IRSTI 29.05.41; 29.05.33; 29.05.09 https://doi.org/10.26577/ijmph.2023.v14.i1.07

N. Beissen[1*], M. Abishev [1], S. Toktarbay[1,2],
T. Yernazarov[1], D. Utepova [3], M. Zhakipova [1]

[1]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[2]Kazakh National Women's Teacher Training University, Almaty, Kazakhstan
[3]Abai Kazakh National Pedagogical University, Almaty, Kazakhstan
*e-mail: Nurzada.Beissen@kaznu.edu.kz
(Received 14 April 2023; received in revised form 17 May 2023; accepted 31 May 2023)

Exploring nonlinear vacuum electrodynamics beyond Maxwell's Equations

Abstract. In this article, we provide a review of the current state of the research on the nonlinear electrodynamics of vacuum in the presence of strong electromagnetic fields. We discuss the parametrized post-Maxwellian formalism that can be facilitate calculations in the weak electromagnetic field approximation. The post-Maxwellian parameters are a set of parameters used in non-linear electrodynamics to describe the behavior of electromagnetic fields in the presence of strong fields. We also discuss the equations for the electromagnetic field in the Born-Infeld electrodynamics and in non-linear Heisenberg-Euler electrodynamics, as well as the exact solution in the form of a plane elliptically polarized wave. We represent the Lagrangian of nonlinear electrodynamics in vacuum in detail. We demonstrate that the Lagrangian can be expressed as a series of integer powers of invariants J_1 and J_2, which allows for the identification of nonlinear electrodynamics that are consistent with experiments carried out in a weak electromagnetic field.
Finally, for methodological purposes, the article emphasizes the importance of the systematic implementation of the calculation of nonlinear electrodynamics effects, regardless of any nonlinear theory. Such an approach can facilitate a deeper understanding of the underlying physics and contribute to the development of novel experimental techniques.
Key words: nonlinear electrodynamics of vacuum, strong magnetic field, post-Maxwelian parameters, Euler-Heisenberg formalism, Born-Infeld formalism.

Introduction

In recent years, significant progress has been made in comprehending the behavior of electromagnetic fields in regimes characterized by strong fields, where the influence of nonlinear effects becomes prominent [1, 2]. The phenomenon of vacuum polarization stands as a firmly established concept within quantum field theory. It arises due to the realization that the vacuum is not truly devoid of particles but rather filled with virtual particles that incessantly appear and disappear. When subjected to a strong electromagnetic field, these virtual particles can be influenced by the field's strength, leading to the creation of observable real particles. Extensive theoretical and experimental investigations have been conducted on this effect, which stands as a well-established prediction of quantum electrodynamics [3]. Its thorough examination has yielded crucial insights into fundamental physical processes, including the Casimir effect, the Lamb shift, and the anomalous magnetic moment of the electron [4].

The Born-Infeld theory is a theoretical framework in physics that was initially proposed by Max Born and Leopold Infeld in 1934 as an endeavor to address certain shortcomings in classical electrodynamics [5]. This theory suggests a modification to the classical Maxwell's equations [6] by introducing a nonlinear term into the electromagnetic field strength tensor. The purpose of this modification is to circumvent the issue of infinite self-energy encountered in classical electrodynamics, which poses a significant problem.

Born-Infeld theory has since been applied to various fields in physics such as particle physics, cosmology, and string theory. In particle physics, it is used to describe the dynamics of electrically charged particles such as electrons and positrons [7]. In cosmology, it has been used to model the behavior of the universe during the early stages of its evolution. In string theory, Born-Infeld theory has been used to describe the dynamics of branes, which are objects that are extended in more than three dimensions [8–10].

Research on Born-Infeld theory has been ongoing for decades, with many physicists working to develop a better understanding of its implications and applications. Some recent research has focused on the use of Born-Infeld theory in the study of black holes [11–15] and the behavior of gravitational waves. Other researchers have investigated the possibility of using Born-Infeld theory to explain the behavior of dark matter and dark energy, which are still poorly understood phenomena in modern physics. The study of Born-Infeld theory continues to be an active area of research in theoretical physics [7, 16–18]. Furthermore, the values of its Post-Maxwellian parameters (PMP) theory were measured with high accuracy in works on the experimental verification [19]. In addition, these PMP values were calculated [20] during the study of light birefringence in a two-dimensional resonator, and the birefringence in a strong magnetic field was evaluated [21–24]. There are several works on the polarization of light in compact objects by using PMP of BI NED of vacuum [12, 18, 25–29].

However, the Born-Infeld theory is just one of the several proposed nonlinear electrodynamics theories of vacuum. In recent years, another proposed theory is the effective Lagrangian of the electromagnetic field due to the polarization effect of the electron-positron vacuum by electromagnetic fields, as predicted by quantum electrodynamics [9, 30–34]. The Heisenberg-Euler (HE) theory has also gained significant attention due to its potential implications in high-energy physics, such as the study of strong electromagnetic fields in the vicinity of black holes or in the early universe [13, 15, 35, 36]. Deflection angles of light in the equatorial plane are computed analytically by considering BI NED and HE theory separately. Additionally, by that the delay times due to light polarization were also calculated [12, 37–40]. In addition to the two main formalisms of vacuum NED, there are also works that review other theories and their applications [18, 39, 41]. These works focus on the main characteristics and features of other models of vacuum NED. The above-mentioned and other scientific works provide a general overview of the theories of nonlinear electrodynamics of the vacuum, and information about the PMP and their characteristics and physics is given in a small way. Therefore, their extraction and characterization of their physics is highly effective for a deeper study of the nonlinear electrodynamics of the vacuum.

