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A.T. Assanova[1], M.A. Mukash[1,2,*], O.M. Stanzhytskyi[3]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan
[3]T.G. Shevshenko Kyiv National University, Kiev, Ukraine
*e-mail: mukashma1983@gmail.com
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A problem with impulse actions for nonlinear ODEs

Abstract. In the paper, we consider problem with impulse actions for system of nonlinear ordinary differential equations (ODEs) in the interval. For solving this problem we use a modification of parameterization method by Dulat Dzhumabaev. In the modification of the method we introduce the parameters as the values of the unknown function at the middle of the subintervals of the partition of the considered interval. The problem with impulse actions transfers to an equivalent problem system of nonlinear ODEs with parameters. Conditions for an existence of solution to the equivalent problem are obtained. Existence theorem for solutions this problem is established by one generalizes a theorem of Hadamard. We also constructed an algorithm for finding of solution to this problem. Finally, we found conditions for solvability to the problem with impulse actions for system of nonlinear ODEs in terms of special matrix composed by the initial data. This method can be applied to various types of nonlinear problems with impulse actions for ODEs.
Key words: problem with impulse actions, nonlinear ODEs, multi-point problem with parameters, modification of parametrization method by Dzhumabaev, solution.

1 Introduction and preliminaries

The problem with impulse actions holds a very significant place in the theory of discontinuous differential equations (see [1-4] and references cited therein). Many authors investigate the existence and uniqueness of problem impulse actions for differential equations by using various methods [5–11]. To the best of our knowledge only monography [4] investigate impulsive system with variable time of the impulse action via numerical-analytic method, Lyapunov direct method and Green's function method.

In present work we consider problem with impulse actions for nonlinear ODEs by parameterization method by Dulat Dzhumabaev [12] proposed for solving BVPs for ODEs and extended to different classes of differential equations [13-20]. We offer a modification of parameterization method for solving to problem with impulse actions for system of nonlinear ODEs.

So, we consider the problem with impulse actions for system of nonlinear ODEs in the following form:

$$\dot{x}(t) = f(t,x),\ t \in (0,T)\backslash\{t_1, t_2, \dots, t_k\}, \quad (1)$$

$$Bx(0) + Cx(T) = d,\ x \in R^n,\ d \in R^n, \quad (2)$$

$$x(t_i + 0) - x(t_i - 0) = s_i, \quad (3)$$

$s_i \in R^n, i = \overline{1,k}$.

Here $f : [0,T] \times R^n \to R^n$ is vector-function with possible discontinuities at the points $t = t_i$, $i = \overline{1,k}$; the $(n \times n)$-matrices B and C are constant matrices;

$0 = t_0 < t_1 < t_2 < \dots < t_k < t_{k+1} = T$,

$\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

Denote $I_k = \{t_1, t_2, \dots, t_k\}$.

Let $PC([0,T]\backslash I_k, R^n)$ be a space of piecewise-continuous vector-functions with norm

$$\|x\|_1 = \max_{i=\overline{0,k}} \sup_{t \in [t_i, t_{i+1})} \|x(t)\|.$$

A solution to problem with impulse actions (1)-(3) is a piecewise-continuously differentiable on $(0,T)\backslash I_k$ a function $x^*(t) \in PC([0,T]\backslash I_k, R^n)$ that satisfies:

• system of nonlinear ODEs (1) (at the same time, at the points $t = 0$, $t = T$, one-sided derivatives $\dot{x}^*_+(0)$, $\dot{x}^*_-(T)$ satisfy to system of nonlinear ODEs (1));

• boundary condition (2);

• conditions for impulse actions (3) at the points of the set I_k.

2 A modification of parameterization method for solving problem (1)-(3)

We split the segment $[0,T]$ on the subintervals as follows:

$$[0,T)=\cup_{r=1}^{k+1}[t_{r-1},t_r).$$

Let $C([0,T],I_k,R^{(k+1)n})$ denote the space of function systems

$x[t]=(x_1(t),x_2(t),\dots,x_k(t),x_{k+1}(t))$, with elements $x_r:[t_{r-1},t_r)\to R^n$ are continuous on $[t_{r-1},t_r)$ and have finite limits $\lim\limits_{t\to t_r-0}x_r(t)$ for all $r=\overline{1,k+1}$. The space is endowed by the norm

$$\|x[\cdot]\|_2=\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}\|x_r(t)\|.$$

The restrictions of $x(t)$ to the partition subintervals, denoted by $x_r(t)$:

$x_r(t)=x(t)$ for $t\in[t_{r-1},t_r)$, $r=\overline{1,k+1}$, satisfy the following multi-point problem

$$\dot{x}_r=\mathrm{f}(t,x_r),\ t\in[t_{r-1},t_r),\ r=\overline{1,k+1}, \quad (4)$$

$$Bx_1(0)+C\lim_{t\to T-0}x_{k+1}(t)=d, \quad (5)$$

$$x_{i+1}(t_i+0)-\lim_{t\to t_i-0}x_i(t)=s_i,\ i=\overline{1,k}. \quad (6)$$

We introduce a parameters as the values of the unknown function at the middle of the subintervals: $\xi_r=x_r\left(\frac{t_r-t_{r-1}}{2}\right)$, $r=\overline{1,k+1}$.

