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Time evolution of an anisotropic universe under Kaluza-Klein framework with a dynamical cosmological constant

Abstract. The objective of the present study is to determine the time evolution of the universe for zero spatial curvature, under Kaluza-Klein framework, in the presence of an anisotropic fluid. To derive the field equations, we have used a power law relation between the normal scale factor and the extra-dimension scale factor. Time-variation of cosmological quantities depends upon the parameter that connects these scale factors. Using an ansatz for a hybrid scale factor (i.e., $a = t^{\alpha}e^{\beta t}$) we have derived expressions for various cosmological parameters in terms of cosmic time and depicted their time dependence graphically. The dynamical cosmological term (Λ) varies extremely slowly at the present time, indicating a slow change in the dark energy content of the universe which is held responsible for cosmic acceleration. The pressure associated with the fifth dimension has been expressed in terms of a skewness parameter (δ). Its behaviour shows that the present universe has a small anisotropy and it becomes smaller with time. This observation is consistent with the behaviour of the anisotropy factor which decreases with time.
Key words: Kaluza-Klein space-time, Anisotropy, Dark energy, Hybrid scale factor, Dynamical cosmological constant

Introduction

Cosmological research throughout the world, based on observations, has proved beyond doubt that the expansion of the universe has been continuing with acceleration. Researchers have been putting enormous efforts to unravel the mysteries of this cosmic acceleration. Since the motions of celestial bodies were known to be governed by gravitation, which is an attractive force, the universe was believed to be undergoing a decelerated expansion. Based on the experimental observations from supernova 1a, a negative pressure is said to be generated by an energy of a strange nature, known as dark energy (DE), which is considered to be causing the accelerated expansion of the universe [1, 2]. Investigations of various kinds are going to determine the nature of DE which is still shrouded in mystery. It has been found from recent research based on supernova data that the process of expansion of the universe has undergone a change of phase from deceleration to acceleration, causing the deceleration parameter to change its sign from positive to negative [3-5]. To find the reasons behind the accelerated expansion of the universe, we generally find two ways of exploration of this field in scientific literature. One of them is to construct dark energy models and study the cosmological phenomena. Another way is to formulate models under the framework of modified theories of gravity (i.e., modified versions of Einstein's theory of general relativity) and study their behaviour. In many theoretical models the cosmological constant (Λ) is regarded as representing the dark energy. To account for the behaviour of dark energy, there are other models such as quintessence, phantom, k-essence and quintom [6-9]. The parameter Λ was initially used in Einstein's theory as a time-independent parameter [10], but it is presently looked upon as a time-varying quantity due to certain shortcomings corresponding to Coincidence Problem and Cosmological Problem [11]. Researchers have made several modifications of Einstein's theory which have generated theories of gravitation like $f(R)$ and $f(R,T)$ [12-14] and scalar tensor theories of gravitations such as Brans-Dicke (BD) and Saez-Ballester (SB) [15, 16]. By formulating cosmological models based on DE, explorations of various kinds have been conducted by researchers [17-20].

In the first half of the last century, Kaluza and Klein proposed a theory for the unification of gravitational force with electromagnetic force, which came to be known as Kaluza-Klein (KK) theory [21, 22]. According to this KK theory, there was an extra dimension (or fifth dimension) which was used to unify the two forces. It was proved by a five dimensional model proposed by Chodos and Detweiler that the fifth dimension undergoes a contraction with cosmic evolution [23]. A phase of multidimensional description is theoretically considered to precede the present four-dimensional phase of the universe. As the universe evolves with time, the extra dimension keeps shrinking in a way such that it can no longer be detected by the available experimental facilities. Many researchers have been motivated by these phenomena to work in the field of theoretical models in higher dimensions. The Kaluza-Klein theory may be regarded as a five dimensional theory of general relativity. Some of the researchers who have made important contributions in the exploration of space-time in five dimensions are Chodos & Detweller [23], Witten [24], Appelquist et al. [25], Appelquist & Chodos [26] and Marchiano [27]. One of the articles that attracted our attention to the formulation of anisotropic DE models, in the framework of KK theory, with a dynamical cosmological constant, is based on an investigation by N. I. Jain. [28].