In this work, we provide a comprehensive review of the post-Maxwellian parameters, their characteristics, and the underlying physics in the context of the Born-Infeld and Heisenberg-Euler theories, which are the primary frameworks within nonlinear vacuum electrodynamics. It is crucial to consider theories or calculations that align with observational results to gain insights into the nature of the geometry surrounding astrophysical black hole candidates and to assess the validity of the theory [40, 42, 43]. A detailed examination of the derivation and solution of the field equations based on the Born-Infeld and Euler-Heisenberg theories, as well as an exploration of the magnitudes and physical implications of the post-Maxwellian parameters within these theories, significantly contributes to the study of nonlinear electrodynamics in vacuum.

In the first part of the article, we focus on the basis of nonlinear electrodynamics and its two formalisms, Lagrangians. In the next second part, we summarize the features and equations of Born-Infeld nonlinear electrodynamics. In the third part, we will focus on the main properties and equations of Heisenberg-Euler formalism and post-Maxwellian formalism. At the end of the section, we review the article and conclude.

The lagrangians of the ned of vacuum

Nonlinear electrodynamics consists of two sections: nonlinear optics, which studies nonlinear-electromagnetic processes occurring in material media, and nonlinear vacuum electrodynamics, which studies similar processes occurring in vacuum in the presence of strong electromagnetic fields. Nonlinear optics and its effects are currently widely used in conducting various theoretical and experimental studies in different fields of physics. Devices based on these effects have found their application in practical uses. Nonlinear vacuum electrodynamics, however, is much less well-known. As is well known, Maxwell's electromagnetism in vacuum is a linear theory. Its predictions on a wide range of questions are constantly being confirmed with increasing accuracy. It was precisely based on the study of phenomena in Maxwell's electromagnetism that the special theory of relativity was created, changing the understanding of space and time that existed in Newtonian mechanics until that time [8]. However, a number of fundamental physical considerations suggest that Maxwell's electromagnetism represents only the first approximation of a more general nonlinear vacuum electrodynamics, applicable in the limit of weak electromagnetic fields, when the magnitude of the electromagnetic fields B and E is significantly smaller than the characteristic quantum electrodynamics value B_q, where m_0 is the electron mass, e - is the modulus of its charge, and h - is the Planck constant. Currently, there are two nonlinear generalizations of Maxwell's equations that are most well-known in the scientific literature. One of them was proposed in the 1930s by Born and Infeld. Born-Infeld nonlinear electrodynamics, based on the ideas used, is a classical theory and is based on a Lagrangian[5, 9, 31, 44–46]:

$$L = -\frac{1}{4\pi a^2}\left[\sqrt{1 + a^2(\mathbf{B^2} - \mathbf{E^2}) - a^4(\boldsymbol{BE})^2} - 1\right] \quad (1)$$

where a - is a constant with dimensions inverse to the dimensions of the magnetic field induction. Another nonlinear generalization of vacuum electrodynamics is a direct consequence of the polarization effect of the electron-positron vacuum by electromagnetic fields. In the first non-vanishing approximation of perturbation theory of quantum electrodynamics, the effective Lagrangian of the electromagnetic field for the case of weak electromagnetic fields has the form [18, 30, 32, 47]:

$$\mathrm{L}=-\frac{1}{8\pi}\left[\mathbf{B^2}\text{-}\mathbf{E^2}\right]+\frac{\alpha}{360\pi^2 \mathrm{B}^2_{\mathrm{q}}}\left\{\left(\mathbf{B^2}\text{-}\mathbf{E^2}\right)^2+7(\mathbf{BE})^2\right\} \quad (2)$$

where $\alpha = e^2/\hbar c = 1/137$ is the fine structure constant. For a long time, nonlinear vacuum electrodynamics had no experimental confirmation and was therefore perceived by many as an abstract theoretical model. Nowadays, its status has significantly changed. Experiments on inelastic scattering of laser photons on gamma rays, carried out at the Stanford Linear Accelerator, confirmed that electrodynamics in vacuum is indeed a nonlinear theory [48]. Therefore, its various predictions that are available for experimental verification deserve the most serious attention.

The born-infeld formalism

Born-Infeld electrodynamics has a number of interesting properties. Firstly, the selfenergy of a point charge in this theory is a finite quantity. Secondly, the speed of electromagnetic signals in this electrodynamics, although dependent on the magnitudes of the fields B^2 and E^2, does not exceed the speed of light in Maxwell's electrodynamics [5,7,16]. And third, this theory is closely related to Einstein's idea of introducing a non symmetric metric tensor $G_{ik} \neq G_{ki}$, the symmetric part of which is the usual metric tensor g_{ik}, and the antisymmetric part is the electromagnetic field tensor F_{ik} [37, 49 −53].