Then, we make a change of functions:

$x_r(t)=y_r(t)+\xi_r$ on each r-th subinterval. We transfer problem (4)-(6) to the equivalent problem with parameters ξ_r :

$$\dot{y}_r=\mathrm{f}(t,y_r+\xi_r),\ t\in[t_{r-1},t_r), \quad (7)$$

$$y_r\left(\frac{t_r-t_{r-1}}{2}\right)=0,\quad r=\overline{1,k+1}, \quad (8)$$

$$By_1(0)+B\xi_1+ +C\lim_{t\to T-0}y_{k+1}(t)+C\xi_{k+1}=d, \quad (9)$$

$$y_{i+1}(t_i+0)+\xi_{i+1}- -\lim_{t\to t_i-0}y_i(t)-\xi_i=s_i,\quad i=\overline{1,k}. \quad (10)$$

A pair $(y^*[t],\xi^*)$, with elements $y^*[t]=\left(y_1^*(t),y_2^*(t),\dots,y_{k+1}^*(t)\right)\in C\left([0,T],I_k,R^{(k+1)n}\right)$ and $\xi^*=(\xi_1^*,\xi_2^*,\dots,\xi_{k+1}^*)\in R^{(k+1)n}$, is called a solution to problem (7)-(10) if the functions $y_r^*(t)$, $r=\overline{1,k+1}$, are continuously differentiable on $[t_{r-1},t_r)$ and satisfy system of nonlinear ODEs (7), conditions (8), and relations (9), (10) with $\xi_r=\xi_r^*$, $r=\overline{1,k+1}$.

The equivalence of problems (1)-(3) and (7)-(10) is understood in the following sense.

Let function $x^*(t)$ be a solution to problem with impulse actions (1)-(3), then the pair $(y^*[t],\xi^*)$, where $y^*[t]=(x^*(t)-x^*(\frac{t_1-t_0}{2}),x^*(t)-x^*(\frac{t_2-t_1}{2}),\dots,$ $x^*(t)-x^*\left(\frac{t_k-t_{k-1}}{2}\right),x^*(t)-x^*(\frac{t_{k+1}-t_k}{2}))$ and $\xi^*=(x^*(\frac{t_1-t_0}{2}),x^*(\frac{t_2-t_1}{2}),\dots,$ $x^*(\frac{t_k-t_{k-1}}{2}),x^*(\frac{t_{k+1}-t_k}{2}))$, is a solution of problem with parameters (7)-(10).

Vice versa, if a pair $\left(\tilde{y}[t],\ \tilde{\xi}\right)$ with elements $\tilde{y}[t]=(\tilde{y}_1(t),\tilde{y}_2(t),\dots,\tilde{y}_{k+1}(t))\in C([0,T],I_k,R^{(k+1)n})$ and $\tilde{\xi}=(\tilde{\xi}_1,\tilde{\xi}_2,\dots,\tilde{\xi}_{k+1})\in R^{(k+1)n}$, be a solution to problem with parameters (7)-(10), then the function $\tilde{x}(t)$ defined as

$$\tilde{x}(t)=\tilde{y}_r(t)+\tilde{\xi}_r,$$
$$t\in[t_{r-1},t_r),\ \ r=\overline{1,k+1},$$

and

$$\tilde{x}(T)=\lim_{t\to T-0}\tilde{y}_{k+1}(t)+\tilde{\xi}_{k+1},$$

is a solution of the original problem with impulse actions (1)-(3).

In contrast to problem (4)-(6), problem with parameters (7)-(10) have conditions (8) for the values of the desired functions in the middle of the subintervals $[t_{r-1},t_r)$, $r=\overline{1,k+1}$.

This is the modification of the parameterization method. We get the problem with parameters (7)-(10), where instead of the initial conditions for the desired function [12], appear the conditions (8) with the values of the desired function in the middle of the subintervals $[t_{r-1},t_r)$, $r=\overline{1,k+1}$.

This allows us to reduce the impact of the length of subintervals $[t_{r-1},t_r)$, $r=\overline{1,k+1}$, on the solvability of the problem with impulse actions (1)-(3).

Assume that ξ_r are known for all $r=\overline{1,k+1}$. Then the problem (7), (8) will be equivalent to the nonlinear Volterra integral equations of the second kind in the following form:

$$y_r(t) = \int_t^{\frac{t_r-t_{r-1}}{2}} f(\tau, y_r(\tau)+\xi_r)d\tau,$$

$$t \in \left[t_{r-1}, \frac{t_r-t_{r-1}}{2}\right), \quad r=\overline{1,k+1}, \qquad (11)$$

$$y_r(t) = \int_{\frac{t_r-t_{r-1}}{2}}^{t} f(\tau, y_r(\tau)+\xi_r)d\tau,$$

$$t \in \left[\frac{t_r-t_{r-1}}{2}, \ t_r\right), \quad r=\overline{1,k+1}. \qquad (12)$$

