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Scale invariance criteria of dynamical chaos

Abstract. This work is devoted to study out the following question: does any qualitative criteria of realization of such universal phenomena as self-organization exist in open systems? Self-organization is also called the appearance of order from chaos under the conditions ofnon-linearity,non-equilibrium and nonclosure. Information entropy and fractal dimension of a set of physical values are usually used as quantitative characteristics of chaos. The more detailed characteristic of dynamical chaos is the Kolmogorov-Sinay entropy. Inhomogeneity of elements of a phase space can be taken into account by use of this characteristic. Technically, precise calculation of Kolmogorov – Sinayentropy can't be realized. Uncertain questions are: What is the minimum of increasingof entropy, how muchitdecreases atself-organization? Also it was not ascertained the connection between entropy criterion of self-similarity and self-affine with fractal dimensions characterized corresponding chaotic processes.

In the paper the values of information at fixed points of probability function of density of information $I_1$and entropyI_2 have been defined. Physical meaning of these values as criteria of self-affinity and self-similarity in chaotic processes have been explained. The Kolmogorov-Sinay entropy and fractal dimensions corresponding to scale-invariant sets have been described also.

It is shownthat self-organizationoccurs whennormalizedinformation entropyStakes valuesin the interval $I_1 \leq S \leq I_2$, where $I_1 = 0.567$, $I_2 = 0.806$. The precision of thesefindings is proved by calculation the valueS. Applications of these resultsin modernscientific and engineering areas are possible.
Keywords: information, entropy, fractal, chaos, self-organization.

Introduction

Because of rapid development of modern technologies we need to know physical regularities of nanostructures, microwave chaotic signals, ensembles of neural networks and so on. In spite of complexity of these objects and processes they have one common property which is the scale invariance. It means that for the description of such objects it isnot necessary to use parameters with a concrete dimension, for example, dimension of length. Self-similarity (object characterized by only one value of similarity factor on different variables) and self-affinity (several similarity factors need for its description) are types of scale invariance. The common property of the processes with different nature is self-organization of matter and its motion. Theory of self-organization is called the synergetics.

Self-organization can be also considered as appearance of order from chaos if described system is open, nonlinear and non-equilibrium. So, for simplicity we shall discuss invariant properties of chaotic processes.

As usual, informational entropy and fractal dimension of a set of physical values are used as quantitative characteristics of chaos. More detailed characteristic of dynamical chaos is the Kolmogorov-Sinay entropy. By use of the value we can take into account heterogeneousness of elements of considered phase space. According to the well-known I. Prigozhin theorem, the derivative of entropy with respect to time tends to its minimal value at self-organization in the system. According to the Yu.Klimontovich theorem, entropy of a system decreases at self-organization if energy of system is constant [1].

But exact calculation of Kolmogorov-Sinay entropy notbe realized. By use of the mentioned above theorems we cannot answer the following questions: how to define a minimal increasing of entropy? What is entropy reduction at self-organization equal to? Also it isnot clear the relation between entropic criteria of self-similarity and self-affinity with fractal dimensions

Informational criteria of scale invariance

Conception of information widely used in cybernetics, genetics, sociology and so on. Development of synergetics and physics of open systems stimulates formulation of a universal definition of information which can be used in different branches of science. The definition of open system contains the conception of information: open system is a system in which energy, matter and information are exchanged with its environment.

As usual, definition of a complex object includes its main properties. Information $I(x)$ for statistical realization of a physical value x is greater than zero and can be defined in non-equilibrium state ($I(x) \neq I(x_0)$,if $x \neq x_0$). Let us consider that $P(x)$ is probability of realization of a variable x. So, expression for the description of quantity of information can be written as

$$I(x) = -\ln P(x). \quad (1)$$

Reiteration and non-equilibrium of a process can be taken into account by the condition $0 < P(x) < 1$. A lot of definitions of information have been suggested in different branches of science, but (1) corresponds with all of them.

