IRSTI 27.31.17, 27.31.44, 27.35.17 https://doi.org/10.26577/ijmph.2023.v14.i2.04

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(*Received 29 November 2023; received in revised form 06 December 2023; accepted 08 December2023*)

On a non-local problem for system of partial differential equations of hyperbolic type in a specific domain

Abstract. The non-local problem for second order system of partial differential equations of hyperbolic type is studied in the specific domain. For solving this problem we use a functional parametrization method. This method is an extension of Dzhumabaev's parametrization method to a partial differential equations of hyperbolic type. We introduce a parameter-function, expressed as the unknown function's value at the characteristics $t = 0$ within the given domain. This transforms the nonlocal problem into an equivalent parameterized problem, involving the Goursat problem for a system of partial differential equations of hyperbolic type and an additional relation based on the functional parameter. Subsequently, starting from the additional condition and the consistency condition, we formulate the Cauchy problem for a system of differential equations with respect to the unknown parameter-function. We develop an algorithm for solving the parameterized problem and demonstrate its convergence. Additionally, we derive conditions for the existence and uniqueness of a solution to the parameterized problem. Unique solvability conditions for the nonlocal problem for second-order system of partial differential equations of hyperbolic type in a specific domain are established in terms of the initial data.

Key words: non-local problem, partial differential equations of hyperbolic type, parameterized problem, Dzhumabaev's parametrization method, solution.

1 Introduction and statement of problem

Nonlocal problems for partial differential equations of hyperbolic type involve integral or nonlocal operators in addition to the usual differential operators. These nonlocal conditions can arise in various physical and mathematical contexts and often have applications in modeling phenomena with memory effects, long-range interactions, or nonlocal interactions (see [1-3] and references cited therein). Non-local problems for hyperbolic equations also have applications in various fields, including physics, biology, and finance.

Solving non-local problems for hyperbolic equations often involves a combination of numerical and analytical methods, as the presence of nonlocal conditions makes direct application of standard numerical techniques more challenging. Here are some methods commonly used for solving nonlocal problems associated with hyperbolic equations: Characteristic methods, Analytical approaches, Finite difference methods, Finite element methods, Spectral methods, Integral equation methods, Operator splitting techniques, Inverse problems techniques and etc. The choice of method depends on the specific form of the non-local condition, the characteristics of the hyperbolic equation, and the desired accuracy of the solution. Often, a combination of methods or hybrid approaches may be necessary to efficiently and accurately solve non-local problems for hyperbolic equations [1–7].

A method of functional parametrization for solving nonlocal problems for hyperbolic equations with mixed derivatives was proposed in works [8-10].

This method is a modification of Dzhumabaev's parametrization method [11], it proposed for solving BVPs for ODEs.

This approach facilitated the determination of conditions under which non-local problems for systems of hyperbolic equations are solvable, expressed in terms of the system coefficients and boundary matrices. Additionally, algorithms for discovering approximate solutions were devised, showcasing their convergence towards the precise solutions of the investigated issues. These findings were then applied to non-local problems with integrals and multi-point conditions [12-16], involving impulsive effects and delayed arguments [17-19], and addressing systems of hyperbolic equations with loads and piecewise-constant arguments [20-24].

In this current article, we will expand the application of the functional parametrization method to address a novel class of non-local problems for second-order hyperbolic equation systems within a specific domain

$\Omega_d = \{(t,x): 0 \le t \le d(x), 0 \le x \le \omega\}$. Here the function $d(x)$ is definite positive continuous on $[0,\omega]$.

We consider the non-local problem for system of partial differential equations of hyperbolic type in the next form:

$$\frac{\partial^2 u}{\partial x \partial t} = A(t,x)\frac{\partial u}{\partial x} + B(t,x)\frac{\partial u}{\partial t} + C(t,x)u + f(t,x), (t,x) \in \Omega_d, \quad (1)$$

$$P(x)\frac{\partial u(0,x)}{\partial x} + S(x)\frac{\partial u(d(x),x)}{\partial x} = \varphi(x), \quad x \in [0,\omega], \quad (2)$$

$$u(t,0) = \psi(t), t \in [0, d(0)]. \quad (3)$$

Here $u(t,x) = col(u_1(t,x), u_2(t,x), \ldots, u_n(t,x))$ is desired function, the matrices $A(t,x), B(t,x)$, $C(t,x)$ on dimension $(n \times n)$ and the vector function $f(t,x)$ on dimension n are continuous on Ω_d, the $(n \times n)$ matrices $P(x)$, $S(x)$ and the n vector function $\varphi(x)$ also are continuous on segment $[0,\omega]$, the n vector function $\psi(t)$ is continuously differentiable on segment $[0, d(0)]$.

