IRSTI 27.33.19, 27.29.25, 27.41.19 https://doi.org/10.26577/ijmph.2023.v14.i1.05

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(Received 02 April 2022; accepted 03 May 2023)

A computational method for solving a boundary value problem for impulsive integro-differential equation

Abstract. In this paper, we are interested in finding a numerical solution of a linear BVP for a Fredholm IDE with a degenerate kernel subjected to impulsive actions. By Dzumabaev's parametrization method the original problem is reduced to a multipoint BVP for the system of Fredholm IDEs with additional parameters. For fixed parameters, the special Cauchy problem for the system of FIDEs on subintervals is obtained and by using a solution to this problem, a system of algebraic equations in parameters is constructed. An algorithm for solving the BVP and its computational implementation is developed. In the algorithm, Cauchy problems for ODEs and the calculation of definite integrals are the main auxiliary problems. By using various numerical methods for solving these auxiliary problems, the proposed algorithm can be implemented in different ways. The program codes were written to solve the problem and all calculations are performed on the Matlab 2018 software platform.
Key words: impulsive integro-differential equation, linear boundar value problem, special Cauhy problem, Dzumabaev's parametrization method, computational method.

1 Introduction and preliminaries

IDEs with impulsive effects have applications to mathematical modeling problems in mechanics, engineering and mathematical biology. In [1-5], BVPs for IDEs with impulse effects were studied by different methods. Obtained the existence of extremal solutions of the nonlinear anti-periodic BVP for IDE with impulse at fixed points, minimal and maximal solutions of BVPs impulsive IDEs of mixed type by upper-lower solution and monotone iterative techniques. Dzhumabaev's parametrization method [6] for investigating and solving a linear BVP for Fredholm IDE with impulsive effects at the fixed times was employed, conditions for unique solvability were obtained [7].

The conditions for the well-posedness of linear BVPs for Fredholm IDEs were established and an algorithm for finding their solutions was proposed [8].

The solution of IDEs with impulsive effects by the upper-lower solution method together with the monotone iterative method was considered [9, 10]. New comparison results were presented and general concepts of upper and lower solutions were included, which were an important tool of the monotone iterative method for determining extreme approximate solutions. It is proved that there is a solution to the BVP for IDEs with an impulsive effect between the upper-lower solutions.

In this paper, we study the linear BVP for the Fredholm IDE with an impulsive effect at a fixed time

$$\frac{dx}{dt} = A(t)x + \int_0^T \varphi(t)\psi(s)x(s)ds + f(t),$$
$$t \in (0,T)\backslash\theta,\ x \in R^n, \tag{1}$$

$$\Delta x(\theta) = d_1, d_1 \in R^n, \tag{2}$$

$$Bx(0) + Cx(T) = d,\ d \in R^n, \tag{3}$$

where $\Delta x(\theta) = x(\theta+0) - x(\theta-0)$, the $n \times n$ matrices $A(t), \varphi(t), \psi(s)$ are continuous on $[0,T]$, $f(t)$ is piecewise continuous on $[0,T]$ with the possible exception of the point $t = \theta$, the $n \times n$ matrices B, C and the vectors d_1, d are constant.

We will denote by $PC([0,T], R^n, \theta)$ the space of functions $x : [0,T] \to R^n$ that are continuous on

$[0,\theta)$, $[\theta,T)$, and have finite limit $\lim_{t\to\theta-0} x(t)$, and $x(T)=\lim_{t\to T-0} x(t)$. This space is equipped with the norm $\|x\|_1=\sup_{t\in[0,T]}\|x(t)\|$.

By a solution to problem (1)-(3) we mean a piecewise continuously differentiable on $(0,T)$ function $x(t)\in PC([0,T],R^n,\theta)$ satisfying equation (1), impulsive input condition (2), and boundary condition (3).

We divide the interval into two parts $[0,T)=\bigcup_{r=1}^{2}[t_{r-1},t_r)$, $t_0=0, t_1=\theta, t_2=T$, and denote it by $\Delta_2(\theta)$.

Let $x_r(t)$ be a restriction of the function $x(t)$ to the rth interval $[t_{r-1},t_r)$, i.e. $x_r(t)=x(t)$, for $t\in[t_{r-1},t_r)$, $r=1,2$.