Information can be defined as

$$I(x/y) = S(x) - S(x/y), \quad (2)$$

where$S(x)$ is absolute information entropy of an event x and $S(x/y)$ is conditional entropy of an event x when another event y is to have occurred. The Eq. (2) can be used for solving of technical problems such as for estimation of circuit grade (in communication channels). Informational entropy $S(x)$ can be defined as mean value of information as

$$S(x) = \sum_i P_i(x) I_i(x) = -\sum_i P_i(x) \ln P_i(x). \quad (3)$$

Here i is number of a cell of x after its segmentation. So, let us use the Eq. (1) as a main definition of information.

A universal definition of information not exist, but information is used for the description of phenomena of different nature. So, another new approach for using of information in the theory of informational phenomena is possible. We can use information as a certain independent variable. Statistical characteristics of a process can be described via information. So, we can try to find new properties of information, for example, its scale-invariant properties.

Therefore, according to the Eq. (1) we shall describe probability of realization of information $P(I)$ as

$$P(I) = e^{-I}. \quad (4)$$

Probability function $f(I)$ can be defined via the following relations:

$$0 \leq P(I) \leq 1, 0 \leq I \leq \infty, \int_0^\infty f(I)dI = 1, \quad P(I) = \int_I^\infty f(I)dI, \; f(I) = P(I) = e^{-I}. \quad (5)$$

Probability function $P(I)$ equals to density of probability distribution function $f(I)$. Information defined via Eq. (1) characterized by property of scale invariance. It means that a whole object and its part have the same law of distribution. Informational entropy $S(I)$ of distribution of information can be defined as mean value of information as

$$S(I) = \int_I^\infty I f(I) dI = (1 + I)e^{-I}. \quad (6)$$

Values of entropy can be normalized to unit, so, for $0 \leq I \leq \infty$ we have $1 \geq S \geq 0$. It is well known that entropy of continuous set tends to infinity at jumping values of variables. Therefore, we must take the integral by use of Lebesgue measure. By choosing information as the measure we have got Eq. (6).

Let us consider that a certain scale-invariant function $g(x)$ justifies to the well-known functional equation as

$$g(x) = \alpha g\big(g(x/\alpha)\big), \quad (7)$$

whereα is a scaling factor. All continuous functions in theirs fixed points justify the Eq. (7). We use $f(I)$ and $S(I)$ as characteristic functions. Fixed points of the functions are

$$f(I) = I, \; e^{-I} = I, \; I = I_1 = 0.567\,, \quad (8)$$

$$S(I) = I, \; (1 + I)e^{-I} = I, \; I = I_2 = 0.806\,. \quad (9)$$

The fixed points are limits of following infinite maps

$$I_{i+1} = f(I_i),\ \lim_{i\to\infty} exp\big(-exp(\dots - exp(I_0)\dots)\big) = \mathrm{I}_1, \quad (10)$$

$$I_{i+1} = S(I_i),\ \lim_{i\to\infty} exp\big(-exp(\dots - exp(\ln(I_0+1) - I_0)\dots)\big) = \mathrm{I}_2, \quad (11)$$

at any initial values I_0. Number of brackets equals to $i+1$.

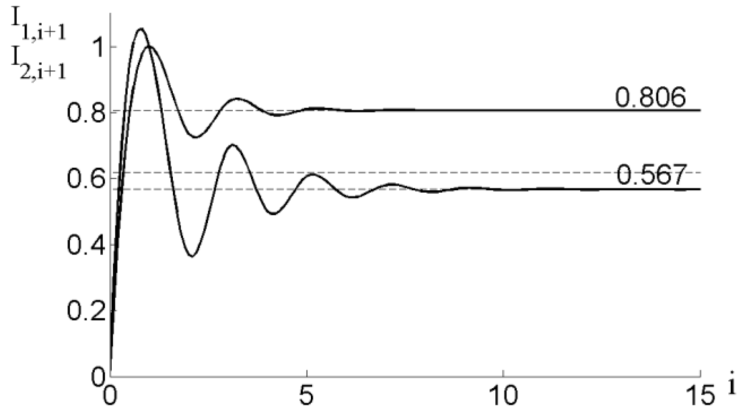


Figure 1 – Establishment of information self-similarity and entropy

Interpretations of physical meaning of numbers $I_1 = 0.567$ and $I_2 = 0.806$ can be different. Probability density is a local (instant) characteristic. Therefore, it can be different for different variables. So, the number I_1 can be used as a criterion of self-affinity. Entropy is an averaged characteristic. So, the number I_2 is the criterion of self-similarity.