$$G_{ik}=g_{ik}+aF_{ik} \quad (3)$$

Using tensor algebra relationships, it can be shown that:

$$G = det \parallel G_{ik} \parallel = \\ = g\left[1 - \frac{a^2}{2}F_{(2)} - \frac{a^4}{2}F_{(4)} + \frac{a^4}{2}F^2{}_{(2)}\right] \quad (4)$$

where g is the determinant of the metric tensor, $F_2 = F_{ik} \cdot F^{ki}, F_4 = F_{ik} \cdot F^{km} \cdot F_{mn} \cdot F^{ni}$ are the invariants of the electromagnetic field tensor [38]. In the absence of a gravitational field and using Cartesian coordinates of an inertial reference frame, the quantities involved in this relationship have the following form [36]:

$$g = -1, F_{(2)} = 2(E^2 - B^2), \\ F_{(4)} = 2(E^2 - B^2)^2 + 4(B \cdot E)^2 \quad (5)$$

Therefore, using (4), the Lagrangian(1)[10, 44] can be written as:

$$L = -\frac{1}{4\pi a^2}\left[\sqrt{-G} - \sqrt{-g}\right] \quad (6)$$

It should also be noted that Born-Infeld electrodynamics can also be derived from more general supersymmetric theories [16]. Thus, the Born-Infeld electrodynamics in many respects is a distinguished theory. At achievable fields in terrestrial laboratories, the values of a^2E^2 and a^2B^2 are significantly smaller than one. In this case, the Lagrangian (1) can be expanded in small parameters $a^2E^2 << 1$ and $a^2B^2 << 1$ [37].

$$L = -\frac{1}{8\pi}(B^2 - E^2) + + \frac{a^2}{32\pi}\{(B^2 - E^2)^2 + 4(B \cdot E)^2\} \tag{7}$$

The first part of this expansion represents the Lagrangian of Maxwell's electrodynamics, while the remaining part is a correction to it, linear in the small parameters mentioned. The equations governing the behavior of the electromagnetic field in nonlinear electrodynamics are comparable to those found in continuum electrodynamics:

$$rot\ H = \frac{1}{c}\frac{\partial D}{\partial t}, \quad div\, D = 0,\ rot\ E = \frac{1}{c}\frac{\partial B}{\partial t},\ div\, B = 0 \tag{8}$$

differing from them in the form of the constitutive equations:

$$D = 4\pi\frac{\partial L}{\partial E},\ H = -4\pi\frac{\partial L}{\partial B} \tag{9}$$

Substituting the Lagrangian of Born-Infeld electrodynamics into these expressions, we obtain [9, 54]:

$$\vec{D} = \frac{\vec{E}+a^2(\vec{B}\vec{E})\vec{B}}{\sqrt{1+a^2(\vec{B}^2-\vec{E}^2)-a^4(\vec{B}\vec{E})^2}}, \quad \vec{H} = \frac{\vec{B}-a^2(\vec{B}\vec{E})\vec{E}}{\sqrt{1+a^2(\vec{B}^2-\vec{E}^2)-a^4(\vec{B}\vec{E})^2}} \tag{10}$$

The most important dynamic invariant of nonlinear electrodynamics of Born-Infeld is the energy-momentum tensor of the free electromagnetic field [26, 51].

$$T_{ik} = \frac{1}{8\pi a^2 Q}\left\{2a^2F_{ik}^{(2)} + g_{ik}\left[2 - a^2J_2 - 2Q\right]\right\} \tag{11}$$

Here, $J_2 = F_{ik}F^{ki}$, $J_4 = F_{ik}F^{kn}F_{nm}F^{mi}$ are the invariants of the electromagnetic field, and $Q = \sqrt{1 - \frac{a^2J_2}{2} - \frac{a^4J_4}{4}}$ is a function of the invariants.

The next is one of the exact solutions of nonlinear electrodynamics of Born-Infeld, where the source is a point electric charge q. It is known that in linear electrodynamics of Maxwell, this problem exhibits a divergence problem of the proper energy and proper mass of the charged point particle. To verify this, let us consider a point particle and calculate the energy of the field created by it. A straightforward calculation yields the following expressions for a point charged particle located at the origin of the coordinates [7, 10, 44]:

$$\rho = q\delta(\vec{r}),\ \vec{E} = \frac{q\vec{r}}{r^3}\ , \varphi = \frac{q}{r} \tag{12}$$

Substituting these expressions into the expression for the energy of the electrostatic field, we arrive at a divergence:

$$\varepsilon = \frac{1}{8\pi}\int_V dV\ \vec{E}^2 = \frac{1}{8\pi}\int_0^\infty r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi \frac{q^2}{r^4} = \frac{q^2}{2}\frac{1}{0} = \infty \tag{13}$$

The same result is obtained when using an alternative formula for the energy of an electrostatic field [36]:

$$\varepsilon = \frac{1}{2}\int \rho(\vec{r})\varphi(\vec{r})dV = = \frac{1}{2}\int_{-\infty}^{\infty} dx\, dy\, dz \frac{\delta(\vec{r})}{r} \Longrightarrow \frac{q^2}{2}\cdot\frac{1}{0} = \infty \tag{14}$$

Since the energy of a particle is proportional to its mass, a point particle should have infinite mass. This result from solving the problem in linear Maxwell's electrodynamics contradicts experimental data. From the expressions for the energy, it can be noticed that the main reason for the divergence of the energy of the electrostatic field of a particle is the rapid growth of the $\varphi \sim \frac{1}{r}$, $E \sim \frac{1}{r^2}$ with respect to $r \rightarrow 0$. Therefore, a very small neighborhood of the point $r = 0$ gives an infinite contribution to the value of integrals (13) and (14).