We introduce the additional parameters $\lambda_r = x_r(t_{r-1})$, replace the function $u_r(t)=x_r(t_r)-\lambda_r$ and get the BVP with impulsive effect

$$\frac{du_r}{dt}=\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\varphi(t)\psi(s)\left(u_j(s)+\lambda_j\right)ds+$$

$$+A(t)(u_r+\lambda_r)+f(t),\ t\in[t_{r-1},t_r), \quad (4)$$

$$u_r(t_{r-1})=0,\ r=1,2, \quad (5)$$

$$\lambda_2-\left(\lambda_1+\lim_{t\to t_1-0}u_1(t)\right)=d_1, \quad (6)$$

$$B\lambda_1+C\lambda_2+C\lim_{t\to T-0}u_2(t)=d. \quad (7)$$

By using the fundamental matrix $X_r(t)$ of the differential equation $\frac{dx}{dt}=A(t)x,\ t\in[t_{r-1},t_r]$ we get a solution to the special Cauchy problem (4), (5):

$$u_r(t)=X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)$$

$$\times\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\varphi(\tau)\psi(s)\left(u_j(s)+\lambda_j\right)dsd\tau+$$

$$+X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)A(\tau)d\tau\lambda_r+$$

$$+X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)f(\tau)d\tau, t\in[t_{r-1},t_r). \quad (8)$$

We introduce the notation:

$$\mu=\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\psi(s)\,u_j(s)ds.$$

Further, by $a_r(P,t)$ we denote the unique solutions to the auxiliary Cauchy problems

$$\frac{dz}{dt}=A(t)z+P(t),\ t\in[t_{r-1},t_r),$$

$$z(t_{r-1})=z_0,\ r=1,2.$$

After making several transformations, we obtain the equation in $\mu\in R^n$:

$$\mu=G\big(\Delta_2(\theta)\big)\mu+\sum_{r=1}^{2}V_r\big(\Delta_2(\theta)\big)\lambda_r+$$
$$+g\big(f,\Delta_2(\theta)\big), \quad (9)$$

where

$$G\big(\Delta_2(\theta)\big)=\sum_{r=1}^{2}\int_{t_{r-1}}^{t_r}\psi(\tau)\,a_r(\varphi,\tau)d\tau,$$

$$V_r\big(\Delta_2(\theta)\big)=\int_{t_{r-1}}^{t_r}\psi(\tau)\,a_r(A,\tau)d\tau+$$

$$+\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\psi(\tau)\,a_j(\varphi,\tau)d\tau\int_{t_{r-1}}^{t_r}\psi(s)ds,$$

$$g\big(f,\Delta_2(\theta)\big)=\sum_{r=1}^{2}\int_{t_{r-1}}^{t_r}\psi(\tau)\,a_r(f,\tau)d\tau.$$

Assuming that the matrix $I-G\big(\Delta_2(\theta)\big)$ is invertible, we determine μ from equation (9) as follows:

$$\mu=\sum_{j=1}^{2}M\big(\Delta_2(\theta)\big)V_j\big(\Delta_2(\theta)\big)\lambda_j+$$
$$+M\big(\Delta_2(\theta)\big)g\big(f,\Delta_2(\theta)\big), \quad (10)$$

where $M\big(\Delta_2(\theta)\big)=\left[I-G\big(\Delta_2(\theta)\big)\right]^{-1}$, I is the identity matrix of dimension n.

By employing (10), we get the expression of functions $u_r(t)$ via additional parameters:

$$u_r(t) = \sum_{j=1}^{2} a_r(\varphi, t) \times$$
$$\times \left(M(\Delta_2(\theta))V_j(\Delta_2(\theta)) + \int_{t_{j-1}}^{t_j} \psi(s)ds \right) \lambda_j +$$
$$+a_r(A,t)\lambda_r + a_r(f,t) +$$
$$+a_r(\varphi,t)M(\Delta_2(\theta))g(f,\Delta_2(\theta)). \quad (11)$$