On the other hand, numbers I_1 and I_2 can be considered as analog to the Fibonacci number $I_{20} = 0.618$ ("golden section" of dynamical measure) for statistical self-affine and self-similar systems correspondently. From Eq. (9) at $I \lesssim 1$ we have

$$(1+I)(1-I) = I,\ I^2 + I - 1 = 0,\ I = I_{20} = 0.618\,, \quad (12)$$

at$I \ll 1$ from the same equation we have $e^{-I} = I$, $I = I_1$. A conclusion is that we can use only Eq. (9) for the description of regularities of self-affinity, dynamical equilibrium and self-similarity of dynamical systems.

Relation between numbers I_1 and I_2 with fractal and multi-fractal dimensions of special sets and with Kolmogorov-Sinay entropy we shall consider apart.

Kolmogorov-Sinai entropy of fractal sets

Let us consider a trajectory $\vec{x}(t) = \{x_1(t), \dots, x_m(t)\}$ of a dynamical system on a strange (fractal) attractor. *m*–dimensionalphase space is subdivided into cells with size equal to l^m. $P_{i_0,\dots,i_n}$is conditional probability of the following events: $\vec{x}(t=0)$ is located in the cell i_1, $\vec{x}(t=\tau)$ is located in the cell $i_2, \dots$, $\vec{x}(t+(n-1)T)$ is located in the cell i_n, τ is time need to measure state of the system. So, Shannon entropy can be defined as

$$S_n = -\sum_{i_1,\dots,i_n} P_{i_1,\dots,i_n} \ln P_{i_1,\dots,i_n}. \quad (13)$$

Kolmogorov-Sinai entropy S (KS-entropy) is defined as average velocity of information losses [2] via the following relation:

$$S = \lim_{\tau\to 0}\lim_{l\to 0}\lim_{N\to\infty}\frac{1}{N\tau}\sum_{n=1}^{N}(S_{n+1}-S_n) =$$
$$= -\lim_{\tau\to 0}\lim_{l\to 0}\lim_{N\to\infty}\frac{1}{N\tau}\sum_{i_1,\dots,i_N} P_{i_1,\dots,i_N}\ln P_{i_1,\dots,i_N}\ . \quad (14)$$

If time is discrete, so $\tau = 1$ and limitation $\tau \to 0$ should not be taken into account. Physical meaning of parameterτ is characteristic time depending on l.

Can KS-entropy be a constant value? This is the question of interest. First of all, it is necessary to define conditions at which $N\tau$ is a constant value for slace-invariant and time phenomena.

By definition of fractal dimension D of a fractal measure $N(l)$ depending on scale of measurement of the self-similar set only we have

$$N(l) = l^{-D},\ \ D = \lim_{l\to 0}\frac{\ln N(l)}{\ln 1/l}. \quad (15)$$

We can use number of variables need for the description of a dynamical system as$N(l)$. Here l is a minimal typical scale of the system. Self-affine sets of variables with different nature (for example, coordinates or time) can be described via Hurst exponent H (H is called the affinity exponent also) as

$$\frac{R}{\sqrt{\langle x^2(t)\rangle}} = \tau^H,\ \ R(\tau) = \max_{1\le t\le\tau} x(t,\tau) - \min_{1\le t\le\tau} x(t,\tau). \quad (16)$$

$H = 1/2$for Brownian motion which is purely stochastic. $1/2 < H < 1$for chaotic phenomena. The well-known formulas

$$D = 1/H,\ \ D = 2 - H, \quad (17)$$

follow from the requirement of scale invariance of generalized (“fractional”) Brownian motion and can be used for the description of self-similar and self-affine sets correspondently [3].

Using the dimensionless variable l as $l = (\langle x^2(t)\rangle)^{1/2}/R$ via Eqs. (15)-(17) we have

$$N\tau = l^{-D}l^{-1/H} = l^{-2D} = N^2, \tau = N. \quad (18)$$

Self-similarity is obtained if number of hierarchical system variables equal to selected dimensionless scales of time. In this case we can accept $\tau = 1$, where will be taking norm N.