For a charged system with spherical symmetry, J.A. Ferrari carried out a study to obtain an approximate solution to Kaluza-Klein's equations [29]. The low-velocity motion of a test particle in the field of a charged object was investigated. It was found that, the corrections to Lorentz's force by Kaluza-Klein's theory could be sufficiently large to be measured. This experimental finding can be used to support the five-dimensional relativity in its present form. According to a study by Kalligas et al. [30], one can experimentally test the possible existence of an extra dimension to spacetime. They derived a set of equations containing terms dependent upon the presence of the extra dimension. Based on the existing observational data from the solar system it is found that, the terms due to the fifth dimension in our region of space must be sufficiently small compared to those due to the usual dimension of spacetime. They have shown that, the parameters belonging to the Kaluza-Klein theory should not be treated as universal constants, and they can vary from place to place depending on local properties of matter. Dzhunushaliev et al. [31] obtained several non-asymptotically flat solutions of 5D Kaluza–Klein gravity which possess both electric and magnetic charges. These solutions were shown to act as quantum virtual handles (wormholes) in spacetime foam models. According to this study, it may be possible to “inflate” these solutions from a quantum to a classical state in the presence of a sufficiently large, external electric or magnetic field. This effect is expected to lead to a possible experimental signal for higher dimensions in multidimensional gravity. According to a recent study by Jean Paul Mbelek [32], it has been shown that, the 5D Kaluza-Klein theory can be improved by incorporating an external scalar field (ψ). Based on that formulation it has been shown that, the results obtained from the theory are in agreement with observational data in the laboratory and also the data obtained in the cosmological or astrophysical contexts. The author has shown that, an experiment shows the evidence of a torque being experienced by a torsion pendulum, which is in conformity with theoretical predictions. A study by Tajmar and Williams [33] presents a novel experimental investigation of the macroscopic interpretation of the fifth dimension of the Kaluza-Klein theory. Their experiment was designed to determine an important feature of the theory, according to which, the motion in the fifth dimension is identified with the electric charge. They tested for a time dilation effect on an electrically charged clock and provided an interpretation of the results in the context of Kaluza-Klein theory. According to their conclusions, a timelike signature in the five-dimensional metric may be required for a classical, macroscopic explanation of the fifth dimension, and the associated absence of a rest frame along the fifth coordinate.

The findings of the article [ref. no. 28] that motivated us to carry out the present study do not show any change of sign of the deceleration parameter (q) in its time-variation, in contrary to what is expected from the inferences of recent astrophysical observations [3-5]. The purpose of the present study is to find the characteristics of the expanding universe, in the presence of an anisotropic fluid, by determining the nature of time evolution of various cosmological quantities and examining their consistency with observations. The entire formulation has been carried out involving a dynamical cosmological term (Λ), in the framework of the anisotropic Kaluza-Klein metric. To obtain the field equations, we have used an empirical relation ($i.e., A = a^n$) between the normal scale factor (a) and the scale factor associated with the extra dimension (A). An *ansatz* for a hybrid scale factor (i.e., $a = Bt^{\alpha}e^{\beta t}$) has been used to find the solutions of the field equations. This scale factor ensures that, the deceleration parameter ($q = -\ddot{a}a/\dot{a}^2$) undergoes a signature flip from positive to negative as it evolvers with time, in consistency with astrophysical observations that suggest that the cosmic expansion process has undergone a change of phase from deceleration to acceleration [3-5]. Based on the hybrid scale factor and the field equations, we have derived expressions for the Hubble parameter (H), deceleration parameter (q), energy density (ρ), cosmological constant (Λ), equation of state (EoS) parameter (ω), skewness parameter (δ) and the anisotropy factor (σ^2/θ). We have showed their time dependence graphically by plotting them with respect to the relative cosmic time (i.e., t/t_0) where t_0 stands for the present age of the universe which is nearly 13.7×10^9 years.

It is observed from the behaviours of both skewness parameter (δ) and the anisotropy factor (σ^2/θ) that, the universe has a small anisotropy in its present space-time composition and it is on its path towards a greater isotropy in future. The dynamical cosmological term (Λ), which is treated as the natural and the most obvious candidate for dark energy, is found to be negative with a very slow rate of change at the present time, although it is found to rise very steeply in the early universe, implying probably a rise in the content of dark energy which is widely held responsible for the accelerated expansion of the universe.

This article is divided into six sections including the *introduction*. It is followed by sections 2 and 3 dealing with the field equations and their solutions respectively. Section 4 deals with the determination of cosmological quantities. Sections 5 and 6 are regarding the results of the study (graphical depiction and interpretation) and the conclusions respectively.