A classical solution to the non-local problem (1)-(3) is a continuous on Ω_d function $u^*(t,x) \in C(\Omega_d, R^n)$ having continuous partial derivatives $\frac{\partial u(t,x)}{\partial x}$, $\frac{\partial u(t,x)}{\partial t}$, $\frac{\partial^2 u(t,x)}{\partial x \partial t}$, that satisfies to system of partial differential equations of hyperbolic type (1) for all $(t,x) \in \Omega_d$, nonlocal condition (2) for all $x \in [0,\omega]$ and condition on the characteristics (3) for all $t \in [0, d(0)]$.

2 Reduction to an equivalent problem

We introduce a parameter-function as follows: $\lambda(x) = u(0,x)$. Then we replace the unknown function $u(t,x)$ by sum of functions:

$$u(t,x) = z(t,x) + \lambda(x) \text{ for all } (t,x) \in \Omega_d.$$

The problem (1)-(3) is reduced to the equivalent problem:

$$\frac{\partial^2 z}{\partial x \partial t} = A(t,x)\frac{\partial z}{\partial x} + B(t,x)\frac{\partial z}{\partial t} + C(t,x)z + A(t,x)\dot{\lambda}(x) + C(t,x)\lambda(x) + f(t,x), \quad (t,x) \in \Omega_d, \quad (4)$$

$$z(0,x) = 0, x \in [0,\omega], \quad (5)$$

$$z(t,0) = \psi(t) - \psi(0), t \in [0, d(0)], \quad (6)$$

$$[P(x) + S(x)]\dot{\lambda}(x) + S(x)\frac{\partial z(d(x),x)}{\partial x} = \varphi(x), \quad x \in [0,\omega]. \quad (7)$$

From consistence condition at the point (0,0) we have

$$\lambda(0) = \psi(0). \quad (8)$$

A solution to parameterized problem (4)-(7) is a functions pair $\{z^*(t,x), \lambda^*(x)\}$ with a continuous on Ω_d vector function $z^*(t,x)$ having a continuous partial derivatives $\frac{\partial z^*(t,x)}{\partial x}$, $\frac{\partial z^*(t,x)}{\partial t}$, $\frac{\partial^2 z^*(t,x)}{\partial x \partial t}$, and with a continuously differentiable on $[0,\omega]$ parameter-function $\lambda^*(x)$, if it fulfills to parameterized system of partial differential equations of hyperbolic type (4) for all $(t,x) \in \Omega_d$, characteristics conditions (5) for all $x \in [0, \omega]$ and (6) for all $t \in [0, d(0)]$, and additional relation (7) for all $x \in [0,\omega]$.

The understanding of the equivalence between problems (1)-(3) and (4)-(7) is as follows.

Let function $u^*(t,x)$ be a solution to non-local problem (1)-(3), then the functions pair $\{z^*(t,x), \lambda^*(x)\}$, where

$$z^*(t,x) = u^*(t,x) - u^*(0,x),$$

$$\lambda^*(x) = u^*(0,x),$$

is a solution of parameterized problem (4)-(7).

Vice versa, if a functions pair $\{\tilde{z}(t,x), \tilde{\lambda}(x)\}$ be a solution to parameterized problem (4)-(7), then the function $\tilde{u}(t,x)$ defined as

$$\tilde{u}(t,x) = \tilde{z}(t,x) + \tilde{\lambda}(x), \quad (t,x) \in \Omega_d,$$

is a solution of the initial non-local problem (1)-(3).

In contrast to non-local problem (1)-(3), parameterized problem (4)-(7) has characteristics condition (5) at the line $t=0$.