For self-affine cases we have

$$N\tau = l^{-D}l^{-1/(2-D)} = l^{-D_*},\ D_* = D + 1/(2-D), \quad (19)$$

physical meaning of D_* is an effective value of fractal dimensions of a self-affine set. Therefore, we have a possibility to relate KS-entropy with fractal and multi-fractal characteristics.

Via the Rényi equation for multi-fractal dimension of measure $N(q,l)$ we have

$$D_q = \lim_{l\to 0}\frac{1}{q-1}\frac{\ln N(q,l)}{\ln l},\ \ N(q,l) = \sum_{i=1}^{N} P_i^q(l)\ , \quad (20)$$

where$q$ is order of multi-fractal moment. Let us take into account transmissions to limits for $i = (i_1,\dots,i_N)$, $i = (i_1,\dots,i_N)$. So, from Eq. (20) at $q \to 1$ we have

$$D_{q\to 1} = D_1 = \lim_{l\to 0}\frac{S(l)}{\ln{}^1/_l} = S_1\ , \quad (21)$$

mathematical expression for $S(l)$ follows from Eq. (14) rewritten without of limitation $l \to 0$. In multi-dimensional cases value of $S(l)$ can be calculated via summation on indexes $i_1,\dots,i_N$.

Due to comparison of S_1 and S_{KS} we must answer the question: can values $N\tau$and $\ln 1/l$. For this aim we must take into account that self-affine states described by a set of values of fractal dimension $D_i, i = 1,2, ...$ Normalized on D_{*_i} value $(N\tau)_*$ can be defined as $l^{-D_*}/\sum_i l^{-D_{*i}}$ If spectrum from D_* is continuous, we have

$$(N\tau)_* = l^{-D_*}/\int l^{-D_*} dD_* = \ln 1/l. \quad (22)$$

The reasoning shows the possibility of equality between $S_{KS}$and S_1.

On the other hand, according to the well-known definition of multi-fractal spectral function $f(\alpha(q))$ we can write

$$f(\alpha(q)) = q\alpha(q) + \tau(q),\ \tau(q) = (1-q)D_q,\ \alpha(q) = -\frac{d\tau(q)}{dq},$$

$$\alpha(q=1) = \alpha_1,\ f(\alpha(q=1)) = f_1,\ D_1 = \alpha_1 = f_1 = S_1, \quad (23)$$

hereα is singularity factor of a cell of a set (the Lipshitz–Hölder factor), $f(\alpha)$ is fractal dimension of a set of cells with equal values of α. Fractal characteristics D_1,α_1, f_1 don't depend on l by definition. Therefore, S_1 is independent on l also.

So, *KS*-entropy defined via normalized value $N\tau_*$ equals to self-similar value (fixed point) of informational entropy, i.e. to number I_2. Criterion I_1 is a stable point of probability distribution function of information ("local entropy"). So, I_1 can be considered as a self-affine parameter of KS-entropy [4].

Informational entropy of two-dimensional objects according to the degree of homogeneity

Criteria of self-organization which we established fornonequilibrium processes (heterogeneous objects) can be differentiated from I_1, I_2. It is necessary to consider the impact of some parameter taking into account the deviation from equilibrium.

In recent years the new generalized statistical mechanics are developed, which can be called the statistics of Tsallis, or quasicanonical Gibbs distribution [5]. At the heart of such theories the exponential function is used, such as:

$$exp_{q-1}[x] = (1 + (q-1)x)^{\frac{1}{q-1}}, \quad (24)$$

where$q$ is the degree of homogeneity, incompleteness parameter of statistical ensemble. In the limit $q \to 1$, we get the usual exponential. Within the meaning of entering

$$|q-1| \sim \frac{1}{M}, M = L - N, M \to \infty, q \to 1, \quad (25)$$

where L is the number of particles of the closed system, N is the number of particles of the subsystem. The completeness of statistics corresponding to the canonical Gibbs's distribution of equilibrium is reached at $q = 1$. The distinction of unit parameter q characterizes the degree of statistical non-equilibrium and the heterogeneity of the system.