Metric and Field equations

In order to obtain the cosmological field equations, we have used the Kaluza-Klein space-time [34] which is expressed as,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2\right] + A^2(t)d\psi^2 \qquad (1)$$

In equation (1), $a(t)$ and $A(t)$ are the fourth and fifth-dimension scale factors respectively. The symbol k is a measure of the spatial curvature, having the values -1, $+1$ and 0 respectively for the open, closed and flat universes. The energy-momentum tensor (T_j^i), for the anisotropic space-time metric represented by equation (1), is given below [35].

$$T_j^i = diag(T_0^0, T_1^1, T_2^2, T_3^3, T_4^4) = diag(-\rho, p, p, p, p_\psi) \qquad (2)$$

In equation (2), p and ρ are respectively the pressure and energy density of the cosmic fluid (dark energy) pervading the universe. The symbol p_ψ denotes the pressure corresponding to the extra dimension. The barotropic equation of state (EoS) parameter for the normal dimensions is $\omega = p/\rho$. Based on some studies on anisotropy, in the framework of Kaluza-Klein theory, we have used the equation, $p_\psi = (\delta + \omega)\rho$, as the directional equation of state for the fifth dimension, where δ is the skewness parameter which represents the deviation from the normal equation-of-state parameter ω [28, 36-41]. The parameter δ serves as measure of deviation from isotropy. Thus, the energy-momentum tensor of equation (2) can be rewritten as,

$$T_j^i = diag(-\rho, \omega\rho, \omega\rho, \omega\rho, (\omega + \delta)\rho) \qquad (3)$$

The time dependence of ω and δ has been investigated in the present study.

Gravitational field equations are obtained from the following equation.

$$G_j^i = R_j^i - \frac{1}{2}R\delta_j^i = -8\pi G T_j^i + \Lambda\delta_j^i \qquad (4)$$

To formulate the field equations, we have used an *ansatz* for the fifth-dimension scale factor (A), which is $A = a^n$ [42]. We have also used $8\pi G = c = 1$. Equation (4) leads to the following field equations.

$$G_0^0 = 3(n+1)\frac{\dot{a}^2}{a^2} + 3\frac{k}{a} = \rho + \Lambda \tag{5}$$

$$G_1^1 = (n+2)\frac{\ddot{a}}{a} + $$
$$+(n^2+n+1)\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -p + \Lambda \tag{6}$$

$$G_4^4 = 3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = -p_\psi + \Lambda \tag{7}$$

Substituting $p = \omega\rho$, $p_\psi = (\omega+\delta)\rho$ and $k = 0$ (i.e., flat space) in equations (5), (6) and (7), we get, respectively, the equations (8). (9) and (10), as given below.

$$3(n+1)\frac{\dot{a}^2}{a^2} = \rho + \Lambda \tag{8}$$

$$(n+2)\frac{\ddot{a}}{a} + (n^2+n+1)\frac{\dot{a}^2}{a^2} = -\omega\rho + \Lambda \tag{9}$$

$$3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} = -(\omega+\delta)\rho + \Lambda \tag{10}$$

Divergence of Einstein's tensor can be expressed as,

$$\left(R_j^i - \frac{1}{2}R\delta_j^i\right)_{;j} = \left(-T_j^i + \Lambda\delta_j^i\right)_{;j} = 0 \tag{11}$$

Based on equation (11), the equation representing energy conservation [35] is given by,

$$\dot{\rho} + 3(\rho+p)\frac{\dot{a}}{a} + n(\rho+p_\psi)\frac{\dot{a}}{a} + \dot{\Lambda} = 0 \tag{12}$$

Substituting the equations of state for the normal dimension and the extra dimension [i.e., $p = \omega\rho$ and $p_\psi = (\omega+\delta)\rho$ respectively] in equation (12), we get,

$$\dot{\rho} + (3+n)(1+\omega)\rho\frac{\dot{a}}{a} + n\delta\rho\frac{\dot{a}}{a} + \dot{\Lambda} = 0 \tag{13}$$

Equation (13) can be written as a sum of two equations which are equations (14) and (15), as given below.

$$\dot{\rho} + (3+n)(1+\omega)\rho\frac{\dot{a}}{a} = Q \tag{14}$$

$$n\delta\rho\frac{\dot{a}}{a} + \dot{\Lambda} = -Q \tag{15}$$

In equations (14) and (15), Q is an arbitrary parameter.

Subtracting equation (10) from equation (9), we get,

$$(n-1)\frac{\ddot{a}}{a} + (n^2+n-2)\frac{\dot{a}^2}{a^2} = \delta\rho \tag{16}$$

Substituting for $\delta\rho$ in equation (16) from equation (15), we get,

$$(n-1)\frac{\ddot{a}}{a} + (n^2+n-2)\frac{\dot{a}^2}{a^2} = -\frac{Q+\dot{\Lambda}}{n(\dot{a}/a)} \tag{17}$$

Solution of Field Equations

To solve the field equations, we have used the following ansatz for the scale factor.

$$a = Bt^\alpha e^{\beta t} \tag{18}$$

where, the constant parameters $B, \alpha, \beta > 0$.