At fixed parameter-function $\lambda(x)$ the parameterized problem (4)-(6) is the Goursat problem for system of partial differential equations of hyperbolic type. For determining an unknown parameter-function $\lambda(x)$ we have additional relation (7).

Let us introduce

$$v(t,x)=\frac{\partial z(t,x)}{\partial x},\ w(t,x)=\frac{\partial z(t,x)}{\partial t}.$$

Assume that parameter-function $\lambda(x)$ is known for all $x\in[0,\omega]$. Then the Goursat problem (4)-(6) will be equivalent to the three system of integral equations in the following form:

$$v(t,x)=\int_0^t\{A(\tau,x)v(\tau,x)+B(\tau,x)w(\tau,x)+C(\tau,x)u(\tau,x)+f(\tau,x)\}d\tau+\int_0^t A(\tau,x)d\tau\,\dot{\lambda}(x)+\int_0^t C(\tau,x)d\tau\ \lambda(x), \quad (9)$$

$$w(t,x)=\int_0^x\{A(t,\xi)v(t,\xi)+B(t,\xi)w(t,\xi)+C(t,\xi)u(t,\xi)+f(t,\xi)\}d\xi+\dot{\psi}(t)+\int_0^x\left[A(t,\xi)\dot{\lambda}(\xi)+C(t,\xi)\lambda(\xi)\right]d\xi, \quad (10)$$

$$z(t,x)=\int_0^t w(\tau,x)\ d\tau+\psi(t)-\psi(0). \quad (11)$$

From (9) we find $v(d(x),x)$ and substitute it in (7) instead of $\frac{\partial z(d(x),x)}{\partial x}$. Then, we have a system of differential equations with respect to $\lambda(x)$:

$$Q(x)\dot{\lambda}(x)=-L(x)\lambda(x)-F(x)-G(x,v,w,z),\ x\in[0,\omega], \quad (12)$$

where

$$Q(x)=P(x)+S(x)+S(x)\int_0^{d(x)}A(\tau,x)d\tau,$$

$$L(x)=S(x)\int_0^{d(x)}C(\tau,x)d\tau,$$

$$G(x,v,w,z)=S(x)\int_0^{d(x)}A(\tau,x)v(\tau,x)d\tau+S(x)\int_0^{d(x)}B(\tau,x)w(\tau,x)d\tau+S(x)\int_0^{d(x)}C(\tau,x)z(\tau,x)d\tau,$$

$$F(x)=S(x)\int_0^{d(x)}f(\tau,x)d\tau-\varphi(x).$$

The system of differential equations (12) and initial condition (8) give us the Cauchy problem for first order differential equations according to parameter-function $\lambda(x)$ for all $x\in[0,\omega]$.

As a result, we derive a closed system that incorporates the Goursat problem for the hyperbolic equations system (4)-(6) and the Cauchy problem for the system of differential equations (12), (8) with the aim of determining the functions pair $\{z(t,x),\lambda(x)\}$.

3 Algorithm

The function $z(t,x)$ with its partial derivatives $v(t,x)$ and $w(t,x)$ are unknown together with parameter-function $\lambda(x)$ and its derivative $\dot{\lambda}(x)$. Problems (4)-(6) and (12), (8) are interconnected. The Goursat problem (4)-(6) is contingent on the parameter-function $\lambda(x)$. The Cauchy problem for system of differential equations (12), (8) is contingent on the unknown functions $z(t,x)$, $v(t,x)$ and $w(t,x)$.

Hence, to find a solution to the parameterized problem (4)-(7), an iterative process is employed, guided by the following algorithm.

Step 1. (a) Assume that $z(t,x)=\psi(t)-\psi(0)$, $v(t,x)=0$ and $w(t,x)=\dot{\psi}(t)$ in the system (12). Further, we suppose that the $(n\times n)$ matrix $Q(x)$ is invertible for all $x\in[0,\omega]$. The Cauchy problem (12), (8) has a unique solution is a first approximation for parameter-function $\lambda(x)$:

$$\lambda^{(1)}(x)=U(x)U^{-1}(0)\psi(0)-U(x)\int_0^x U^{-1}(\xi)Q^{-1}(\xi)F(\xi)d\xi-U(x)\int_0^x U^{-1}(\xi)Q^{-1}(\xi)G\left(\xi,0,\dot{\psi}(t),\psi(t)-\psi(0)\right)d\xi,\quad x\in[0,\omega], \quad (13)$$