In the Born-Infeld nonlinear electrodynamics, the nonlinearity of the equations is "switched on" in the region of strong fields and suppresses the unlimited growth of the field. Indeed, consider a point particle. Since the problem has spherical symmetry, in this case the Born-Infeld equations of nonlinear electrodynamics take the form [5, 10, 45, 55]:

$$\frac{1}{r^2}\frac{d}{dr}[r^2 D_r] = 4\pi q\delta(\vec{r}),\ rot\,\vec{E} = 0 \quad (15)$$

From the first equation of this system, it follows that $D_r = q / r^2$. The material equation in the static spherical symmetric case is: $D_r = \frac{E_r}{\sqrt{1-a^2E_r^2}}$. Solving this equation for E, we get: $E_r = \frac{D_r}{\sqrt{1-a^2D_r^2}}$. Substituting the explicit expression for D_r into the right side, we will have: $E_r = \frac{q}{\sqrt{r^2+a^2q^2}}$. Let's explore this expression. At $r >> a|q|$, it takes on the Coulomb form $E_r = \frac{q}{r^2}$, while remaining finite at $r \to 0$: $E_r = q/(|q|a)$. Therefore, the energy of the electrostatic field of a point charged particle in nonlinear Born-Infeld electrodynamics is also a finite quantity.

Heisenberg-euler and post-maxwellian formalism

Next, we consider the nonlinear electrodynamics of vacuum, which is a consequence of quantum electrodynamics, called Heisenberg-Euler electrodynamics. In this theory, the effective Lagrangian of the field has the form in (2). The electromagnetic field equation in non-linear Heisenberg-Euler electrodynamics without sources is similar to the macroscopic equation of continuum electrodynamics [30, 44, 57, 58]:

$$rot\,\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t},\ div\,\vec{D} = 0, \vec{D} = \frac{4\pi\partial L}{\partial\vec{E}}$$
$$rot\,\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t},\ div\,\vec{B} = 0,\ \vec{H} = \frac{4\pi\partial L}{\partial\vec{B}}. \quad (16)$$

Using expression (2), it is easy to obtain expansions of the vectors D and H in powers of B/B_q and E/B_q with a first-order post-Maxwellian accuracy [41, 46]:

$$\vec{D} = \vec{E} + \frac{\alpha}{45\pi B_q^2}\{2(\vec{E}^2 - \vec{B}^2)\vec{E} + 7(\vec{B}\vec{E})\vec{B}\},$$
$$\vec{H} = \vec{B} + \frac{\alpha}{45\pi B_q^2}\{2(\vec{E}^2 - \vec{B}^2)\vec{B} + 7(\vec{B}\vec{E})\vec{E}\}. \quad (17)$$

The system of electromagnetic field equations in nonlinear electrodynamics of Heisenberg Euler has an exact solution in the form of a plane elliptically polarized wave [49]:

$$\vec{E} = \vec{E}_1\cos(\Omega t - \vec{K}\vec{r}) + \vec{E}_2\sin(\Omega t - \vec{K}\vec{r}),$$
$$(\vec{E}_1\vec{E}_2) = 0, \quad \vec{B} = \frac{c}{\Omega}[\vec{K}\,\vec{E}], \quad \frac{\Omega^2}{c^2} = \vec{K}^2,$$
$$(\vec{K}\,\vec{E}_1) = (\vec{K}\,\vec{E}_2) = 0. \quad (18)$$

The invariants of this wave are equal to zero.

The parametrized post-Maxwellian electrodynamics of vacuum is considered as a generalization of nonlinear models in the case of weak fields [9,39]. There are other models of nonlinear electrodynamics. Modern experiment cannot decide which theory is the most adequate to nature. To choose nonlinear electrodynamics, the most adequate to nature, it is necessary to calculate nonlinear effects in various theories and compare their predictions with the results of the corresponding experiments. To facilitate such calculations in the weak electromagnetic field approximation, we propose to use the parametrized post-Maxwellian formalism, which in a certain sense is similar to the parametrized post-Newtonian formalism in the theory of gravity, used to calculate various gravitational effects in the weak field of the solar system [9,38,41]. We assume that the main premise of such a formalism is that the Lagrangian of nonlinear electrodynamics in vacuum is an analytic function of the invariants $J_1 = (\vec{E}^2 - \vec{B}^2)/B_q^2$ and $J_2 = (\vec{E}\vec{B})^2/B_q^4$, at least in the vicinity of their zero values. Therefore, in the case of a weak electromagnetic field $J_1 \ll 1, J_2 \ll 1$ this Lagrangian can be expanded in a series in integer powers of these invariants [10, 26, 38]:

$$\mathcal{L} = \frac{B_q^2}{8\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L_{nm}\, J_1^n\, J_2^m \quad (19)$$

Since at $J_1 \to 0, J_2 \to 0$ the theory with the Lagrangian (2) must pass into Maxwell's electrodynamics, then $L_{00} = 0, L_{10} = 1$. With this approach, each nonlinear electrodynamics will correspond to a well-defined set of post-Maxwellian parameters L_{mn}. From the point of view of experiments performed in a weak electromagnetic field, one nonlinear electrodynamics will differ from another only in the values of these parameters.