Introduce the notation

$$D_{r,j}(\Delta_2(\theta)) = a_r(\varphi, t_r) \times$$
$$\times \left(M(\Delta_2(\theta))V_j(\Delta_2(\theta)) + \int_{t_{j-1}}^{t_j} \psi(s)ds \right),$$

$$D_{r,r}(\Delta_2(\theta)) = a_r(\varphi, t_r) \times$$
$$\times \left(M(\Delta_2(\theta))V_r(\Delta_2(\theta)) + \int_{t_{r-1}}^{t_r} \psi(s)ds \right)$$
$$+a_r(A,t_r),$$

$$F_r(\Delta_2(\theta)) = a_r(A,t_r)\, M(\Delta_2(\theta)) \times$$
$$\times g(f,\Delta_2(\theta)) + a_r(f,t_r)\,, \quad r = 1,2.$$

Then from (11) we have

$$\lim_{t\to t_r-0} u_r(t) =$$
$$= \sum_{j=1}^{2} D_{r,j}(\Delta_2(\theta))\lambda_j + F_r(\Delta_2(\theta)). \quad (12)$$

Taking into account expressions (12), (6), (7), we obtain the following equation in parameters λ_r, $r = 1, 2$:

$$-\left(I + D_{1,1}(\Delta_2(\theta))\right)\lambda_1 + \left(I - D_{1,2}(\Delta_2(\theta))\right)\lambda_2 =$$
$$= d_1 + F_1(\Delta_2(\theta)), \quad (13)$$

$$\left(B + CD_{2,1}(\Delta_2(\theta))\right)\lambda_1 +$$
$$+C\left(I + D_{2,2}(\Delta_2(\theta))\right)\lambda_2 =$$
$$= d - CF_2(\Delta_2(\theta)). \quad (14)$$

System of equations (13), (14) we can rewrite as follow:

$$Q_*(\Delta_2(\theta))\lambda = -F_*(\Delta_2(\theta)), \lambda \in R^{2n}, \quad (15)$$

where $F_*(\Delta_2(\theta)) = \begin{cases} -d_1 - F_1(\Delta_2(\theta)) \\ -d + CF_2(\Delta_2(\theta)). \end{cases}$

Solving equation (15), we find λ_r, $r = 1, 2$ and substitute it into (11) to calculate $u_r(t)$. Finally, we find the desired function $x(t)$ by substituting $u_r(t) = x_r(t) - \lambda_r$ on each rth interval.

2 An algorithm for solving problem (1)-(3) and its computational implementation

Assuming that the matrix $[I - G(\Delta_2(\theta))]$ is invertible, for solving problem (1)-(3), we offer the following

Algorithm.

I. By solving the Cauchy problems

$$\frac{dz}{dt} = A(t)z + A(t), z(t_{r-1}) = 0,$$
$$t \in [t_{r-1}, t_r], r = 1,2,$$

$$\frac{dz}{dt} = A(t)z + \varphi(t), z(t_{r-1}) = 0,$$
$$t \in [t_{r-1}, t_r], r = 1,2,$$

$$\frac{dz}{dt} = A(t)z + f(t), z(t_{r-1}) = 0,$$
$$t \in [t_{r-1}, t_r], r = 1,2,$$

the matrix functions $a_r(A,t)$, $a_r(\varphi,t)$, $a_r(f,t)$ are found.

II. Evaluate the integrals

$$\hat{\psi}(r,A) = \int_{t_{r-1}}^{t_r} \psi(\tau)a_r(A,\tau)d\tau,$$
$$\hat{\psi}(r,\varphi) = \int_{t_{r-1}}^{t_r} \psi(\tau)a_r(\varphi,\tau)d\tau,$$

$$\hat{\psi}(r,f) = \int_{t_{r-1}}^{t_r} \psi(\tau)a_r(f,\tau)d\tau,$$

$$\hat{\psi}(r) = \int_{t_{r-1}}^{t_r} \psi(\tau)d\tau.$$

III. Define the matrices

$$G(\Delta_2(\theta)) = \sum_{r=1}^{2} \hat{\psi}(r,\varphi),$$

$$M(\Delta_2(\theta)) = [I - G(\Delta_2(\theta))]^{-1},$$

$$V_r(\Delta_2(\theta)) = \hat{\psi}(r,A) + \sum_{j=1}^{2} \hat{\psi}(j,\varphi)\, \hat{\psi}(r),$$