The empirical scale factor, expressed by equation (18), is a hybrid of power-law and exponential functions of time. It has been used in several recent cosmological studies [43-49]. The reason for using this scale factor is that it leads to a deceleration parameter (given by equation no. 20) which (with suitable parameter values) undergoes a change of sign (as a function of time) from positive to negative which is in accordance with the recent astrophysical observations [ref. nos. 3-5] demonstrating a change of phase of the expanding universe from deceleration to acceleration. If we choose a scale factor of either the exponential form (i.e., $a = Be^{\beta t}$) or the power law form (i.e., $a = Bt^\alpha$), the deceleration parameters obtained from them do not have the required property of signature flip. There are some studies where hyperbolic functions of time have been used as empirical scale factors [50-55], having the same property (i.e., deceleration-to-acceleration transition) of cosmic expansion. The parameter B in the expression for the scale factor (eqn. 18) does not appear in the expressions for the Hubble parameter (eqn. 19) and deceleration parameter (eqn. 20) because of their expressions, i.e., $H = \frac{\dot{a}}{a}$ and $q = -\frac{\ddot{a}\,a}{\dot{a}^2}$ respectively. All functions of H and q are therefore independent of B. The left-hand sides of equations (8), (9) and (10) are such functions of the scale factor (a) and its derivatives that they are also independent of the parameter B. This is the reason why the parameter B is not found any expression of the present article except that of the scale factor (eqn. 18).

Using equation (18) we get the following expression for the Hubble parameter (H).

$$H = \frac{\dot{a}}{a} = \alpha t^{-1} + \beta \tag{19}$$

Using equation (18) we get the following expression for the deceleration parameter (q).

$$q = -\frac{\ddot{a}\, a}{\dot{a}^2} = \frac{\alpha}{(\alpha+\beta t)^2} - 1 \tag{20}$$

Combining equation (19) with equation (20) we get,

$$q = \frac{\alpha}{(Ht)^2} - 1 \tag{21}$$

Putting $t = t_0$, $q = q_0$ and $H = H_0$ in equation (21) and rearranging the terms we get,

$$\alpha = (q_0 + 1)(H_0 t_0)^2 \tag{22}$$

The symbols, q_0 and H_0, denote respectively the values of q and H at the present time (i.e., $t = t_0$). Here t_0 stands for the present age of the universe which is nearly 13.7×10^9 years.

Putting $t = t_0$ and $H = H_0$ in equation (19) we get,

$$t_0 = \frac{H_0 t_0 - \alpha}{\beta} \tag{23}$$

Using equation (22) in equation (23), one obtains,

$$t_0 = \frac{H_0 t_0 - (q_0+1)(H_0 t_0)^2}{\beta} \tag{24}$$

Both q_0 and $H_0 t_0$ are dimensionless quantities. α and t_0 are expressed in terms of these quantities in equations (22) and (24) respectively. β has the dimension of time inverse (i.e., 1/Time).

Determination of Cosmological Parameters

Using equations (18) in equation (17), we get,

$$\dot{\Lambda} = -Q + (-n^3 - 2n^2 + 3n)(\alpha^3 t^{-3} + 3\alpha^2\beta t^{-2} + 3\alpha\beta^2 t^{-1} + \beta^3) + (n^2 - n)(\alpha^2 t^{-3} + \alpha\beta t^{-2}) \tag{25}$$

Solving equation (25) we get,

$$\Lambda = -Qt - n(n-1)(n+3)\left(-\frac{\alpha^3 t^{-2}}{2} - 3\alpha^2\beta t^{-1} + 3\alpha\beta^2 \ln t + \beta^3 t\right) + \\ + n(n-1)\left(-\frac{\alpha^2 t^{-2}}{2} - \alpha\beta t^{-1}\right) + C \tag{26}$$

where C is the constant of integration.

Using equations (18) and (26) in equation (8) we get the following expression for the energy density.

$$\rho = 3(n+1)(\alpha t^{-1} + \beta)^2 + Qt - n(n-1)(n+3)\left(-\frac{\alpha^3 t^{-2}}{2} - 3\alpha^2\beta t^{-1} + 3\alpha\beta^2 \ln t + \beta^3 t\right) - \\ - n(n-1)\left(-\frac{\alpha^2 t^{-2}}{2} - \alpha\beta t^{-1}\right) - C \tag{27}$$

Using equations (18), (26) and (27) in equation (10), we get,

$$\omega = \frac{f_1(t)+f_2(t)+C}{f_3(t)-f_1(t)-f_2(t)-C} + \frac{f_4(t)+f_5(t)}{f_3(t)-f_1(t)-f_2(t)-C} \tag{28}$$

Using equations (18) and (27) in equation (16) we get,

$$\delta = \frac{(n-1)(-\alpha t^{-2}) + (n^2+2n-3)(\alpha t^{-1}+\beta)^2}{f_3(t)-f_1(t)-f_2(t)-C} \tag{29}$$