For finding the values of unknown functions $y_r(t)$, we will use representations (11) or (12) depending on the location of the points

$$t=t_r, \quad r=\overline{1,k+1}.$$

$$y_1(0) = \int_0^{\frac{t_1}{2}} f(\tau, y_1(\tau)+\xi_1)d\tau, \qquad (13)$$

$$\lim_{t\to T-0} y_{k+1}(t) = $$

$$= \int_{\frac{t_{k+1}-t_k}{2}}^{T} f(\tau, y_{k+1}(\tau)+\xi_{k+1})d\tau, \qquad (14)$$

$$y_{i+1}(t_i+0) =$$

$$= \int_{t_i}^{\frac{t_{i+1}-t_i}{2}} f(\tau, y_{i+1}(\tau)+\xi_{i+1})d\tau, \qquad (15)$$

$$\lim_{t\to t_i-0} y_i(t) =$$

$$= \int_{\frac{t_i-t_{i-1}}{2}}^{t_i} f(\tau, y_i(\tau)+\xi_i)d\tau, \qquad (16)$$

$$i=\overline{1,k}.$$

Substituting representations (13)-(16) in relations (9) and (10) instead of corresponding values $y_1(0)$, $\lim_{t\to T-0} y_{k+1}(t)$, $y_{i+1}(t_i+0)$, $\lim_{t\to t_i-0} y_i(t)$, $i=\overline{1,k}$, we have

$$B\xi_1 + B\int_0^{\frac{t_1}{2}} f(\tau, y_1(\tau)+\xi_1)d\tau + C\xi_{k+1} +$$

$$+C\int_{\frac{t_{k+1}-t_k}{2}}^{T} f(\tau, y_{k+1}(\tau)+\xi_{k+1})d\tau - d = 0, \qquad (17)$$

$$\int_{t_i}^{\frac{t_{i+1}-t_i}{2}} f(\tau, y_{i+1}(\tau)+\xi_{i+1})d\tau + \xi_{i+1} -$$

$$-\int_{\frac{t_i-t_{i-1}}{2}}^{t_i} f(\tau, y_i(\tau)+\xi_i)d\tau - \xi_i - s_i = 0,$$

$$i=\overline{1,k}. \qquad (18)$$

For known $y_r(t)$ $(r=\overline{1,k+1})$, the system of equations (17), (18) is a system of equations for the parameters $(\xi_1, \xi_2, \dots, \xi_{k+1})$. Let us write down the system of equations (17), (18) in the following form

$$Q(\xi, y) = 0, \qquad (19)$$

$$\xi = (\xi_1, \xi_2, \dots, \xi_{k+1}) \in R^{(k+1)n}.$$

Condition *E*. *For choosing* I_k *the system of nonlinear equations* $Q(\xi, 0) = 0$ *has a solution*

$$\xi^{(0)} = \left(\xi_1^{(0)}, \xi_2^{(0)}, \dots, \xi_{k+1}^{(0)}\right) \in R^{(k+1)n}.$$

Let Condition *E* be satisfied. Suppose that the problem

$$\dot{y}_r = f\left(t, y_r + \xi_r^{(0)}\right), \ t \in [t_{r-1}, t_r), \qquad (20)$$

$$y_r\left(\frac{t_r-t_{r-1}}{2}\right) = 0, \quad r=\overline{1,k+1}, \qquad (21)$$

has a solution

$y_r^{(0)}(t)$, $t \in [t_{r-1}, t_r)$, $r=\overline{1,k+1}$, and system function

$$y^{(0)}[t] = \left(y_1^{(0)}(t), y_2^{(0)}(t), \dots, y_{k+1}^{(0)}(t)\right)$$

belongs to $C\left([0,T], I_k, R^{(k+1)n}\right)$.

Given the pair $(y^{(0)}[t], \xi^{(0)})$ we define a piecewise continuous function on $[0,T]$:

$$x^{(0)}(t) = y_r^{(0)}(t) + \xi_r^{(0)},$$

$$t \in [t_{r-1}, t_r), \ r=\overline{1,k+1},$$

and

$$x^{(0)}(T) = \lim_{t\to T-0} y_{k+1}^{(0)}(t) + \xi_{k+1}^{(0)}.$$

We choose numbers $\rho_\xi > 0$, $\rho_y > 0$, $\rho_x > 0$, and define the sets:

$$S(\xi^{(0)}, \rho_\xi) = \{\xi = (\xi_1, \xi_2, \dots, \xi_{k+1} \in R^{(k+1)n}:$$
$$\|\xi - \xi^{(0)}\| = \max_{r=\overline{1,k+1}} \left\|\xi_r - \xi_r^{(0)}\right\| < \rho_\xi \},$$
$$S(y^{(0)}[t], \rho_y) =$$
$$=\{y[t] \in C([0,T], I_k, R^{(k+1)n}):$$
$$\|y[\cdot] - y^{(0)}[\cdot]\|_2 < \rho_y \},$$
$$S(x^{(0)}(t), \rho_x) = \{x(t) \in PC([0,T]\backslash I_k, R^n):$$
$$\|x - x^{(0)}\|_1 < \rho_x \},$$
$$G_f(\rho_x) =$$
$$=\{(t,x): t \in [0,T], x \in S(x^{(0)}(t), \rho_x)\}.$$

Let Condition *E* hold.