$$g(f,\Delta_2(\theta)) = \sum_{r=1}^{2} \hat{\psi}(r,f).$$

IV. Define the matrices

$$D_{r,j}(\Delta_2(\theta)) = a_r(\varphi,t_r) \times \\ \times \left(M(\Delta_2(\theta))V_j(\Delta_2(\theta)) + \hat{\psi}(j)\right),$$

$$D_{r,r}(\Delta_2(\theta)) = a_r(\varphi,t_r) \times \\ \times \left(M(\Delta_2(\theta))V_r(\Delta_2(\theta)) + \hat{\psi}(r)\right) \\ +a_r(A,t_r),$$

$$F_r(\Delta_2(\theta)) = a_r(A,t_r)\, M(\Delta_2(\theta)) \times \\ \times g(f,\Delta_2(\theta)) + a_r(f,t_r), \quad r = 1,2.$$

V. Form the system

$$\begin{bmatrix} -\left(I + D_{1,1}(\Delta_2(\theta))\right) & I - D_{1,2}(\Delta_2(\theta)) \\ B + CD_{2,1}(\Delta_2(\theta)) & C\left(I + D_{2,2}(\Delta_2(\theta))\right) \end{bmatrix} \lambda = \\ = \begin{bmatrix} d_1 + F_1(\Delta_2(\theta)) \\ d - CF_2(\Delta_2(\theta)) \end{bmatrix}.$$

Solving the system, find $\lambda^* = (\lambda_1^*, \lambda_2^*) \in R^{2n}$.

VI. Find the vector

$$\mu^* = \sum_{j=1}^{2} M(\Delta_2(\theta))V_j(\Delta_2(\theta))\lambda_j^* + \\ +M(\Delta_2(\theta))g(f,\, \Delta_2(\theta)),$$

and construct

$$\mathcal{F}^*(t) = \varphi(t)\left(\mu^* + \sum_{r=1}^{2} \hat{\psi}(r)\lambda_r^*\right) + f(t).$$

Solving the Cauchy problems

$$\frac{dz}{dt} = A(t)z + \mathcal{F}^*(t), z(t_{r-1}) = \lambda_r^*, \\ t \in [t_{r-1}, t_r], r = 1,2,$$

find the desired solution.

Since the coefficient matrix $A(t)$ is variable, it is impossible to construct the fundamental matrix $X_r(t)$. Consequently, we can offer a computational implementation of the algorithm based on numerical methods for solving Cauchy problems and quadrature formulas for calculating definite integrals. By using various methods for solving these problems, the proposed algorithm can be implemented in different ways.

3 Example

Let us consider the BVP (1) - (3), where

$$T = 1,\ \theta = \tfrac{1}{3},\ A(t) = \begin{pmatrix} t & t^2-1 \\ 1 & 2t \end{pmatrix},$$

$$B = C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ d = \begin{pmatrix} \frac{1}{20} \\ -\frac{5}{6} \end{pmatrix}, d_1 = \begin{pmatrix} -\frac{7}{30} \\ \frac{17}{18} \end{pmatrix},$$

$$\varphi(t) = \begin{pmatrix} 1 & t \\ t-1 & 2 \end{pmatrix}, \psi(s) = \begin{pmatrix} s & 0 \\ 1 & 1 \end{pmatrix}, n = 2.$$

$$f(t) = \begin{cases} \begin{pmatrix} \frac{49t^2}{20} - t^3 - t^4 + \frac{511t}{216} - \frac{4459}{3240} \\ \frac{14657}{3240}t - t^2 - 2t^3 + \frac{2293}{3240} \end{pmatrix}, t \in [0;\theta), \\ \begin{pmatrix} \frac{t^3}{6} - t^4 + \frac{5t^2}{3} + \frac{1349t}{1080} - \frac{1921}{3240} \\ \frac{4t^2}{3} - 2t^3 + \frac{7799t}{3240} - \frac{265}{648} \end{pmatrix}, t \in [\theta;T). \end{cases}$$

We divide the interval into two parts: $[0,1) = \left[0,\frac{1}{3}\right) \cup \left[\frac{1}{3},1\right)$ and show the result of the computational implementations of the algorithm by dividing the subintervals with stepsize $h_1 = h_2 = \frac{1}{60}$. The accuracy of the solution to the BVP depends on the accuracy of methods applied to solve the Cauchy problems and on the calculation of definite integrals. The Cauchy problems are solved by the fourth-order Runge-Kutta method and the Bulirsch-Stoer method [11, 12], and definite integrals are calculated by the Sympson's method (Table 1).