Thus, post-Maxwellian formalism, abstracting from the details of this or that nonlinear electrodynamics, from its equations, hypotheses and postulates, in a word, from everything that makes up its complete theoretical scheme, takes only the final result: the expansion of the Lagrangian, which, according to this theory, is valid in the weak electromagnetic field approximation. Further analysis of theories and elucidation of the correspondence of their predictions with the results of experiments is of a general nature and formally comes down to answering two questions: what are the values of the post-Maxwellian parameters of the theory under study and what are these parameters according to the results of the corresponding experiments. Therefore, one of the tasks facing this formalism is the task of identifying those nonlinear electrodynamics that are consistent with experiments carried out in a weak electromagnetic field [18, 26].

However, no less important, from our point of view, is the opportunity provided by this formalism for the systematic implementation of the calculation of nonlinear electrodynamic effects, regardless of any nonlinear theory: the task of theory and experiment in this case should not only be the search for one or another effect that will refute that or other non-linear electrodynamics, but also systematic calculation and setting up experiments in order to test the hypothesis about the analyticity of the Lagrangian in the vicinity of $J_1 = 0, J_2 = 0$ and subsequent determination with the required accuracy of the values of all post-Maxwellian parameters [9, 36, 51].

As a result of the implementation of this formalism, a generalized post-Maxwellian theory of the electromagnetic field can be constructed, capable of describing all experiments in a weak electromagnetic field. It is quite obvious that this theory cannot answer many questions about the properties of nonlinear electromagnetic interaction, and its main purpose will be only to describe one of the limiting cases of exact nonlinear electrodynamics - the weak electromagnetic field approximation [18, 57].

After the successful implementation of this formalism, any nonlinear electrodynamics that claims to be an adequate description of reality will have to pass into this post-Maxwellian theory in the limit of a weak electromagnetic field. Since it is quite obvious that with the current level of development of experimental technology, such a program can be implemented only with respect to the first few coefficients of expansion (19), we write the expression, restricting ourselves to only the accuracy necessary for our purposes [9, 45, 47, 55].

$$\mathcal{L} = \frac{1}{8\pi}\left[\vec{E}^2 - \vec{B}^2\right] + \xi\left[\eta_1\left(\vec{E}^2 - \vec{B}^2\right)^2 + 4\eta_2\left(\vec{E}\vec{B}\right)^2\right] + \xi^2\left[\eta_3\left(\vec{E}^2 - \vec{B}^2\right)^3 + \eta_4\left(\vec{E}^2 - \vec{B}^2\right)\left(\vec{E}\vec{B}\right)^2\right] + \\ + \xi^3\left[\eta_5\left(\vec{E}^2 - \vec{B}^2\right)^4 + \eta_6\left(\vec{E}^2 - \vec{B}^2\right)^2\left(\vec{E}\vec{B}\right)^2 + \eta_7\left(\vec{E}\vec{B}\right)^4\right] \quad (20)$$

wher e $\xi = 1/B_q^2$, and $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7$ – dimensionless post-Maxwellian parameters. In the Born-Infeld theory, these parameters have the form [5, 26, 47, 49]:

$$\eta_1 = \frac{a^2 B_q^2}{32\pi}, \quad \eta_2 = \frac{a^2 B_q^2}{32\pi}, \quad \eta_3 = -\frac{a^4 B_q^4}{64\pi}, \\ \eta_4 = -\frac{a^4 B_q^4}{16\pi}, \quad \eta_5 = \frac{5a^6 B_q^6}{512\pi}, \quad \eta_6 = \frac{3a^6 B_q^6}{64\pi}, \\ \eta_7 = -\frac{a^6 B_q^6}{32\pi} \quad (21)$$

In nonlinear Heisenberg-Euler electrodynamics, only the following parameters are known [30, 37, 39]:

$$\eta_1 = \frac{\alpha}{360\pi^2}, \; \eta_2 = \frac{\alpha}{90\pi^2}. \quad (22)$$

and the rest can be found using higher approximations of the perturbation theory of quantum electrodynamics. If we insert equation (20) into the equation of nonlinear electrodynamics and look for the solution further, we get the values of the post-Maxwellian parameters up to the 4th power of ξ. We get two solutions of the resulting eikonal equation. Therefore, due to the nonlinear electrodynamics of the vacuum, it is determined that in the presence of a homogeneous field, light propagates in the form of two possible plane electromagnetic waves with two different frequencies [38]:

$$\omega_1 = ck\left\{1 + \frac{16\pi\eta_1\xi}{k^2}\left[2k\left(\vec{k}\left[\vec{E}_0\vec{B}_0\right]\right) - \left[\vec{k}\vec{B}_0\right]^2 - \left[\vec{k}\vec{E}_0\right]^2\right]\right\}, \\ \omega_2 = ck\left\{1 + \frac{16\pi\eta_2\xi}{k^2}\left[2k\left(\vec{k}\left[\vec{E}_0\vec{B}_0\right]\right) - \left[\vec{k}\vec{B}_0\right]^2 - \left[\vec{k}\vec{E}_0\right]^2\right]\right\}, \quad (23)$$

which, in general, are different, not additive, proportional ξ. Proceeding to the analysis of the obtained relations (19), we note, first of all, that at $\eta_1 = \eta_2$ the expressions for ω_1 and ω_2 coincide with the accuracy of the terms proportional to ξ^2. In this case, electromagnetic waves of only one type can propagate in each direction. This means that nonlinear electrodynamics, whose post-Maxwellian parameters satisfy the relation $\eta_1 = \eta_2$, is in some sense a separate theory among other nonlinear electrodynamics. An example of such a theory, in particular, is electrodynamics of Born-Infeld, the post-Maxwellian parameters of which satisfy the specified relation. If we used the exact expression (2) for the Lagrangian [5], and not its post-Maxwell expansion, it would lead to the exact dispersion equation [12, 18, 44, 58, 59]:

$$\omega_1 = \omega_2 = \\ =ck\left\{1 + \frac{a^2}{2k^2}\left[2k\left(\vec{k}\left[\vec{E}_0\vec{B}_0\right]\right) - \left[\vec{k}\vec{B}_0\right]^2 - \left[\vec{k}\vec{E}_0\right]^2\right]\right\} \quad (24)$$

which shows that in Born-Infeld electrodynamics, in the presence of constant and homogeneous electromagnetic fields, only one type of electromagnetic waves can propagate in each direction [37].

Nonlinear vacuum electrodynamics is a theory created to solve some fundamental problems in Maxwell's electrodynamics. For example, in Maxwell's theory, the electromagnetic mass of a point charge, due to the infinite value of the energy of the electrostatic field, will be infinite, which contradicts the experiment. This problem can be solved by adding additional nonlinear terms to the Lagrangian linear theory. However, there is a certain arbitrariness in the choice of nonlinear Lagrangian terms, so it is considered optimal to consider the socalled parametrized post-Maxwellian formalism of nonlinear vacuum electrodynamics, where a certain set of parameters corresponds to each selected type of nonlinearity. Thus, it becomes possible to classify and group various variants of the theory according to the emerging effects and degree of complexity of nonlinear processes in them, depending on the number and magnitude of PMPs. PMPs usually correspond to the coefficients before the nonlinear terms of the Lagrangian. At the same time, the physics of nonlinear electrodynamic processes becomes more complicated as it approaches the critical value of the magnetic field B_q caused by quantum electrodynamics.

In the simplest Born-Infeld (BI) nonlinear theory, $\eta_1 = \eta_2 \sim 4.1 \cdot 10^{-6}$, other parameters are equal to zero. This is explained by the fact that the nonlinear effects of the theory solve a minimal number of problems inherent in the linear theory. In the case of NED of HE, there are two parameters, and this theory is already consistent with quantum electrodynamics in the one-loop approximation, and birefringence also appears there. In any case, it is necessary to test the nonlinear post-Maxwellian theory on the basis of experiments or astrophysical observations, the essence of which is to check the values of the post-Maxwellian parameters.

Summary and Discussion

So, in this work, we analyzed two main theories of nonlinear vacuum electrodynamics. First, we considered their Lagrangians and focused on their main characteristics. Based on the type of Lagrangians, we took the basic field equations of each theory and determined the features of the post-Maxwellian parameters PMPs in them. We have shown that the PMPs are equal according to BI theory and that it is a simple theory. We have shown that the PMPs in the GE theory, which is a consequence of quantum electrodynamics, have two different values and the solutions of the field equation according to this theory. At the same time, we determined that the basic laws and parameters in the post-Maxwellian formalism depend on the nature of the chosen theory. At the same time, we have shown that the physics of post-Maxwellian parameters depends on their magnitude and the type of that formalism. In the future, the information in this work can be used as a necessary object for further in-depth study of nonlinear electrodynamics.

Acknowledgments:

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP13067667).