Let us construct a sequence of pairs $(y^{(m)}[t], \xi^{(m)})$, $m \in \mathrm{N}$, according to the following algorithm.

3 Algorithm

Stage 1.

(1) From system of equations
$$Q(\xi, y^{(0)}) = 0$$
we find
$$\xi^{(1)} = \left(\xi_1^{(1)}, \xi_2^{(1)}, \dots, \xi_{k+1}^{(1)}\right) \in R^{(k+1)n};$$
(2) Solving problem (7), (8) for
$$\xi_r = \xi_r^{(1)},\ r = \overline{1, k+1},$$
we find an elements of system functions
$$y^{(1)}[t] = \left(y_1^{(1)}(t), y_2^{(1)}(t), \dots, y_{k+1}^{(1)}(t)\right).$$

Stage 2.

(1) From system of equations
$$Q(\xi, y^{(1)}) = 0$$
we find
$$\xi^{(2)} = \left(\xi_1^{(2)}, \xi_2^{(2)}, \dots, \xi_{k+1}^{(2)}\right) \in R^{(k+1)n};$$
(2) Solving problem (7), (8) for
$$\xi_r = \xi_r^{(2)},\ r = \overline{1, k+1},$$
we find an elements of system functions
$$y^{(2)}[t] = \left(y_1^{(2)}(t), y_2^{(2)}(t), \dots, y_{k+1}^{(2)}(t)\right).$$

And, continuing this process, we get

Stage *m*.

(1) From system of equations
$$Q(\xi, y^{(m-1)}) = 0$$
we find
$$\xi^{(m)} = \left(\xi_1^{(m)}, \xi_2^{(m)}, \dots, \xi_{k+1}^{(m)}\right) \in R^{(k+1)n};$$
(2) Solving problem (7), (8) for
$$\xi_r = \xi_r^{(m)},\ r = \overline{1, k+1},$$
we find an elements of system functions
$$y^{(m)}[t] = \left(y_1^{(m)}(t), y_2^{(m)}(t), \dots, y_{k+1}^{(m)}(t)\right).$$
$m = 1,2, \dots$.

Condition *C*. *The function $f(t,x)$ is continuous and has a uniformly continuous partial derivative $f_x'(t,x)$ in $G_f(\rho_x)$ and there exists a number $L > 0$ such that*
$$\|f_x'(t,x)\| \le L$$
for all $(t,x) \in G_f(\rho_x)$.

4 Main result

Introduce the notation
$$h = \max_{i=\overline{1,k+1}} \{ \sup_{t \in \left[t_{i-1}, \frac{t_i - t_{i-1}}{2}\right)} \left(\frac{t_i - t_{i-1}}{2} - t\right),$$
$$\sup_{t \in \left[\frac{t_i - t_{i-1}}{2}, t_i\right)} \left(t - \frac{t_i - t_{i-1}}{2}\right)\}.$$

Theorem 1. *Let for I_k, $\rho_\xi > 0$, $\rho_y > 0, \rho_x > 0$, Condition E and Condition C be fulfilled, and Jacobi matrix $\frac{\partial Q(\xi,y)}{\partial \xi} : R^{(k+1)n} \to R^{(k+1)n}$ is invertible for all $(y[t], \xi) \in S(y^{(0)}[t], \rho_y) \times S(\xi^{(0)}, \rho_\xi)$, and the inequalities are valid:*

i) $\left\|\left(\frac{\partial Q(\xi,y)}{\partial \xi}\right)^{-1}\right\| \le \beta,\ \beta - const;$

ii) $q = \beta \max\{2, h\|B\| + h\|C\|\}$
$$\max_{i=\overline{1,k+1}} \{e^{L\left(\frac{t_i - t_{i-1}}{2}\right)} - 1 - L\left(\frac{t_i - t_{i-1}}{2}\right)\} < 1;$$

iii) $\frac{\beta}{1-q}\|Q(\xi^{(0)}, y^{(0)})\| \le \rho_\xi;$

iv) $\frac{\beta}{1-q} \max_{i=\overline{1,k+1}} \left\{e^{L\left(\frac{t_i - t_{i-1}}{2}\right)} - 1\right\}$
$$\|Q(\xi^{(0)}, y^{(0)})\| \le \rho_y;$$

v) $(1 + Lh)\rho_\xi + Lh\rho_y \le \rho_x$.

Then the sequence of pairs $(y^{(k)}[t], \xi^{(k)})$, $k \in N$, determined by the algorithm belongs to $S(y^{(0)}[t], \rho_y) \times S(\xi^{(0)}, \rho_\xi)$, converges to $(y^*[t], \xi^*)$ is an isolated solution of problem (7)-(10) in $S(y^{(0)}[t], \rho_y) \times S(\xi^{(0)}, \rho_\xi)$ and the following estimates hold:

$$\|\xi^* - \xi^{(k)}\| \le \frac{q^k}{1-q}\beta\|Q(\xi^{(0)}, y^{(0)})\|, \qquad (20)$$

$$\left\|y_r^*(t) - y_r^{(k)}(t)\right\| \le \|\xi^* - \xi^{(k)}\|$$

$$\max_{i=\overline{1,k+1}}\left\{e^{L\left(t-\frac{t_i-t_{i-1}}{2}\right)}-1, e^{L\left(\frac{t_i-t_{i-1}}{2}-t\right)}-1\right\}. \quad (21)$$

Proof.