The expressions for the functions (i.e., f_1, f_2, f_3, f_4, f_5) used in equations (28) and (29) are given below.

$$f_1(t) = -Qt - n(n-1)(n+3)\left(-\frac{\alpha^3 t^{-2}}{2} - 3\alpha^2\beta t^{-1} + 3\alpha\beta^2 \ln t + \beta^3 t\right),$$

$$f_2(t) = n(n-1)\left(-\frac{\alpha^2 t^{-2}}{2} - \alpha\beta t^{-1}\right),$$

$$f_3(t) = 3(n+1)(\alpha t^{-1} + \beta)^2,$$

$$f_4(t) = -(n+2)(-\alpha t^{-2}),$$

$$f_5(t) = -(n^2 + 2n + 3)(\alpha t^{-1} + \beta)^2.$$

The expansion scalar (θ) and the shear scalar (σ^2) are given by the following equations.

$$\theta = 3\frac{\dot{a}}{a} + \frac{\dot{A}}{A} = (n+3)H \tag{30}$$

$$\sigma^2 = \frac{3}{8}\left(\frac{\dot{a}}{a} - \frac{\dot{A}}{A}\right)^2 = \frac{3}{8}(1-n)^2 H^2 \tag{31}$$

Using equations (30) and (31), the anisotropy factor (σ^2/θ) can be expressed as,

$$\frac{\sigma^2}{\theta} = \frac{3(1-n)^2}{8(n+3)} H = \frac{3(1-n)^2}{8(n+3)} [\alpha t^{-1} + \beta] \tag{32}$$

Equation (32) shows that, σ^2/θ decreases with time.

Results and Discussion

In the present study, we have investigated the characteristics of the expanding universe for directional EoS parameters for the normal and extra dimensions, given by $p = \omega\rho$ and $p_\psi = (\omega + \delta)\rho$ respectively. We have graphically depicted the time dependence of various cosmological quantities for $n = 0.5$, $B = 1$, $\beta = 0.04$, $Q = 0$ and $C = -0.3$. There are restrictions regarding the value of the parameter n. For $n = 0$ & 1, the cosmological term (Λ) becomes independent of time. For $n = -1$, the energy density (ρ) increases with time which is not admissible in an expanding universe.

Figure 1 depicts the time dependence of the scale factor (a). It increases monotonically with time, which is natural for an expanding universe.

Figure 2 shows the time dependence of the Hubble parameter (H) which is found to be positive in this plot and it is decreasing gradually with time. Its positive value is consistent with the characteristics of an expanding universe. Its decrease with time indicates that the scale factor (a) increases faster with time in comparison to the expansion rate ($\dot{a}$). One can also say that, the time-rate of fractional change of the scale factor (a) decreases with time.

Figure 3 depicts the variation of the deceleration parameter (q) as a function of time. The sign of this parameter changes from positive to negative, implying a change of phase of the universe from decelerated expansion to accelerated expansion, in complete agreement with the conclusions drawn from recent astrophysical observations [3-5].

Figure 4 shows the time evolution of the energy density (ρ). It is found to be decreasing with time, which is consistent with the fact that the universe is expanding with time.

Figure 5 depicts the time dependence of the cosmological constant (Λ), which represents the dark energy which is regarded as playing the major role in causing the accelerated expansion of the universe. It rises very fast in the early universe from a large negative value to a smaller negative value and continues to be less negative with time. This behaviour is similar to the findings of some recent studies [48, 56-58].

Figure 6 shows the variation of the EoS parameter (ω) with time. It is negative and it becomes more negative with time at a gradually decreasing rate. Its values show that the universe is dominated by *quintessence* dark energy (i.e., $\omega > -1$) at the early stage and also at the present time (i.e., $t = t_0$), but it will be dominated by vacuum fluid (i.e., $\omega = -1$) at a later stage [59]. This is consistent with the findings of some studies based on four-dimensional space-time geometries which are completely different from the Kaluza-Klein metric used for the present study [60, 61]. As per SN Ia data we have $-1.67 < \omega < -0.6237$ while the range obtained by a combination of galaxy clustering statistics and SN Ia data (with CMB anisotropy) and is $-1.33 <$

$\omega < -0.79$ [62, 63]. The values of ω at $t = t_0$, as obtained from equation (28), are consistent with these ranges obtained experimentally.

Figure 7 depicts the time-variation of delta (δ). Its value is negative and it approaches zero with time. Its present value is sufficiently small (-0.02 approximately), indicating a small anisotropy in the present universe, which becomes smaller with time. This behaviour is consistent with the observation of a recent study on anisotropic dark energy model in the Kaluza-Klein framework [28].