The exact solution to the given problem is

$$x^*(t) = \begin{cases} \begin{pmatrix} t^2 - \frac{9}{20}t + \frac{1}{20} \\ t^2 - 1 \end{pmatrix}, t \in [0;\theta), \\ \begin{pmatrix} t^2 - t \\ t^2 - \frac{7}{6}t + \frac{1}{3} \end{pmatrix}, t \in [\theta;T), \end{cases}$$

and the following evaluation is performed

$$\max_{j=\overline{1,62}} \|x^*(\hat{t}_j) - \tilde{x}(\hat{t}_j)\| \leq \leq 2.447310000000000E-09,$$

$$\max_{j=\overline{1,62}} \|x^*(\hat{t}_j) - \hat{x}(\hat{t}_j)\| \leq \leq 6.632227100844550E-09.$$

Here $(\tilde{x}_1(t), \tilde{x}_2(t))$ and $(\hat{x}_1(t), \hat{x}_2(t))$ are numerical solutions to the BVP, where the Cauchy problems were solved by the fourth-order Runge-Kutta method and the Bulirsch-Stoer method, respectively.

Results obtained from the Runge-Kutta method give an improved answer compared to the Bulirsh-Stoer method since the right-hand side of system are not smooth. The smoothness of the right-hand side is necessary for the Bulirsh-Stoer method to work. In the case of a smooth system, the Bulirsh-Stoer method allows achieving significantly greater accuracy than the fourth-order Runge-Kutta method.

However, Bulirsch-Stoer method showed better performance in terms of computation time 0.113143 seconds versus the Runge-Kutta method 0.130946 seconds, which is 0.017803 seconds faster.

Table 1 – The difference between the exact and numerical solutions of the problem (1) - (3)

$\hat{t}$	$\lvert x_1^*(\hat{t}) - \tilde{x}_1(\hat{t})\rvert$	$\lvert x_2^*(\hat{t}) - \tilde{x}_2(\hat{t})\rvert$	$\lvert x_1^*(\hat{t}) - \hat{x}_1(\hat{t})\rvert$	$\lvert x_2^*(\hat{t}) - \hat{x}_2(\hat{t})\rvert$
0	2.069257001857670E-09	8.098699488812140E-10	5.033313002500430E-09	6.632220106439490E-09
$\frac{1}{60}$	2.112360779349400E-09	7.727358752873670E-10	5.075584785796390E-09	6.615344827487490E-09
$\frac{1}{30}$	2.153450112674090E-09	7.329690188129230E-10	5.108957111532590E-09	6.588437018173470E-09
$\frac{1}{20}$	2.192396004324640E-09	6.906359928393610E-10	5.133272005380940E-09	6.551323927794780E-09
$\frac{1}{15}$	2.229072450926630E-09	6.457965273654050E-10	5.148361448964470E-09	6.503804494961680E-09
$\frac{1}{12}$	2.263353449105670E-09	5.985154594156940E-10	5.154054447120690E-09	6.445658451426080E-09
$\frac{1}{10}$	2.295112998956790E-09	5.488599574832160E-10	5.150171998938300E-09	6.376642991412500E-09
$\frac{7}{60}$	2.324227119793190E-09	4.969020750422710E-10	5.136529120444710E-09	6.296493104684940E-09
$\frac{2}{15}$	2.350571785825210E-09	4.427157529462990E-10	5.112931786288580E-09	6.204920799390830E-09
$\frac{3}{20}$	2.374025011298020E-09	3.863799280523490E-10	5.079180011476060E-09	6.101616989440120E-09
$\frac{1}{6}$	2.394464781833860E-09	3.279777560649680E-10	5.035061781573370E-09	5.986243833966400E-09
$\frac{11}{60}$	2.411772118170520E-09	2.675981658484260E-10	4.980358118316170E-09	5.858441176620490E-09