Here the $(n\times n)$ matrix $U(x)$ is a fundamental matrix to system of differential equations

$$\dot{\lambda}(x)=-Q^{-1}(x)L(x)\lambda(x),\ x\in[0,\omega].$$

We can also determine the first approximation for $\dot{\lambda}\ (x)$:

$$\dot{\lambda}^{(1)}(x) = -Q^{-1}(x)L(x)\lambda^{(1)}(x) - Q^{-1}(x)\,F(x) - -Q^{-1}(x)G\left(x, 0, \dot{\psi}\,(t), \psi(t) - \psi(0)\right), \quad x \in [0, \omega], \tag{14}$$

(b) Using the founded $\dot{\lambda}^{(1)}(x)$ and $\lambda^{(1)}(x)$, we solve the Goursat problem (4)-(6) for $\dot{\lambda}(x) = \dot{\lambda}^{(1)}(x)$ and $\lambda(x) = \lambda^{(1)}(x)$.

We have systems of integral equations (9)-(11) and find a first approximations for $z(t,x), v(t,x)$ and $w(t,x)$ are the functions $z^{(1)}(t,x)$, $v^{(1)}(t,x)$ and $w^{(1)}(t,x)$ for all $(t,x) \in \Omega_d$.

Step 2. (a) Assume that $z(t,x) = z^{(1)}(t,x), v(t,x) = v^{(1)}(t,x)$ and $w(t,x) = w^{(1)}(t,x)$ in the system (12). The Cauchy problem (12), (8) has a unique solution is a second approximation for parameter-function $\lambda(x)$:

$$\lambda^{(2)}(x) = U(x)U^{-1}(0)\psi(0) - -U(x)\int_0^x U^{-1}(\xi)Q^{-1}(\xi)F(\xi)d\xi - U(x)\int_0^x U^{-1}(\xi)Q^{-1}(\xi)G\big(\xi, v^{(1)}, w^{(1)}, z^{(1)}\big)d\xi, \quad x \in [0, \omega], \tag{15}$$

We can also determine the second approximation for $\dot{\lambda}\ (x)$:

$$\dot{\lambda}^{(2)}(x) = -Q^{-1}(x)L(x)\lambda^{(2)}(x) - Q^{-1}(x)\,F(x) - -Q^{-1}(x)G\big(x, v^{(1)}, w^{(1)}, z^{(1)}\big), \quad x \in [0, \omega], \tag{16}$$

(b) Using the founded $\dot{\lambda}^{(2)}(x)$ and $\lambda^{(2)}(x)$, we solve the Goursat problem (4)-(6) for $\dot{\lambda}(x) = \dot{\lambda}^{(2)}(x)$ and $\lambda(x) = \lambda^{(2)}(x)$.

We have systems of integral equations (9)-(11) and find a second approximations for $z(t,x), v(t,x)$ and $w(t,x)$ are the functions $z^{(2)}(t,x)$, $v^{(2)}(t,x)$ and $w^{(2)}(t,x)$ for all $(t,x) \in \Omega_d$.

And so on.

Step k. (a) Assume that $z(t,x) = z^{(k-1)}(t,x), v(t,x) = v^{(k-1)}(t,x)$ and $w(t,x) = w^{(k-1)}(t,x)$ in the system (12). The Cauchy problem (12), (8) has a unique solution is a *k*th approximation for parameter-function $\lambda(x)$:

$$\lambda^{(k)}(x) = U(x)U^{-1}(0)\psi(0) - -U(x)\int_0^x U^{-1}(\xi)Q^{-1}(\xi)F(\xi)d\xi - U(x) * \int_0^x U^{-1}(\xi)Q^{-1}(\xi)G\big(\xi, v^{(k-1)}, w^{(k-1)}, z^{(k-1)}\big)d\xi, \quad x \in [0, \omega], \tag{17}$$

We can also determine the *k*th approximation for $\dot{\lambda}\ (x)$:

$$\dot{\lambda}^{(k)}(x) = -Q^{-1}(x)L(x)\lambda^{(k)}(x) - Q^{-1}(x)\,F(x) - -Q^{-1}(x)G\big(x, v^{(k-1)}, w^{(k-1)}, z^{(k-1)}\big), \quad x \in [0, \omega], \tag{16}$$

(b) Using the founded $\dot{\lambda}^{(k)}(x)$ and $\lambda^{(k)}(x)$, we solve the Goursat problem (4)-(6) for $\dot{\lambda}(x) = \dot{\lambda}^{(k)}(x)$ and $\lambda(x) = \lambda^{(k)}(x)$.