Using points of impulse actions I_k, we divide the interval $[0,T]$.

Let us transfer from problem (1)-(3) to the equivalent problem with parameters (7)-(10).

Take any pair $(y[t],\xi)\in S(y^{(0)}[t],\rho_y)\times S(\xi^{(0)},\rho_\xi)$, then

$$\left\|y_r(t)-y_r^{(0)}(t)+\xi_r-\xi_r^{(0)}\right\|\le$$
$$\le\left\|y_r(t)-y_r^{(0)}(t)\right\|+\left\|\xi_r-\xi_r^{(0)}\right\|<$$
$$\le \rho_\xi+\rho_y\le\rho_x,$$
$$t\in[t_{r-1},t_r),\qquad r=\overline{1,k+1}. \quad (22)$$

Using Condition *C*, for all $r=\overline{1,k+1}$, we obtain the following inequalities:

$$\|\xi_r+\int_t^{\frac{t_r-t_{r-1}}{2}}f(\tau,y_r(\tau)+\xi_r)d\tau-\xi_r^{(0)}-$$
$$-y_r^{(0)}(t)\|\le\left\|\xi_r-\xi_r^{(0)}\right\|+$$
$$\|\int_t^{\frac{t_r-t_{r-1}}{2}}f(\tau,y_r(\tau)+\xi_r)d\tau-$$
$$-\int_t^{\frac{t_r-t_{r-1}}{2}}f\left(\tau,y_r^{(0)}(\tau)+\xi_r^{(0)}\right)d\tau\|\le$$
$$\le\left[1+L\left(\frac{t_r-t_{r-1}}{2}-t\right)\right]\left\|\xi_r-\xi_r^{(0)}\right\|+$$
$$\int_t^{\frac{t_r-t_{r-1}}{2}}L\left\|y_r(\tau)-y_r^{(0)}(\tau)\right\|d\tau\le$$
$$\le\left[1+L\left(\frac{t_r-t_{r-1}}{2}-t\right)\right]\rho_\xi+$$
$$+L\left(\frac{t_r-t_{r-1}}{2}-t\right)\rho_y\le(1+Lh)\rho_\xi+Lh\rho_y\le\rho_x,$$
$$t\in\left[t_{r-1},\frac{t_r-t_{r-1}}{2}\right),\quad r=\overline{1,k+1}, \quad (23)$$

$$\|\xi_r+\int_{\frac{t_r-t_{r-1}}{2}}^t f(\tau,y_r(\tau)+\xi_r)d\tau-\xi_r^{(0)}-$$
$$-y_r^{(0)}(t)\|\le\left\|\xi_r-\xi_r^{(0)}\right\|+$$
$$\|\int_{\frac{t_r-t_{r-1}}{2}}^t f(\tau,y_r(\tau)+\xi_r)d\tau-$$
$$-\int_{\frac{t_r-t_{r-1}}{2}}^t f\left(\tau,y_r^{(0)}(\tau)+\xi_r^{(0)}\right)d\tau\|\le$$
$$\left[1+L\left(t-\frac{t_r-t_{r-1}}{2}\right)\right]\left\|\xi_r-\xi_r^{(0)}\right\|+$$
$$+\int_{\frac{t_r-t_{r-1}}{2}}^t L\left\|y_r(\tau)-y_r^{(0)}(\tau)\right\|d\tau\le$$
$$\left[1+L\left(t-\frac{t_r-t_{r-1}}{2}\right)\right]\rho_\xi+L\left(t-\right.$$
$$\left.-\frac{t_r-t_{r-1}}{2}\right)\rho_y\le(1+Lh)\rho_\xi+Lh\rho_y\le\rho_x,$$
$$t\in\left[\frac{t_r-t_{r-1}}{2},\ t_r\right),\qquad r=\overline{1,k+1}. \quad (24)$$

In view of (23), (24) and inequality (*v*) of the Theorem 1, the pairs $(t,y_r(t)+\xi_r)$, $(t,\int_t^{\frac{t_r-t_{r-1}}{2}}f(\tau,y_r(\tau)+\xi_r)d\tau+\xi_r)$ and $(t,\int_{\frac{t_r-t_{r-1}}{2}}^t f(\tau,y_r(\tau)+\xi_r)d\tau+\xi_r)$ belong to $G_f(\rho_x)$, where $(y[t],\xi)\in S(y^{(0)}[t],\rho_y)\times S(\xi^{(0)},\rho_\xi)$ for $t\in[t_{r-1},t_r)$, $r=\overline{1,k+1}$.

A solution of problem (7)-(10) will be found by the proposed algorithm. Taking the pair $(y^{(0)}[t],\xi^{(0)})$ from Condition E as the initial approximation, we find the next approximation with respect to the parameter from equations

$$Q(\xi,y^{(0)})=0,\quad \xi\in R^{(k+1)n}. \quad (25)$$

By virtue of the conditions of the Theorem, the operator $Q(\xi,y^{(0)})$ in $S(\xi^{(0)},\rho_\xi)$ satisfies all assumptions of Theorem 1 in [13, p. 41].

