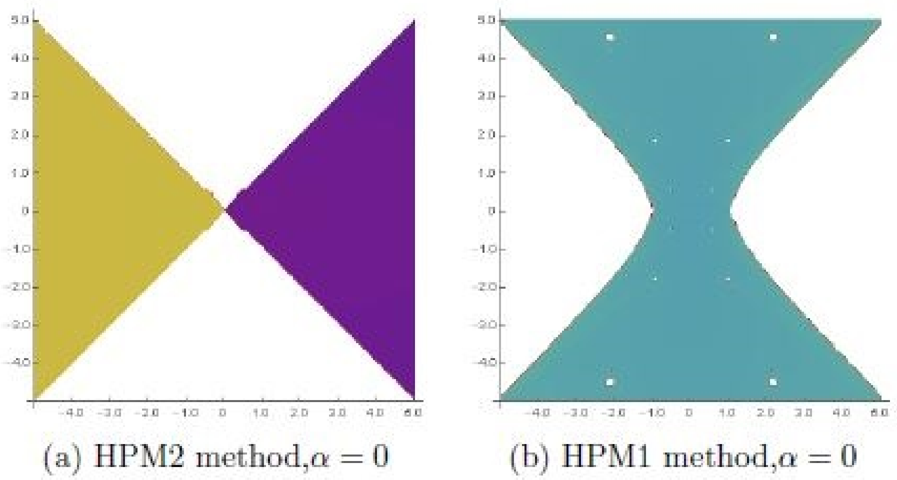


Figure 8 – Finding the roots of the polynomial $p_3(z) = z^3 - z$

At the end of this section, we have compared the attraction basin of the proposed methods with one-step methods Newton (NM), Steffensen (SM) and Abbasbandy's method (AM) [19]. The dynamical behavior two-step Kung-Traub's method (KTM) [20], Fomtini-Sormani's method (FSM) [21] and Maheshwari's method (MM) [22] are given. Also, the dynamical planes were obtained through three steps of Chun-Lee's methods (CLM) [23].