We define the full entropy $S(x,y)$ according to given degree of homogeneity of object. Let us take as a one-dimensional variables and conditionnal probabilities $z_1 = P(x), z_2 = P(y/x)$. According to the Eq. (24) we have

$$exp_{q-1}[z_1]exp_{q-1}[z_2] = exp_{q-1}[z_1 + z_2 + (q-1)z_1 z_2]. \quad (26)$$

Introducing the left side as "logarithm $q-1$" of the product, we obtain

$$\ln_{q-1} z_1 z_2 = \ln_{q-1} z_1 + \ln_{q-1} z_2 + |q-1| \ln_{q-1} z_1 \ln_{q-1} z_2. \quad (27)$$

From the Eq. (27) followed the expression for the non-additive "S_{q-1} −entropy":

$$S_{q-1}(x,y) = S_{q-1}(x) + S_{q-1}(y/x) + |q-1| S_{q-1}(x) S_{q-1}(y/x). \quad (28)$$

In the limit of $q \to 1$ we have additive entropy $S(x,y) = S(x) + S(x/y)$.

According to the definition of q the Eq. (25) it can be determined from experimental data. In order to describe the heterogeneity of geometric objects we enter the small parameter εthrough $q \approx 1$. The algorithm for determining q accept as

$$q = \frac{N - <m>n(\delta)}{N} = 1 - \frac{<m>n(\delta)}{N} = 1 - \varepsilon, \quad (29)$$

$$\varepsilon = \frac{<m>n(\delta)}{N}, 0 < \varepsilon < 1, \quad (30)$$

where N is the total number of points (samples), $n(\delta)$is the number of cells with the scale of measurement δ, in which there is at least one point, $< m >$ is the average number of points in the cell. From the generalization of the theory of Tsallis can be used module of $(q - 1)$. According to the Eq. (29), we have a choice

$$q = 1 + \varepsilon. \quad (31)$$

Using the expression (24), we define the q isdependence of information entropy, which is the only measure of complexity, uncertainty of non-equilibrium system. For quasi-equilibrium process, characterized by the parameter q, the information defined in the form

$$I = -\ln_{q-1} P. \quad (32)$$

Hence represent the probability as a function of information:

$$P(I) = exp_{q-1}[-I] = (1 - (q-1)I)^{\frac{1}{q-1}}. \quad (33)$$

The density function of the probability distribution implementation of information $p(I)$ is defined as

$$f(I) = \frac{d}{dI} exp_{q-1}[-I] = (1 - (q-1)I)^{\frac{2-q}{q-1}}. \quad (34)$$

Entropy is defined as the average value of the information:

$$S_q(I) = \int_I^{\infty} If(I)dI = (1 + I)exp_{q-1}(-I) = (1 + I)(1 - (q-1)I)^{\frac{1}{q-1}}. \quad (35)$$

Self-similar values $f(I_1) = I_1$ and $S(I_2) = I_2$ find as fixed points of mappings

$$I_{1q,i+1} = \left(1 - (q-1)I_{1,i}\right)^{\frac{2-q}{q-1}}, \quad (36)$$

$$I_{2q,i+1} = \left(1 + I_{2,i}\right)\left(1 - (q-1)I_{2,i}\right)^{\frac{1}{q-1}}, \quad (37)$$

$$I_{1q,0} = I_{2q,0} = 0, \infty; \qquad i = 0,1,2 \ldots.$$

Thus, the value of q can characterize the deviation of the state from self-similarity and self-affinity through the values of information and informational entropy. In multifractal analysis some q parameter is set in the range $(-\infty, \infty)$, but its physical meaning is staying unclear. However, we note that under the condition

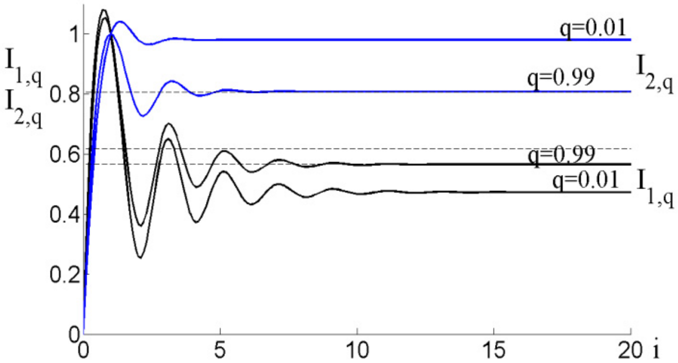
$$\sum_i (P_i^q - P_i) \ll 1, \quad (38)$$

Tsallis's entropy defined by the Eq. (24) coincides with the Rényi's entropy:

$$S_{R,q} = -\frac{1}{q-1} \ln \sum_i P_i^q. \quad (39)$$

This fact gives the physical meaning of the parameter q in certain cases. Forexample, inmultifractalanalysis$q$is aninversetemperature.