References

1. Okumura, T., Azuma, T., and et.al Bennett, "Proof-of-Principle Experiment for Testing StrongField Quantum Electrodynamics with Exotic Atoms: High Precision X-Ray Spectroscopy of Muonic Neon." *Phys. Rev. Lett.* Volume130 (2023):L17.
2. Gerold, Schellstede O., Volker Perlick, Claus Lämmerzahl, "Testing nonlinear vacuum electrodynamics with Michelson interferometry." *Phys. Rev.* Volume 92 (2015):L2.
3. Feynman, R. P., "Mathematical formulation of the quantum theory of electromagnetic interaction." *Phys. Rev.* Volume 80 (1950):L3.
4. Schwartz, M.D. Quantum Field Theory and the Standard Model. United Kingdom: Cambridge University Press, 2014.
5. Born, M., Infeld, L. "Foundations of the New Field Theory." *Proceedings of the Royal Society of London Series A* Volume 144 (1934):L852.
6. Jackson, J. D., Ronald, F., "Fox. Classical Electrodynamics, 3rd ed." *American Journal of Physics*, Volume 67, Pages 841–842 (1999):L9.
7. Born, M., "Modified Field Equations with a Finite Radius of the Electron." *Nature (London)* Volume 132 (1933):L3329.
8. Fradkin, E. S. , Tseytlin, A. A., "Non-linear electrodynamics from quantized strings." *Physics Letters B* Volume 163, Pages 123–130 (1985):L1-4.
9. Adler, S. L., "Photon splitting and photon dispersion in a strong magnetic field." *Annals of Physics* Volume 67, Pages 599–647 (1971).
10. Kruglov, S. I., "Notes on Born-Infeld-type electrodynamics." *Modern Physics Letters A* Volume 32 (2017):L36.
11. Kim, J.Y., "Bending of electromagnetic wave in an ultra-strong magnetic field." *Journal of Cosmology and Astroparticle Physics* Volume 2012 (2012).
12. Kim, J.Y., "Deflection of light by a Coulomb charge in Born-Infeld electrodynamics." *European Physical Journal C* Volume 81, Number 508 (2021).
13. Toshmatov, B., Stuchlík, Z., Schee, J., Ahmedov, B., "Electromagnetic perturbations of black holes in general relativity coupled to nonlinear electrodynamics." *Physical Review D* Volume 97 (2018):L8.
14. Abishev, M., Beissen, N., Belissarova, F., Boshkayev, K., Mansurova, A., Muratkhan, A., Quevedo, H., Toktarbay, S., "Approximate perfect fluid solutions with quadrupole moment." *International Journal of Modern Physics D* Volume 30 (2021):L13.
15. Gunasekaran, S., Kubizňák, D., Mann, B.R., "Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization." *Journal of High Energy Physics* Volume 2012, Number 110 (2012).
16. Born, M., "On the Quantum Theory of the Electromagnetic Field." *Proceedings of the Royal Society of London Series A* Volume 143, Pages 410–437 (1934):L849.
17. Bandos, I., Lechner, K., Sorokin, D., Townsend, P.K., "Nonlinear dualityinvariant conformal extension of Maxwell's equations" *Physical Review D* Volume 102 (2020):L12.
18. Kruglov, S. I., "Born-Infeld-type electrodynamics and magnetic black holes." *Annals of Physics* Volume 383, Pages 550–559 (2017).
19. Denisov, V.I., Denisova, I.P., "Verifiable Post-Maxwellian Effect of the Nonlinear Electrodynamics in Vacuum." *Optics and Spectroscopy* Volume 90, Pages 282–287 (2001):L2.
20. Zavattini, G., Gastaldi, U., Pengo, R., Ruoso, G., Valle, F. D., Milotti, E., "Measuring the Magnetic Birefringence of Vacuum: the Pvlas Experiment." *International Journal of Modern Physics A* Volume 27 (2012): L15.
21. Valle, F.D., Gastaldi, U., Messineo, G., Milotti, E., Pengo, R., Piemontese, L., Ruoso, G., Zavattini. G., "Measurements of vacuum magnetic birefringence using permanent dipole magnets: the PVLAS experiment." *New Journal of Physics* Volume 15 (2013): L5.
22. Valle, F.D., Di Domenico, G., Gastaldi, U., Milotti, E., Pengo, R., Ruoso, G., Zavattini, G., "Towards a direct measurement of vacuum magnetic birefringence: PVLAS achievements." *Optics Communications*, Volume 283 (2010): L5.
23. Cadene, A., Fouche, M., Rivere, A., Battesti, R., Coriani, S., Rizzo, A., Rizzo, C., "Circular and linear magnetic birefringences in xenon at $\lambda = 1064$ nm." *J. Chem. Phys.* Volume 142 (2015): L12.
24. Fan, X., Kamioka, S., and et.al. "The OVAL experiment: a new experiment to measure vacuum magnetic birefringence using high repetition pulsed magnets." *European Physical Journal D* Volume 71, Number 308 (2017): L11.
25. Kim, J.Y., "Deflection of light by magnetars in the generalized Born-Infeld electrodynamics." *European Physical Journal C* Volume 82, Number 485 (2022): L5.

26. Denisov, V. I., Denisova, I. P., Krivchenkov, I. V., Vshivtseva, P. A., "Beam curvature in the magnetic field of a neutron star for an arbitrary angle between the magnetic dipole moment and incident beam." *Physics - Doklady*, Volume 48, Pages 657–659 (2003): L12.

27. Valle, F. D., Ejlli, A., Gastaldi, U., Messineo, G., Milotti, E., Pengo, R., Ruoso, G., Zavattini, G., "The PVLAS experiment: measuring vacuum magnetic birefringence and dichroism with a birefringent Fabry-Perot cavity." *European Physical Journal C* Volume 76, Number 24, (2016).

28. Event Horizon Telescope Collaboration, Akiyama, K., and et.al. "First M87 Event Horizon Telescope Results. VI. The Shadow and Mass of the Central Black Hole." *The Astrophysical Journal Letters* Volume 875 (2019) :L6.

29. Yajima H., Tamaki, T., "Black hole solutions in Euler-Heisenberg theory." *Phys. Rev. D* Volume 63 (2001): L6.

30. Heisenberg, W.,. Euler, H., "Folgerungen aus der Diracschen Theorie des Positrons." *Zeitschrift fur Physik* Volume 98, Pages 714–732 (1936): L11-12.

31. Berestetskii, V. B., Lifshits, E. M., Pitaevskii, L. P., "Quantum electrodynamics 2nd revised edition". *Moscow Izdatel Nauka Teoreticheskaia Fizika* Volume 4 (1980).

32. Gitman, D.M., Shabad, A.E., "A note on "Electron self-energy in logarithmic electrodynamics" by P. Gaete and J. Helay¨el-Neto." *European Physical Journal C* Volume 74 (2014).

33. King, B., Di Piazza, A., Keitel, C.H., "A matterless double slit." *Nature Photonics* Volume 4, Pages 92–94 (2010): L2.

34. Cho, Y.M., Pak, D.G., Walker, M.L., "Light propagation effects in QED: Effective action approach." *Phys. Rev. D* Volume 73 (2006):L6.

35. Beissen, N., Utepova, D., Abishev, M., Quevedo, H., Khassanov, M., Toktarbay, S., "Gravitational refraction of compact objects with quadrupoles." *Symmetry* Volume 15 (2023): L3.