Table continuation

$\hat{t}$	$\lvert x_1^*(\hat{t})-\tilde{x}_1(\hat{t})\rvert$	$\lvert x_2^*(\hat{t})-\tilde{x}_2(\hat{t})\rvert$	$\lvert x_1^*(\hat{t})-\hat{x}_1(\hat{t})\rvert$	$\lvert x_2^*(\hat{t})-\hat{x}_2(\hat{t})\rvert$
$\frac{1}{5}$	2.425826000000000E-09	2,053310854677190E-10	4.914840000000000E-09	5.717819995254560E-09
$\frac{13}{60}$	2.436511447787520E-09	1.412754357943410E-10	4.838265447787080E-09	5.563963401122860E-09
$\frac{7}{30}$	2.443710454694620E-09	7.553235814583560E-11	4.750385454790900E-09	5.396425417636410E-09
$\frac{1}{4}$	2.447310000000000E-09	8.208989044078410E-12	4.650933000000000E-09	5.214730980362960E-09
$\frac{4}{15}$	2.447197118343390E-09	6.058087365090610E-11	4.539632118350300E-09	5.018371163068020E-09
$\frac{17}{60}$	2.443260767748950E-09	1.307182140308780E-10	4.416190768109780E-09	4.806802733625660E-09
$\frac{3}{10}$	2.435394018396320E-09	2.020770217825430E-10	4.280302018204450E-09	4.579448931174570E-09
$\frac{19}{60}$	2.423488786387760E-09	2.745261795666920E-10	4.131644785847210E-09	4.335693803980690E-09
$\frac{1}{3}$	2.407443112917300E-09	3.479257992822230E-10	3.969877112805500E-09	4.074881210236470E-09
$\frac{1}{3}$	2.407443799867790E-09	3.479264168437800E-10	3.969876788412210E-09	4.074881577997850E-09
$\frac{7}{20}$	2.387155029204280E-09	4.221299559903360E-10	4.073173021090780E-09	4.142549039909400E-09
$\frac{11}{30}$	2.362525786114840E-09	4.969820180389380E-10	4.164952771557040E-09	4.200523977260230E-09
$\frac{23}{60}$	2.333461118775300E-09	5.723204410390540E-10	4.244998130786830E-09	4.248563563458150E-09
$\frac{2}{5}$	2.299870016653220E-09	6.479713293705560E-10	4.313078005857560E-09	4.286400671904560E-09
$\frac{5}{12}$	2.261663439862010E-09	7.237556838568530E-10	4.368951450617330E-09	4.313741315542520E-09
$\frac{13}{30}$	2.218757455541010E-09	7.994805113070710E-10	4.412359450034490E-09	4.330268489269320E-09
$\frac{9}{20}$	2.171073015810880E-09	8.749426391729820E-10	4.443031997869440E-09	4.335636362082230E-09
$\frac{7}{15}$	2.118535125106290E-09	9.499303461890650E-10	4.460682129225900E-09	4.329470654380880E-09
$\frac{29}{60}$	2.061073756109980E-09	1.024217410969950E-09	4.465005754017430E-09	4.311367588893250E-09
$\frac{1}{2}$	1.998624987731290E-09	1.097565000000000E-09	4.455683988435770E-09	4.280890000000000E-09
$\frac{31}{60}$	1.931129756105320E-09	1.169721057861510E-09	4.432374772767030E-09	4.237567942377840E-09
$\frac{8}{15}$	1.858537129750810E-09	1.240419472865740E-09	4.394718117195500E-09	4.180896526553850E-09
$\frac{33}{60}$	1.780803005058120E-09	1.309377357407160E-09	4.342332993179100E-09	4.110330642688450E-09
$\frac{17}{30}$	1.697888468710220E-09	1.376296653900210E-09	4.274813475424470E-09	4.025288345699260E-09
$\frac{7}{12}$	1.609765487575740E-09	1.440860419692210E-09	4.191729463531860E-09	3.925143579955330E-09
$\frac{3}{5}$	1.516415076840970E-09	1.502735598284120E-09	4.092625072171780E-09	3.809224401185580E-09
$\frac{37}{60}$	1.417828188188250E-09	1.561566246102710E-09	3.977014190192920E-09	3.676812754434270E-09