Figure 8 shows the variation of the ratio σ^2/θ (anisotropy factor) as a function of time. Its present value is small and it is found to decrease with time, indicating the fact that the present universe has a small anisotropy which is gradually becoming smaller with time. Thus, the condition for isotropy, i.e., $\sigma^2/\theta \to 0$ as $t \to \infty$, is satisfied. This observation is consistent with the findings by Shamir *et al*, based on Bianchi type III space-time [64, 65].

To express some of the cosmological parameters (depicted in the above-mentioned figures) in proper units, scaling constants (constant multipliers) are needed, and they can be determined by using the values (obtained from observations) of these parameters at the present time (i.e., at $t = t_0$). Based on recent scientific literature [66-72], we have obtained the following values of Hubble parameter, deceleration parameter and energy density: $H_0 = 72.20\, km\, s^{-1}\, Mpc^{-1}$, $q_0 = -0.55$, $\rho_0 = 9.83 \times 10^{-27}\, Kg\, m^{-3}$. Figure 2 (i.e., Hubble parameter versus time) is based on equations (19) and (24). The value of H for $t/t_0 = 1$, as per these equations, is 0.075. The scaling constant for Figure 2 is thus, $H_0/0.075$ whose value comes out to be 962.67 $km\, s^{-1}\, Mpc^{-1}$. The data along the vertical axis of Figure 2 have to be multiplied by this constant to obtain the Hubble parameter values in proper units. Figure 4 (i.e., energy density versus time) is based on equations (24) and (27). The value of ρ for $t/t_0 = 1$, as per these equations, is 0.321. The scaling constant for Figure 4 is thus, $\rho_0/0.321$ whose value comes out to be $3.062 \times 10^{-26}\, Kg\, m^{-3}$. The data along the vertical axis of Figure 4 have to be multiplied by this constant to obtain the energy density values in proper units. Figure 8 shows the time-variation of σ^2/θ (anisotropy factor) which has the dimension of the Hubble parameter (H) according to equation (32). The value of σ^2/θ for $t/t_0 = 1$, based on equations (24) and (32), is 0.002. Based on equation (32), the scaling constant for Figure 8 is therefore, $\frac{3(1-n)^2}{8(n+3)} \frac{H_0}{0.002}$ which is $13.4\, H_0$ having the value of 966.96 $km\, s^{-1}\, Mpc^{-1}$. The data along the vertical axis of Figure 8 have to be multiplied by this constant to obtain the values of the anisotropy factor in the standard units of Hubble parameter. Figure 5 (i.e., cosmological parameter Λ versus time) is based on equations (24) and (26). The value of Λ for $t/t_0 = 1$, as per these equations, is -0.296. One of the values of Λ, estimated from observational data, is $1.25 \times 10^{-52}\, m^{-2}$, according to recent scientific literature [73-76]. But its theoretical and observational estimates lie over a very long range of values spread over several orders of magnitude [76-78]. So, it has not been possible for us to calculate a scaling constant for Λ. The purpose of calculating Λ in our study is to find its nature of time-variation for the present formulation which involves an anisotropic Kaluza-Klein model where a hybrid scale factor (eqn. 18) has been used for calculations. Our findings in this regard are consistent with the nature of time-variation of Λ obtained from some recent studies based on various theoretical models [48, 56-58]. Due to constraints of various natures, time-variations of different cosmological parameters have been shown graphically in arbitrary units in most of the articles that we have gone through and cited in the present study.