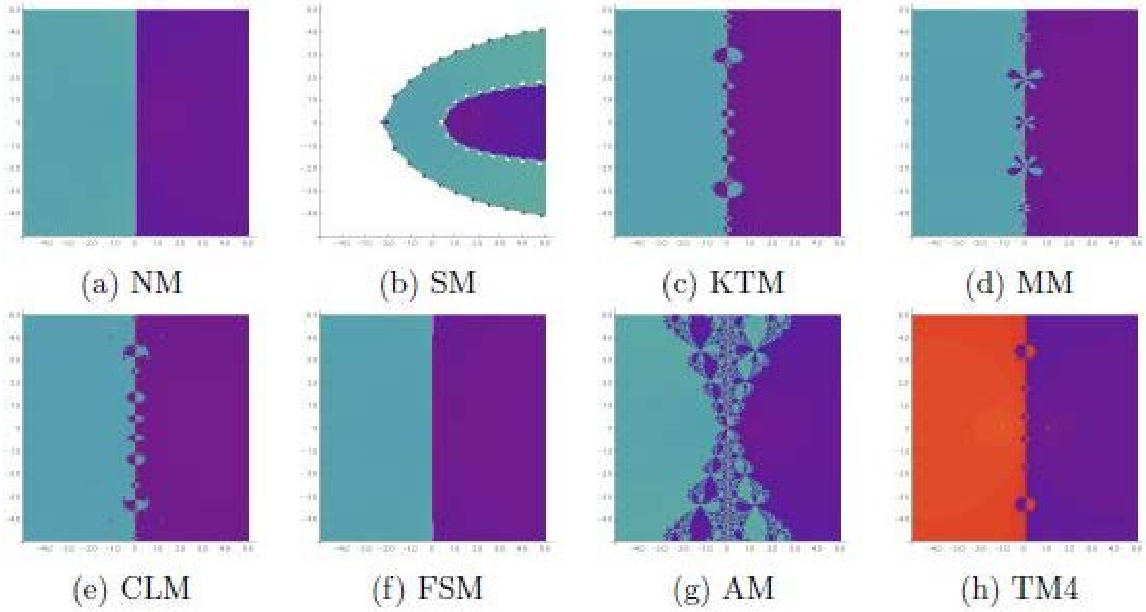


Figure 9 – Finding the roots of the polynomial $p_1(z) = z^2 - 1$

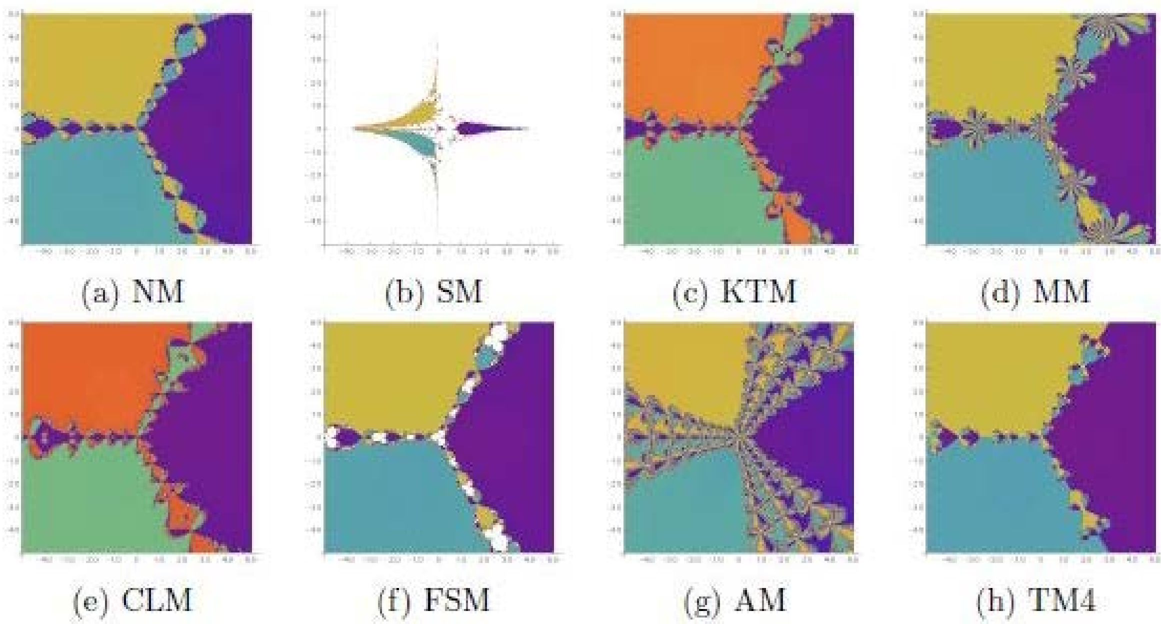


Figure 10 – Finding the roots of the polynomial $p_2(z) = z^3 - 1$

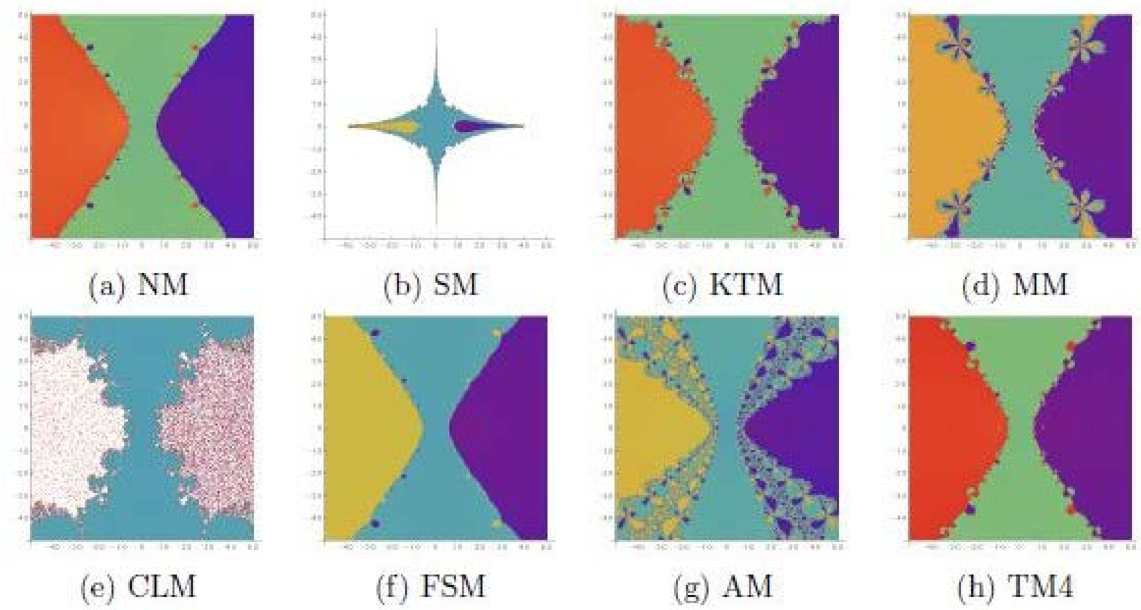


Figure 11 – Finding the roots of the polynomial $p_3(z) = z^3 - z$

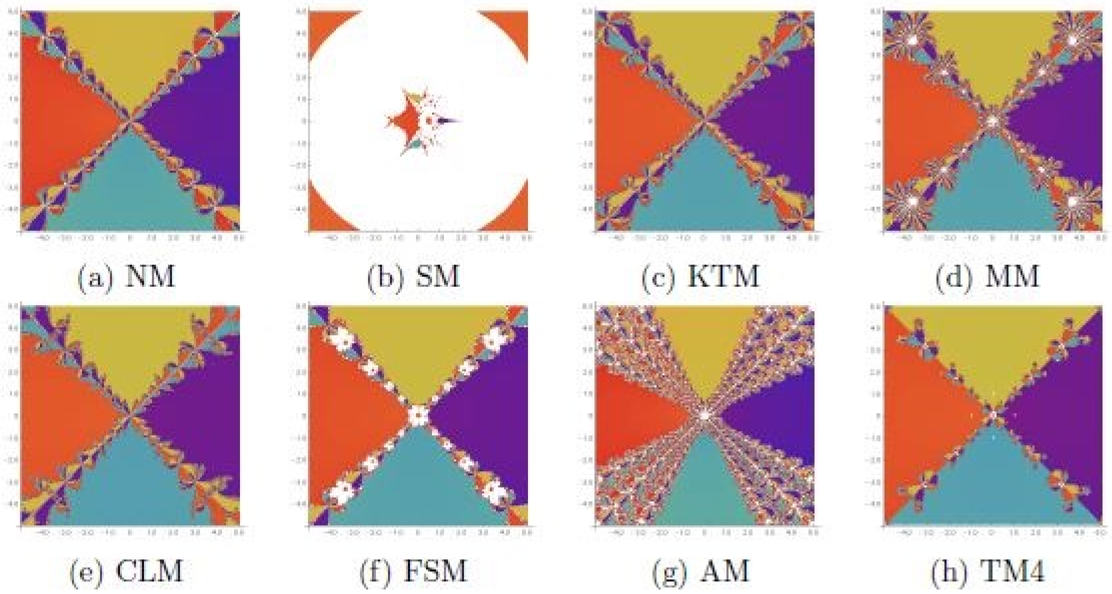


Figure 12 – Finding the roots of the polynomial $p_4(z) = z^4 - 1$

6 Conclusion

This third-parametric Hansen-Patrick's family does not need any derivative. The convergence orders of the new derivative-free methods with memory have increased from 4 to 6, 7, 7.53, 7.94, 7.99, and 8. We were able to improve the convergence order of optimal two-step methods up to 100% without any additional functional evaluations. The efficiency index of the proposed adaptive family with memory is $\sqrt[3]{8}=2$, which is much better than that of optimal methods without memory, and all the methods mentioned in the references [24, 25, 26,27,28,29,30]. Comparison improvement of convergence order of the proposed method with other schemes is presented in Table 5 They perform better than the other well-organized efficient iterative methods in all the considered problems. The proposed iterative methods not only have a higher order of convergence but also a stable behavior in the complex plane, with global convergence to the simple roots in many cases or wide areas of convergence in the rest of them. In addition, our methods have not only minimum residual error corresponding to considered test function f but also have smaller error differences between two consecutive iterations.

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