Universal entropic regularities of evolution of open systems to self-similarity and self-affinity modes according to the Eq. (36) and (37) are shown in Fig. 2, where we take some characteristic time as $t = i$. Fluctuation from equilibrium leads to qualitatively different patterns establish the scale invariance for $q <$ 1and $q > 1$.



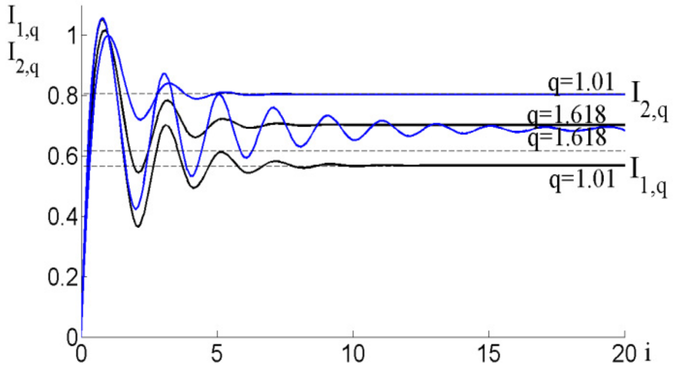


Figure 2 – Establishing the evolution of information ($I_{1,q}$) and entropy ($I_{2,q}$) for different 'q'

Applications of criteria of self-similarity and self-affinity

Normalized values of Shannon entropy $I_1 \leq S_n/S_{n_{max}} \leq I_2$ describe self-organized process including regimes of self-affinity and self-similarity. Which isshown in Figure 2 for $q > 1$. We can simplify physical analyses of chaotic signals in nanoelectronics, astrophysics, biophysics, etc., by taking into account the described fact. In one-dimensional case value of $S_{n_{max}}$ equals to entropy of isosceles triangular impulse. In multi-dimensional case instead of S_n we can use normalized multi-fractal spectral function $f(\alpha_1) = \alpha_1$. It is possible, because $f_{max}(\alpha) = D_0$ and Hausdorff-Besikovitch dimension D_0 is calculated by use standard algorithms.

Numbers I_1, I_2 define fractal measures of self-similar and self-affine phenomena. We consider measure as quantitative characteristic of any additive and measurable set. Fractal measure defines as

$$M = M_0 l^{-(D-d)} = M_0 l^{-\gamma}, \quad (40)$$

where$d$ is topological dimension of measure carrier, M_0 is non-fractal (regular) measure. Let us define values of $\gamma = D - d$ via $I_1$and I_2. We shall designate maximal integer value D asd_{max}, $D_{max} = d_{max} + 1$and $I_* = (I_1, I_2)$. It is possible that

$$D = d_{max} + I_*, \gamma = D - d = d_{max} - d + I_*, d = 0,1,2,3. \quad (41)$$

Common number of possible values of γ in Eq. (41) can be defined as

$$N_\gamma = \sum_{i=1}^{4} \sum_{j\leq i}^{4} \sum_{k=1}^{2} (d_{max,i} - d_j) = 2\sum_{i\geq j=1}^{4} (d_{max,i} - d_j) = 2\sum_{n=1}^{4} n = 2 \cdot \frac{4(4+1)}{2} = 20. \quad (42)$$

Also we can use value of Fibonacci number for calculation of N_γ. In this case we must accept $I_* = (I_1, I_2, I_{20})$. Result of the calculation is $N_\gamma = 30$. Physical meaning of I_{20} is a transition value of I_* at transitions from $I_1$to I_2.