table continuation

$\hat{t}$	$\lvert x_1^*(\hat{t}) - \tilde{x}_1(\hat{t})\rvert$	$\lvert x_2^*(\hat{t}) - \tilde{x}_2(\hat{t})\rvert$	$\lvert x_1^*(\hat{t}) - \hat{x}_1(\hat{t})\rvert$	$\lvert x_2^*(\hat{t}) - \hat{x}_2(\hat{t})\rvert$
$\frac{19}{30}$	1.314003850172440E-09	1.616975472788770E-09	3.844378843043120E-09	3.527138526561450E-09
$\frac{13}{20}$	1.204956134870370E-09	1.668564946800950E-09	3.694171135659730E-09	3.359378053079810E-09
$\frac{2}{3}$	1.090709911277800E-09	1.715912000000000E-09	3.525804898041330E-09	3.172649000000000E-09
$\frac{41}{60}$	9.713042592451870E-10	1.758565466424690E-09	3.338655268914080E-09	2.966006533693800E-09
$\frac{21}{30}$	8.467931911226860E-10	1.796048290127190E-09	3.132058196131380E-09	2.738442709572520E-09
$\frac{43}{60}$	7.172486771178650E-10	1.827854582969750E-09	2.905301665911960E-09	2.488876417461320E-09
$\frac{11}{15}$	5.827567317595590E-10	1.853443178448380E-09	2.657626729796190E-09	2.216152822018260E-09
$\frac{3}{4}$	4.434263234376350E-10	1.872242461758990E-09	2.388224340510450E-09	1.919035538117120E-09
$\frac{23}{30}$	2.993854608668300E-10	1.883640105004860E-09	2.096225470982200E-09	1.596202895554070E-09
$\frac{47}{60}$	1.507872160466660E-10	1.886989106636430E-09	1.780701225362340E-09	1.246239896002250E-09
$\frac{4}{5}$	2.192468429029760E-12	1.881595684816160E-09	1.440658536910890E-09	8.676343116187370E-10
$\frac{49}{60}$	1.593496445906340E-10	1.866724622101580E-09	1.075028344121880E-09	4.587663721133950E-10
$\frac{5}{6}$	3.204492782238330E-10	1.841589034046190E-09	6.826697263040420E-10	1.790297077253200E-11
$\frac{51}{60}$	4.852262758259230E-10	1.805350688388610E-09	2.623527228795550E-10	4.568116857006640E-10
$\frac{13}{15}$	6.533778518225120E-10	1.757117021328190E-09	1.872418609050190E-10	9.673630230189190E-10
$\frac{53}{60}$	8.245636812231890E-10	1.695932727585610E-09	6.675306835068890E-10	1.515879727298500E-09
$\frac{9}{10}$	9.984000848284320E-10	1.620779011002330E-09	1.180042077941760E-09	2.104641017020010E-09
$\frac{11}{12}$	1.174458946762160E-09	1.530564536222910E-09	1.726416953840020E-09	2.736089535493310E-09
$\frac{14}{15}$	1.352260178832940E-09	1.424127649141130E-09	2.308426179375990E-09	3.412843646577460E-09
$\frac{19}{20}$	1.531268875587520E-09	1.300221208477840E-09	2.927977874045420E-09	4.137711229201810E-09
$\frac{29}{30}$	1.710889929729120E-09	1.157511919647190E-09	3.587130932569680E-09	4.913702944531200E-09
$\frac{59}{60}$	1.890463563469780E-09	9.945734513738810E-10	4.288108566147250E-09	5.744052455902700E-09
1	2.069257000000000E-09	8.098750836627030E-10	5.033314000000000E-09	6.632227100844550E-09

4 Conclusion

We developed an algorithm for finding a solution to linear BVP (1)-(3) and its computational implementation. This approach can be applied to quasi-linear BVPs for impulsive Fredholm IDE.

5 Acknowledgement

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP13268824).

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