The scale factor (a) is proportional to the parameter B according to equation (18). If we change the value of the parameter B from 1 to 2, the values of the scale factor (a) and its time derivative (da/dt) become twice their values (for $B = 1$) of a and da/dt respectively, without affecting the values of the Hubble parameter ($H = \frac{\dot{a}}{a}$) and deceleration parameter ($q = -\frac{\ddot{a}\, a}{\dot{a}^2}$), because their expressions make them independent of B for the scale factor that we have chosen. As per equation (19), H has a linear dependence upon β. As β increases, the deceleration parameter (q) approaches the value of -1 faster, according to equation (20). Based on equation (22) we get $\alpha = 0.484$. Since the universe is undergoing an expansion, we must get $H > 0$ from theoretical calculations. This requirement leads to $\beta > -\alpha/t$ (using eqn. 19), implying that β has a negative limiting value. Taking $t = t_0 = 13.7 \times 10^9$ years $= 4.32 \times 10^{17} sec$ here, we get $\beta > -1.12 \times 10^{-18}\, sec^{-1}$. Since the universe is presently expanding with acceleration, the expansion process will certainly continue beyond the present time (t_0). So, the negative limiting value of β is surely closer to zero than $-1.12 \times 10^{-18} sec^{-1}$. Thus, it would be theoretically logical to use positive values of β for the present formulation. Calculations using equation (27) show that, if we choose Q to be negative, the energy density (ρ) becomes negative beyond a certain time (t) depending on the value of Q. Thus, one should take $Q \geq 0$, and we have used its lowest permissible value, i.e., zero, in the present study. In the expression for ρ (eqn. 27), one of the terms has a linear dependence upon t, where the value of Q determines the degree of this time-dependence. Based on equation (8) we have, $\Lambda = 3(n+1)H^2 - \rho$. Thus, Λ has a similar dependence upon Q, which is also evident from equation (26). The larger the value of Q, the faster would be the change in Qt (which is present in both Λ and ρ) with time, but this effect is small in comparison to the effect of variation of other terms. The parameter C is present in the expression for Λ (eqn. 26) as a constant of integration. Due to the inter-dependence of Λ and ρ (through eqn. 8), C is also present in the expression for ρ (eqn. 27). In order to keep ρ positive for all values of t, one needs to choose a negative value for C. The time-variations of ω and δ depend upon the parameters β, Q and C, governed by equations (28) and (29) respectively. If one increases the parameter β, it causes a faster change of the scale factor with time (as per eqn. 18), causing a faster change of all the cosmological quantities whose time-variations have been depicted graphically in the present study. The parameter values used for the present study are, $B = 1, \beta = 0.04, Q = 0$ and $C = -0.3$. This combination of values is a result of an optimization carried out through trial and error to meet some requirements for ensuring the authenticity of predictions regarding anisotropy based on this model. These requirements are: 1) a, H and ρ are positive, 2) ρ decreases with time, which is expected for an expanding universe, 3) q undergoes a change of sign from positive to negative to be consistent with the deceleration-to-acceleration transition of cosmic expansion, 4) the values of q and ω, for $t/t_0 = 1$, are sufficiently consistent with those obtained from recent astrophysical observations.