36. Denisov, V.I., Denisova, I.P., Svertilov,S.I., "Nonlinear Electrodynamic Effect of Ray Bending in the Magnetic-Dipole Field." *Physics - Doklady* Volume 46, Pages 705–707 (2001): L10.

37. Abishev, M.E., Toktarbay, S., Beissen, N.A., Belissarova, F.B., Khassanov, M.K., Kudussov, A.S., Abylayeva, A. Zh., "Effects of non-linear electrodynamics of vacuum in the magnetic quadrupole field of a pulsar." *Monthly Notices of the Royal Astronomical Society* Volume 481, Pages 36–43 (2018).

38. Denisov, V.I., Dolgaya, E.E., Sokolov, V.A., "Nonperturbative QED vacuum birefringence." *Journal of High Energy Physics* Number 105 (2017).

39. Heyl, J., Caiazzo, I., "Strongly Magnetized Sources: QED and X-ray Polarization." *Galaxies* Volume 6 (2018): L3.

40. Yernazarov, T., Abishev, M., Aimuratov, Y., "Correspondence of gamma radiation coming from grbs and magnetars based on the effects of nonlinear vacuum electrodynamics" (2021).

41. Kruglov, S.I., "Magnetized black holes and nonlinear electrodynamics." *International Journal of Modern Physics A* Volume 32 (2017): L23-24.

42. Beissen, N.A., Utepova, D., Muratkhan, A., Orazymbet, A., Khassanov, M., Toktarbay, S., "Application of gbt theorem for gravitational deflection of light by compact objects." *Recent Contributions to Physics* Volume 84, Pages 15–21 (2023): L1.

43. Muratkhan, A., Orazymbet, A., Zhakipova, M., Assylbek, M., Toktarbay, S., "A shadows from the static black hole mimickers." *International Journal of Mathematics and Physics* Volume 13, Pages 44–49 (2023): L2.

44. Denisov, D.M., "Effects of nonlinear electrodynamics in the magnetic field of a pulsar." *Canadian Journal of Physics* Volume 92, Number 11 (2014).

45. Abishev, M., Aimuratov, Y., Aldabergenov, Y., Beissen, N., Bakytzhan, Z., Takibayeva, M., "Some astrophysical effects of nonlinear vacuum electrodynamics in the magnetosphere of a pulsar." *Astroparticle Physics* Volume 73, Pages 8–13 (2016).

46. Heyl, J. S., Shaviv, N.J., Lithwick, Y., "The Optics of Neutron-Star Magnetospheres." In *AAS/High Energy Astrophysics Division #4* Volume 4 of *AAS/High Energy Astrophysics Division* (1999).

47. Heisenberg, W., Euler, H., "Consequences of Dirac Theory of the Positron." *arXiv e-prints* (2006).

48. Meyerhofer, D. D., "High-intensity-laser-electron scattering." *IEEE Journal of Quantum Electronics* Volume 33 (1997): L11.

49. Denisov, V.I., Denisova, I.P., Pimenov, A.B., Sokolov, V.A., "Rapidly rotating pulsar radiation in vacuum nonlinear electrodynamics." *European Physical Journal C* Volume 76, Number 612 (2016).

50. Kruglov, S. I., "On generalized ModMax model of nonlinear electrodynamics." *Physics Letters B* Volume 822 (2021).

51. Abishev, M. E., Ivashchuk, V. D., Malybayev, A. N., Toktarbay, S., "Dyon-Like Black Hole Solutions in the Model with Two Abelian Gauge Fields." *Gravitation and Cosmology* Volume 25, Pages 374–382 (2019).

52. Heyl, J.S., Hernquist, L., "Birefringence and dichroism of the QED vacuum." *Journal of Physics A Mathematical General* Volume 30, Pages 6485–6492 (1997): L18.

53. Heyl, J.S., Hernquist, L., "QED one-loop corrections to a macroscopic magnetic dipole." *Journal of Physics A Mathematical General* Volume 30, Pages 6475–6483 (1997): L18.

54. Denisov, V. I., Sokolov, V.A., "Analysis of regularizing properties of nonlinear electrodynamics in the Einstein-Born-Infeld theory." *Soviet Journal of Experimental and Theoretical Physics* Volume 113, Pages 926–933 (2011): L6.

55. Denisov, V.I., Shvilkin, B.N., Sokolov, V.A., Vasili'ev, M. I., "Pulsar radiation in postMaxwellian vacuum nonlinear electrodynamics." *Phys. Rev. D* Volume 94 (2016): L4.

56. Kruglov, S.I., "Nonlinear Electrodynamics and Magnetic Black Holes." *Annalen der Physik* (2017).

57. Heyl, J.S., Hernquist, L., "Analytic form for the effective Lagrangian of QED and its application to pair production and photon splitting." *Phys. Rev. D* Volume 55, Pages 2449–2454 (1997).

58. Abishev, M.E., Denisov, V.I., Denisova, I.P., Kechkin, O.V., "The evaluation of electromagnetic forward radiations during the propagation of gravitational waves through the dipole field of the magnetar." *Astroparticle Physics* Volume 103, Pages 94–97 (2018).

59. Kim, C.M., Kim, S.P., "Magnetars as Laboratories for Strong Field QED." *arXiv e-prints* (2021).