We choose a number $\varepsilon_0>0$ satisfying the inequalities

$$\varepsilon_0\beta\le\frac{1}{2},\quad \frac{\beta}{1-\varepsilon_0\beta}\left\|Q\left(\xi^{(0)},y^{(0)}\right)\right\|<\rho_\xi.$$

Then, using the uniform continuity of the Jacobi matrix $\frac{\partial Q(\xi,y^{(0)})}{\partial\xi}$ in $S(\xi^{(0)},\rho_\xi)$,

we find $\delta_0\in(0,\frac{1}{2}\rho_\xi)$ such that for any $\xi,\tilde\xi\in S(\xi^{(0)},\rho_\xi)$, satisfying the inequality

$$\|\xi-\tilde\xi\|<\delta_0$$

is true that

$$\left\|\frac{\partial Q(\xi,y^{(0)})}{\partial \xi}-\frac{\partial Q(\tilde{\xi},y^{(0)})}{\partial \xi}\right\|<\varepsilon_0.$$

We choose
$\alpha \geq \alpha_0 = \max\{1, \frac{\beta}{\delta_0}\|Q(\xi^{(0)},y^{(0)})\|\}$ and construct the following iterative process:

$$\xi^{(1,0)}=\xi^{(0)},$$
$$\xi^{(1,s+1)}=\xi^{(1,s)}-$$
$$-\frac{1}{\alpha}\left(\frac{\partial Q(\xi^{(1,s)},y^{(0)})}{\partial \xi}\right)^{-1} Q(\xi^{(1,s)},y^{(0)}), \quad (26)$$
$$s=0,1,2,\dots .$$

By Theorem 1 in [10, p. 41], the iterative process (26) converges to $\xi^{(1)} \in S(\xi^{(0)},\rho_\xi)$, is an isolated solution of the equation

$$Q(\xi,y^{(0)})=0$$

and

$$\|\xi^{(1)}-\xi^{(0)}\| \leq \beta\|Q(\xi^{(0)},y^{(0)})\|<\rho_\xi. \quad (27)$$

Under our assumptions, the Cauchy problem (7), (8) with $\xi_r = \xi_r^{(1)}$ on $[t_{r-1},t_r)$ has a unique solution $y_r^{(1)}(t)$ and it satisfies the inequality

$$\left\|y_r^{(1)}(t)-y_r^{(0)}(t)\right\| \leq$$
$$\leq \left\|\int_t^{\frac{t_r-t_{r-1}}{2}} L\left(\left\|\xi_r^{(1)}-\xi_r^{(0)}\right\| + \left\|y_r^{(1)}(\tau)-y_r^{(0)}(\tau)\right\|\right)d\tau,\right.$$
$$t\in\left[t_{r-1},\frac{t_r-t_{r-1}}{2}\right), \quad r=\overline{1,k+1}, \quad (28)$$

$$\left\|y_r^{(1)}(t)-y_r^{(0)}(t)\right\| \leq$$
$$\leq \left\|\int_{\frac{t_r-t_{r-1}}{2}}^{t} L\left(\left\|\xi_r^{(1)}-\xi_r^{(0)}\right\| + \left\|y_r^{(1)}(\tau)-y_r^{(0)}(\tau)\right\|\right)d\tau,\right.$$
$$t\in\left[\frac{t_r-t_{r-1}}{2}, \ t_r\right), \quad r=\overline{1,k+1}. \quad (29)$$

Using the Gronwall-Bellman inequality, we have

$$\left\|y_r^{(1)}(t)-y_r^{(0)}(t)\right\| \leq$$
$$\leq \left(e^{L\left(\frac{t_r-t_{r-1}}{2}-t\right)}-1\right)\left\|\xi_r^{(1)}-\xi_r^{(0)}\right\|,$$
$$t\in\left[t_{r-1},\frac{t_r-t_{r-1}}{2}\right), \quad r=\overline{1,k+1}, \quad (30)$$

$$\left\|y_r^{(1)}(t)-y_r^{(0)}(t)\right\| \leq$$
$$\leq \left(e^{L\left(t-\frac{t_r-t_{r-1}}{2}\right)}-1\right)\left\|\xi_r^{(1)}-\xi_r^{(0)}\right\|,$$
$$t\in\left[\frac{t_r-t_{r-1}}{2}, \ t_r\right), \quad r=\overline{1,k+1}. \quad (31)$$