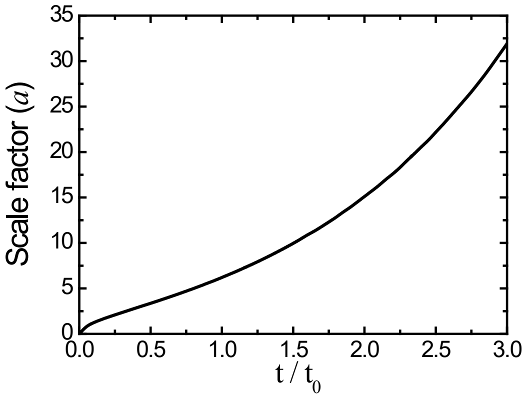

Figure 1 – Plot of scale factor versus time

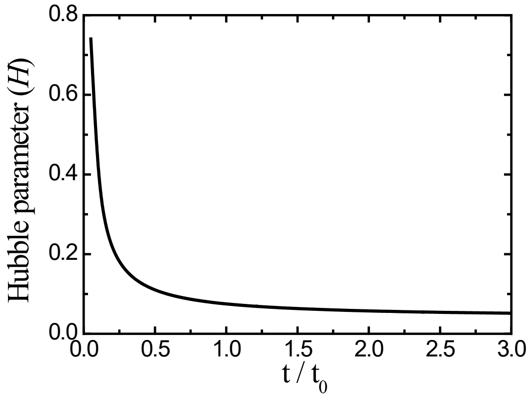

Figure 2 – Plot of Hubble parameter versus time

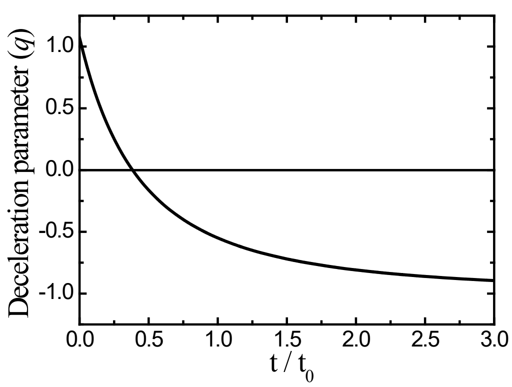

Figure 3 – Plot of deceleration parameter versus time

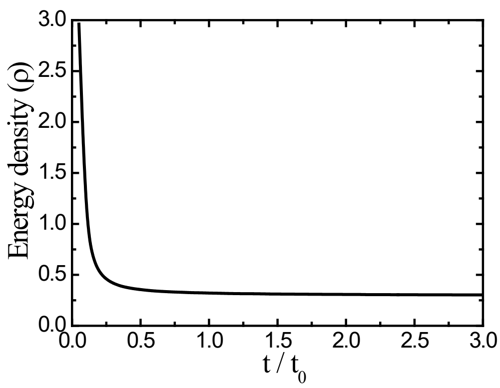

Figure 4 – Plot of energy density versus time

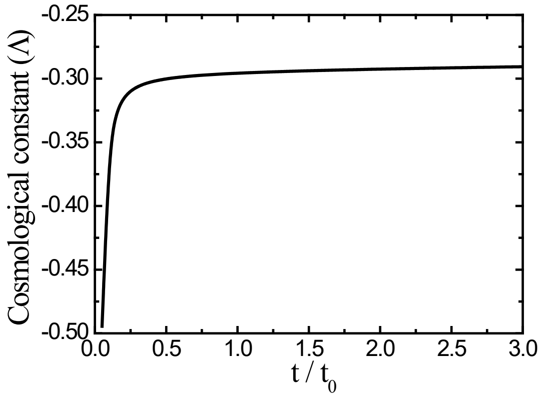

Figure 5 – Cosmological constant versus time

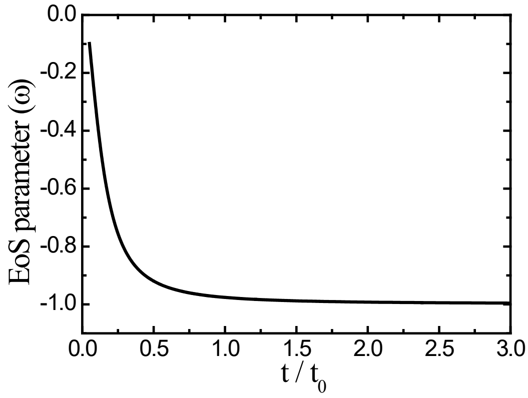

Figure 6 – EoS parameter versus time

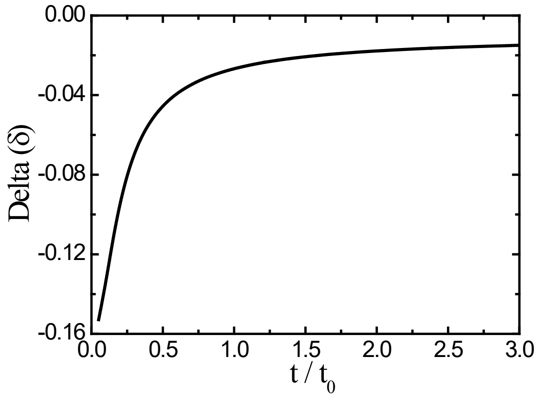

Figure 7 – Delta versus time

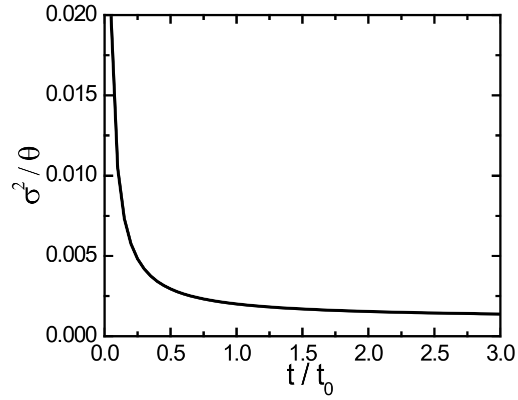

Figure 8 – σ^2/θ versus time

Conclusion

In the present study, we have carried out a theoretical investigation in flat space under the framework of Kaluza-Klein metric, with a dynamical cosmological constant (Λ), to find the time-variation of some cosmological quantities. It is observed that the Hubble parameter decreases with time, indicating that the time-rate of fractional change of the scale factor (a) decreases with time. The deceleration parameter (q) shows a signature flip as time goes on, indicating a change of phase from decelerated expansion to accelerated expansion, in agreement with astrophysical observations. The energy density decreases with time, indicating the effect of the expansion of the universe. The cosmological constant (Λ), which represents dark energy, rises very rapidly in the early universe but its change becomes extremely slow with time. The time-variation of the EoS parameter (ω) shows that the universe has been in the quintessence regime (i.e., $\omega > -1$) of dark energy since its early stage and it is gradually approaching a vacuum fluid dominated stage (i.e., $\omega = -1$). The parameter δ has a small negative value at the present time (t_0), showing the presence of anisotropy. Its value becomes closer to zero as times goes on, implying a decrease in cosmic anisotropy with time. This prediction regarding a gradual transition towards isotropy is also supported by the observation that, as $t \to \infty$, $\sigma^2/\theta \to 0$. As a future extension of this work, we have plans to carry out the entire formulation with $Q \neq 0$.

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