We again have systems of integral equations (9)-(11) and find the *k*th approximations for $z(t,x), v(t,x)$ and $w(t,x)$ are the functions $z^{(k)}(t,x)$, $v^{(k)}(t,x)$ and $w^{(k)}(t,x)$ for all $(t,x) \in \Omega_d$. $k = 1,2,\ldots$.

4 Main result

Introduce a notations

$$\alpha(x) = \max_{t\in[0,d(x)]}\|A(t,x)\|, \delta = \max_{x\in[0,\omega]} d(x),$$

$$M = \max_{(t,x)\in\Omega_d}(\|A(t,x)\| + \|B(t,x)\| + \|C(t,x)\|,$$

$$\gamma(x) = \|Q^{-1}(x)\|, \beta = \max_{x\in[0,\omega]}\|Q^{-1}(x)L(x)\|.$$

Theorem 1. *Assume that the matrix $Q(x)$ on dimension (n×n) is reversible for all $x \in [0,\omega]$ and the next inequality is hold:*

$$q(x) = \gamma(x)\,\|S(x)\|\,M\max\{\delta, T, \delta T\} * \omega\{e^{\alpha(x)\delta} - 1 - \alpha(x)\delta\}e^{\beta\omega} \le \zeta < 1,$$

where $\zeta - const$.

Then the functional sequence of pairs $\{z^{(k)}(t,x), \lambda^{(k)}(x)\}$, $k \in N$, determined by the algorithm, converges to $\{z^*(t,x), \lambda^*(x)\}$ is a solution to problem (4)-(7).

Proof of the Theorem 1 is based on algorithm above and is similar to proof of Theorem 1 in [16].

From equivalence of problem (1)-(3) and (4)-(7), we obtain

Theorem 2. *Assume that the matrix $Q(x)$ on dimension (n×n) is reversible for all $x \in [0,\omega]$ and the next inequality is hold:*

$$q(x) = \gamma(x)\,\|S(x)\|\,M\max\{\delta, T, \delta T\} * \omega\{e^{\alpha(x)\delta} - 1 - \alpha(x)\delta\}e^{\beta\omega} \leq \zeta < 1,$$

where $\zeta - const.$

Then the non-local problem for partial differential equations of hyperbolic type (1)-(3) has a unique solution $u^*(t,x)$

determined by the equality

$u^*(t,x)$=$z^*(t,x) + \lambda^*(x)$, $(t,x) \in \Omega_d$,

where the pair $\{z^*(t,x), \lambda^*(x)\}$ is the solution to problem (4)-(7).

4 Conclusion

In this article, we developed the functional parametrization method to solve the non-local problem for the system of partial differential equations of hyperbolic type in the special domain. We established sufficient conditions for a unique solvability to nonlocal problem (1)-(3). These results can be extend to various non-local problems for partial differential equations of hyperbolic type in the specific domain.

5 Acknowledgement

This research is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP19675358).

References

1. Ptashnyck B.I. *Ill-posed boundary value problems for partial differential equations.* Kiev: Naukova Dumka, 1984 (in Russian).

2. Mitropol'skii Yu.A., Khoma G.P., Gromyak M.I. *Asymptotical methods of research quasi-wave equations of hyperbolic type.* Kiev: Naukova Dumka, 1991. (in Russian)

3. Samoilenko A.M., Tkach B.P. *Numerical-analytical methods in the theory periodical solutions of equations with partial derivatives*. Kiev: Naukova Dumka, 1992 (in Russian).

4. Kiguradze T.I. "Some boundary value problems for systems of linear partial differential equations of hyperbolic type". *Mem. Differential Equations Math. Phys.* 1 (1994): 1-144.