From the inequalities (23), (24) we get that
$y^{(1)}[t]=\left(y_1^{(1)}(t),y_2^{(1)}(t),\dots,y_{k+1}^{(1)}(t)\right) \in S(y^{(0)}[t],\rho_y)$.
Using the structure of the operator $Q(\xi,y)$ and the equality $Q(\xi^{(1)},y^{(0)})=0$, we have

$$\beta\|Q(\xi^{(1)},y^{(1)})\| =$$
$$=\beta\|Q(\xi^{(1)},y^{(1)})-Q(\xi^{(1)},y^{(0)})\| \leq$$
$$\leq \beta \max\{2, h\|B\|+h\|C\|\}$$
$$\max\left(\left\|\int_t^{\frac{t_r-t_{r-1}}{2}} L\left\|y_r^{(1)}(\tau)-y_r^{(0)}(\tau)\right\|d\tau,\right.\quad \int_{\frac{t_r-t_{r-1}}{2}}^{t} L\left\|y_r^{(1)}(\tau)-y_r^{(0)}(\tau)\right\|d\tau\right).$$

Substituting instead of $\left\|y_r^{(1)}(\tau)-y_r^{(0)}(\tau)\right\|$ the right-hand sides of (30), (31) and calculating integrals, we have

$$\beta\|Q(\xi^{(1)},y^{(1)})\| \leq q\|\xi^{(1)}-\xi^{(0)}\|. \quad (32)$$

We take $\rho_1=\beta\|Q(\xi^{(1)},y^{(1)})\|$.

If $\xi \in S(\xi^{(1)},\rho_1)$, then by virtue of inequalities *(ii), (iii)* of Theorem and (27), (32), we have the estimate

$$\|\xi-\xi^{(0)}\| \leq \|\xi-\xi^{(1)}\|+\|\xi^{(1)}-\xi^{(0)}\|<$$
$$<\beta\|Q(\xi^{(1)},y^{(1)})\|+\|\xi^{(1)}-\xi^{(0)}\| \leq$$
$$\leq (q+1)\|\xi^{(1)}-\xi^{(0)}\|<$$
$$<\frac{\beta}{1-q}\|Q(\xi^{(0)},y^{(0)})\|<\rho_\xi,$$

i.e. $S(\xi^{(1)},\rho_1)\subset S(\xi^{(0)},\rho_\xi)$.

The operator $Q(\xi,y^{(1)})$ in $S(\xi^{(1)},\rho_1)$ satisfies all conditions of Theorem 1 [13, p. 41]. Therefore, the iterative process

$$\xi^{(2,0)}=\xi^{(1)},$$
$$\xi^{(2,s+1)}=\xi^{(2,s)}-$$
$$-\frac{1}{\alpha}\left(\frac{\partial Q(\xi^{(2,s)},y^{(1)})}{\partial \xi}\right)^{-1} Q(\xi^{(2,s)},y^{(1)}), \quad (33)$$
$$s=0,1,2,\dots ,$$

converges to $\xi^{(2)} \in S(\xi^{(1)}, \rho_1)$, is an isolated solution of the equation

$$Q(\xi, y^{(1)}) = 0$$

and

$$\|\xi^{(2)} - \xi^{(1)}\| \le \beta \|Q(\xi^{(1)}, y^{(1)})\|. \quad (34)$$

From here and from (32) it follows that

$$\|\xi^{(2)} - \xi^{(1)}\| \le q \|\xi^{(1)} - \xi^{(0)}\|. \quad (35)$$

Assuming that the pair $(y^{(m)}[t], \xi^{(m)}) \in S(y^{(0)}[t], \rho_y) \times S(\xi^{(0)}, \rho_\xi)$ is defined and the following estimates

$$\|\xi^{(m-1)} - \xi^{(m-2)}\| \le q^{m-1} \|\xi^{(1)} - \xi^{(0)}\|, \quad (36)$$

$$\beta \|Q(\xi^{(m-1)}, y^{(m-1)})\| \le \le q \|\xi^{(m-1)} - \xi^{(m-2)}\|, \quad (37)$$

are hold.

The mth approximation with respect to the parameter $\xi^{(m)}$ can be found from the equation $Q(\xi, y^{(m-1)}) = 0$.

Using (36), (37) and the equality

$$Q(\xi^{(m-1)}, y^{(m-2)}) = 0,$$

similarly to (32) we establish the inequality

$$\beta \|Q(\xi^{(m-1)}, y^{(m-1)})\| \le \le q^{m-1} \|\xi^{(1)} - \xi^{(0)}\|. \quad (38)$$

We take $\rho_{m-1} = \beta \|Q(\xi^{(m-1)}, y^{(m-1)})\|$ and show that $S(\xi^{(m-1)}, \rho_{m-1}) \subset S(\xi^{(0)}, \rho_\xi)$.

Indeed, in view of (36)-(38) and inequality (*iii*) of Theorem, we have

$$\|\xi - \xi^{(0)}\| \le \|\xi - \xi^{(m-1)}\| + +\|\xi^{(m-1)} - \xi^{(m-2)}\| + \dots + + \|\xi^{(1)} - \xi^{(0)}\| < \rho_{m-1} + +q^{m-2}\|\xi^{(1)} - \xi^{(0)}\| + \dots + \|\xi^{(1)} - \xi^{(0)}\| \le \le (q^{m-1} + \dots + q + 1)\|\xi^{(1)} - \xi^{(0)}\| < <\frac{\beta}{1-q} \|Q(\xi^{(0)}, y^{(0)})\| < \rho_\xi.$$

Since $Q(\xi, y^{(m-1)})$ in $S(\xi^{(m-1)}, \rho_{m-1})$ satisfies all conditions of Theorem 1 [13, p. 41], then there exists $\xi^{(m)} \in S(\xi^{(m-1)}, \rho_{m-1})$ is a solution of the equation

$$Q(\xi, y^{(m-1)}) = 0$$

and the estimate

$$\|\xi^{(m)} - \xi^{(m-1)}\| \le \beta \|Q(\xi^{(m-1)}, y^{(m-1)})\|. \quad (39)$$

Solving the Cauchy problem (7), (8) for $\xi_r = \xi_r^{(m)}$, we find the functions $y_r^{(m)}(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, k+1}$.