Therefore, we have a possibility to use 20 exact values of fractal dimension D. Determination of numerical value of fractal dimension is a stand-alone scientific problem. Well-known methods for its determination via experimental data supply accuracy of results less than one percent, but using of I_1 and I_2 considerably simplify solving of the problem. Conclusions of the present work can be checked by calculation of Kolmogorov-Sinai entropy of different fractals. Each generation of a fractal can be corresponded to an evolutional level. Because of the fact geometrical fractals considered as models of dynamical systems can be used for such calculations.

Let us select two methods of comparison of theory with numerical experiment. the first one is based on calculation of Kolmogorov-Sinai entropy only for one part of fractal according to Eq. (6):

$$S = (1+I)e^{-I} = (1 + D\ln {}^{1}/_{l})e^{-D\ln {}^{1}/_{l}}$$
$$I = -\log_{1/l} P(l) \ln {}^{1}/_{l} = \log_{1/l} N(l) \ln {}^{1}/_{l} = D\ln {}^{1}/_{l}, \quad (43)$$

where minimum information feedback (connection) to the fractal dimension have been used.

The second method is based on calculation of K-entropy via Eq. (14) for n generations of a fractal. For this situation to take into account the influence of two dimensional objects heterogeneity, we should use the Eq. (28).

Calculation have shown that Eq. (43) gives the deviation of $S$from the interval $[I_1, I_2]$ when$D \geq 1,5$. Fig. 3 shows the dependence of $S(D)$ according to Eq. (28). Fractals are generally accepted [6]. Information entropy normalized values of all the above fractals are in the range of self-organization $[I_1, I_2]$.

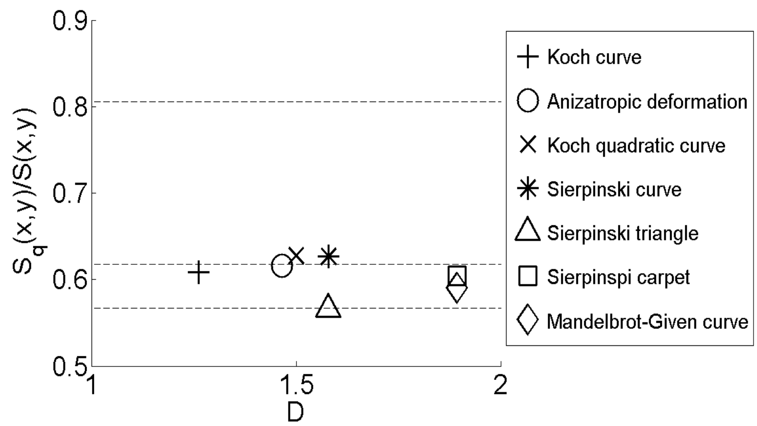


Figure 3 – Informationentropyoffractals(x, y) dimensionsconsideringthehomogeneousdegree

To compare the resultsof Eq. (28) and (37), choice of the values of$q = 1 - \varepsilon, q = 1 + \varepsilon$is determinedfrom the following considerations. TheEq. (35) is given conditions of minimum$S_q(I)$in the form $q = 1 + {}^{1}/_{I_*}, I_* > 1. \varepsilon = {}^{1}/_{I_*} = 0.618$follows from Fig. 2.By (43) Eq.$I_* = D_* \ln {}^{1}/_{\delta} = {}^{1}/_{0.618}$.It follows that the quantitative change $S_q(I)$occurs when $D > D_* \approx 1.5$. On the other hand, when $I \to 0$is taken $S_q(I) \to 1$,i.e. entropy increases when $I < I_*$and$I > I_*$.Consequently, in order to take into accountthe effect ofqualitatively differentparameter $q$should we choose$q = 1 - \varepsilon$when$I < I_*$.

Conclusions

Wehaveshownthatchaoticprocessesandobjects canbequantitativelyclassifiedintoself-affinean-dself-similar. Relevant criteria I_1, I_2 are the values of Kolmogorov – Sinai's entropy, determine the characteristic fractal measures. The existence of these criteria of scale – invariance is also proved by appropriate treatment of results of experimental studies of chaotic processes of different nature (turbulence, Solar radio emission, optical radiation of variable stars, etc.).

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