5. Kolesov A.Yu., Mishchenko E.F., Rozov N.Kh. "Asymptotic methods of investigation of periodic solutions of nonlinear hyperbolic equations". *Trudy MIAN* 222 (1998): 1-191.

6. Nakhushev A.M. *Problems with shift for a partial differential equations*. Moskow: Nauka, 2006 (in Russian).

7. Kharibegashvili S.S. "Some multidimensional problems for hyperbolic partial differential equations and system". *Mem. Differential Equations Math. Phys.* 37 (2006): 1-136.

8. Asanova A.T., Dzhumabaev D.S., "Unique solvability of the boundary value problem for systems of hyperbolic equations with data on the characteristics". *Comp. Math. Math. Phys.* 42(2002): 1609-1621.

9. Asanova A.T. Dzhumabaev D.S. "Unique solvability of nonlocal boundary value problems for systems of hyperbolic equations". *Differ. Equ.* 39 (2003): 1414-1427.

10. Asanova A.T. "A nonlocal boundary value problem for systems of quasilinear hyperbolic equations". *Doklady Math.* 74 (2006): 787-791.

11. Dzhumabaev D.S. "Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation." *USSR Comput. Math. Math. Phys.* 1 (1989): 34–46.

12. Asanova A.T. Dzhumabaev D.S. "Well-posedness of nonlocal boundary value problems with integral condition for the system of hyperbolic equations". *J. Math. Anal. Appl.* 402 (2013): 167-178.

13. Asanova A.T. "On solvability of nonlinear boundary value problems with integral condition for the system of hyperbolic equations". *Electr. J. Qualit. Theory Differ. Equ.* 2015 (2015): 63, 1-13.

14. Assanova A.T., Imanchiev A.E. "On conditions of the solvability of nonlocal multi-point boundary value problems for quasi-linear systems of hyperbolic equations". *Eurasian Math. J.* 6-4 (2015): 19-28.

15. Assanova A.T. "Solvability of a nonlocal problem for a hyperbolic equation with integral conditions". *Electr. J. Differ. Equ.* 2017 (2017): 170, 1-12.

15. Assanova A.T. "An integral-boundary value problem for a partial differential equation of second order". *Turkish J. Math.* 43 (2019): 1967-1978.

16. Assanova A.T. "On the solvability of a nonlocal problem for the system of Sobolev-type differential equations with integral condition". *Georgian Math. J.* 28 (2021): 49-57.

17. Asanova A.T. On a nonlocal boundary-value problem for systems of impulsive hyperbolic equations". *Ukrainian Math. J.* 65 (2013): 349-365.

18. Asanova A.T. "Well-posed solvability of a nonlocal boundary-value problem for the systems of hyperbolic equations with impulse effects". *Ukrainian Math. J.* 67 (2015): 333-346.

19. Assanova A.T. "On the solvability of nonlocal boundary value problem for the systems of impulsive hyperbolic equations with mixed derivatives". *J. Discontinuity, Nonlinearity and Complexity.* 5 (2016): 153-165.

20. Assanova A.T., Kadirbayeva Zh.M. "Periodic problem for an impulsive system of the loaded hyperbolic equations." *Electr. J. Differ. Equ.* 2018 (2018): 72, 1-8.

21. Assanova A.T., Kadirbayeva Zh.M., Bakirova E.A. "About of an unique solvability of a nonlocal boundary value problem for the loaded systems of hyperbolic equations with impulse effects". Ukrainian Math. J. 69 (2017): 1011-1029.

22. Assanova A.T., Iskakova N.B., Orumbayeva N.T. "On the well-posedness of periodic problems for the system of hyperbolic equations with finite time delay". *Math. Methods Appl. Sci.* 43 (2020): 881–902.

23. Assanova A.T., Uteshova R.

"Solution of a nonlocal problem for hyperbolic equations with piecewise constant argument of generalized type". *Chaos, Solitons and Fractals* 165 (2022): 112816.

24. Assanova A.T. "A generalized integral problem for a system of hyperbolic equations and its applications". *Hacettepe J. Math.& Stat.* 56 (2023): 1513–1532.