If $\rho_m = \beta \|Q(\xi^{(m)}, y^{(m)})\| = 0$, then

$$Q(\xi^{(m)}, y^{(m)}) = 0.$$

Hence, taking into account that $y_r^{(m)}(t)$ is a solution to the Cauchy problem (7), (8) with $\xi_r = \xi_r^{(m)}$ on $[t_{r-1}, t_r)$, $r = \overline{1, k+1}$, we obtain an equalities

$$B y_1^{(m)}(0) + B\xi_1^{(m)} + +C \lim_{t \to T-0} y_{k+1}^{(m)}(t) + C\xi_{k+1}^{(m)} = d,$$

$$y_{i+1}^{(m)}(t_i + 0) + \xi_{i+1}^{(m)} - - \lim_{t \to t_i - 0} y_i^{(m)}(t) - \xi_i^{(m)} = s_i, \quad i = \overline{1, k},$$

i.e. the pair $(y^{(m)}[t], \xi^{(m)})$ is a solution to problem (7)-(10).

Using (38), (39) and the Gronwall-Bellman inequality, we set the estimates

$$\|\xi^{(m)} - \xi^{(m-1)}\| \le q \|\xi^{(m-1)} - \xi^{(m-2)}\|, \quad (40)$$

$$\left\|y_r^{(m)}(t) - y_r^{(m-1)}(t)\right\| \le \le \left(e^{L\left(\frac{t_r - t_{r-1}}{2} - t\right)} - 1\right) \left\|\xi_r^{(m)} - \xi_r^{(m-1)}\right\|, \quad t \in \left[t_{r-1}, \frac{t_r - t_{r-1}}{2}\right), \quad r = \overline{1, k+1}, \quad (41)$$

$$\left\|y_r^{(m)}(t) - y_r^{(m-1)}(t)\right\| \le \le \left(e^{L\left(t - \frac{t_r - t_{r-1}}{2}\right)} - 1\right) \left\|\xi_r^{(m)} - \xi_r^{(m-1)}\right\|, \quad t \in \left[\frac{t_r - t_{r-1}}{2}, \ t_r\right), \quad r = \overline{1, k+1}. \quad (42)$$

From inequalities (40)-(42) and $q < 1$ it follows that the sequence of pairs $(y^{(m)}[t], \xi^{(m)})$ as $m \to \infty$ converges to $(y^*[t], \xi^*)$ is a solution to problem (7)-(10). Moreover, by virtue of inequalities *iii)* and *iv)* Theorem 1, the pairs $(y^{(m)}[t], \xi^{(m)})$, $m \in N$, and $(y^*[t], \xi^*)$ belong to $S(y^{(0)}[t], \rho_y) \times S(\xi^{(0)}, \rho_\xi)$.

Passing to the limit as $l \to \infty$ in the following inequalities

$$\|\xi^{(m+l)} - \xi^{(m)}\| \le \frac{q^m}{1-q} \beta \|Q(\xi^{(0)}, y^{(0)})\|,$$

$$\left\|y_r^{(m+l)}(t) - y_r^{(m)}(t)\right\| \le \le \left(e^{L\left(\frac{t_r - t_{r-1}}{2} - t\right)} - 1\right) \left\|\xi_r^{(m+l)} - \xi_r^{(m)}\right\|,$$

$$t \in \left[t_{r-1}, \frac{t_r - t_{r-1}}{2}\right),$$

$$\left\| y_r^{(m+l)}(t) - y_r^{(m)}(t) \right\| \leq$$

$$\leq \left(e^{L\left(t - \frac{t_r - t_{r-1}}{2}\right)} - 1 \right) \left\| \xi_r^{(m+l)} - \xi_r^{(m)} \right\|,$$

$$t \in \left[\frac{t_r - t_{r-1}}{2}, \; t_r\right), \qquad r = \overline{1, k+1}\,,$$

we obtain the estimates (20), (21).

The isolation of the solution is proved similarly proof in Theorem 1 [13, p. 42].

The Theorem 1 is proved.

Since problem (1)-(3) and problem (7)-(10) are equivalent, the following assertion holds.

Theorem 2. *Assume that the conditions of Theorem 1 are fulfilled.*

Then problem (1)-(3) has an isolated solution in $S\big(x^{(0)}(t), \rho_x\big)$.

5 Conclusion

In this paper, we offered a modification of the parameterization method by Dzhumabaev for solving problem with impulse actions for system of nonlinear ODEs in the interval. This technique can be applied to various kinds of differential equations with discontinuities [18-20].

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