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Numerical Analysis of Thin Cylindrical Shell Vibrations with a Weak Nonlinearity

Abstract. In this paper, nonlinear vibrations of an infinite thin cylindrical shell as a limiting case of a nanotube are studied. The main relations of Sanders-Koiter's nonlinear shell theory and the Hamilton variation principle are applied to obtain a nonlinear mathematical model of the shell vibrations and allow fully accounting for the influence of nonlinear effects. Using the method of multiple scales with specification of fast and slow times, high-order asymptotic relations taking into account quadratic and cubic nonlinearities are found. Based on the solution of the asymptotic scheme at fourth order and application of the inextensibility condition for the semi-membrane shell theory, the numerical analysis of tangential and radial displacements of the cylindrical shell at leading order relative to the Fourier coefficients is conducted. The impact of the wave number, polar angle, radius, and wall thickness of the shell on the amplitude and period of the arising vibrations is investigated. Numerical illustrations of the obtained solutions are presented for several cases.
Key words: mathematical model, nonlinearity, cylindrical shell, vibration, asymptotic.

Introduction

Theoretical studies of various phenomena at nanoscale are an important step for experimental and technological progress in this direction. They allow not only deepening our knowledge of the fundamental principles underlying nano-objects, but also can be the ground for creating new and improving existing materials and structures with properties not available at the macrolevel [1].

When studying cylindrical-shaped nano-objects with large length, one can neglect the conditions at their ends since the nature of wave formation upon loss of stability and, thus, the shape of the approximating deflection functions practically do not depend on the boundary conditions. Then it is expedient to model such nano-objects in the form of an infinite thin circular cylindrical shell, the limiting case of which is a thin circular ring. The history of this question was deeply reviewed in the works of Evensen [2, 3]. In particular, Evensen theoretically and experimentally showed the importance of accounting for the ring strain caused by nonlinearity. Absence of this factor resulted in incorrect results even at the qualitative level.

Research of low-frequency vibration modes specific for thin shells, in particular carbon nano-objects, based on the nonlinear theories of cylindrical shells was conducted in a number of works, see e.g. [1, 4]. It was noted that the derivation of nonlinear equations for low-frequency vibrations of carbon nanotube and the analysis of their bifurcation states were useful for understanding the process of energy exchange in nanotubes and the transition between different vibration modes. Comparison of the natural frequencies of carbon nanotubes, obtained on the basis of the beam and shell models, is presented in [5]. A linear asymptotic theory valid in the vicinity of the lowest cut-off frequency for a circular cylindrical shell was derived by Kaplunov et al. [6]. A high-order asymptotic scheme resulting in a system of nonlinear equations, which takes into consideration both cubic and quadratic nonlinearity, and allowing determining a nonlinear correction to the lowest cut-off frequencies of a thin cylindrical shell was obtained in [7].

In many works, the authors introduce special physical hypotheses to obtain the models of nonlinear vibrations of single-walled carbon nanotubes. Amongst recent publications in this research field, one can highlight the works [8, 9], in which the resonant interaction of normal vibration modes of single-walled nanotubes was investigated. The authors utilized the semi-inverse asymptotic method and carried out the comparison of analytical and numerical models to estimate the nonlinear phenomenon of energy localization. In the work [10], free and forced nonlinear vibrations of the isotropic piezoelectric/viscoelastic Euler-Bernoulli nano-beam were studied using the consistent couple stress theory. The frequency response of the nano-beam for different damping and size-effect coefficients and various values of forcing function amplitude was analyzed.

Despite the breadth of research carried out for development of mathematical models of such structures, a large class of dynamic problems of cylindrical nano-objects taking into account nonlinear factors is still poorly investigated. One of the effective ways to solve the problems of the nano-object dynamics is the use of asymptotic theories and methods. They allow conducting a preliminary analysis of the studied problem, discard "small terms" and introduce new slowly varying variables that can be accurately calculated [11]. Asymptotic methods also provide excellent results in the field of extreme parameters.

In this paper, the results of [7] are extended by conducting the numerical analysis of circumferential (tangential) and radial displacements of an infinite cylindrical shell at leading order of asymptotic expansion, the coefficients of which are determined through the fourth-order asymptotic scheme including five angular modes.

Nonlinear Mathematical Model

Let us consider a weakly nonlinear setup for vibrations of an infinite thin circular cylindrical shell with thickness h, the transverse cross-section of which is illustrated in Fig. 1. Define dimensionless time $\tau = t/t_0$ with scaling $t_0 = \sqrt{\rho R^2\left(1-\nu^2\right)/E}$ as in [7]. Here ρ is the mass density, R is the shell mid-surface radius, ν is Poisson's ratio, and E is Young's modulus.

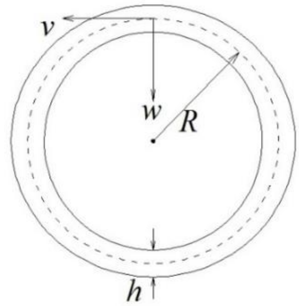

Figure 1 – Cross-section of an infinite circular cylindrical shell

It is known that the problem of shell vibrations in the general case is characterized by an infinite number of natural frequencies, with each frequency corresponding to a certain vibration mode. When studying natural linear vibrations, the displacement amplitudes of the system do not depend on the frequency and are conditioned only by the initial conditions. The presence of nonlinearity in a dynamical system, when the time and spatial coordinates are independent variables, significantly complicates the solution of the problem. In this case, the shell is considered as a system with a finite number of degrees of freedom, and its curved surface is approximated in a certain manner [12].

Circumferential and radial mid-surface displacements v and w of the shell are nondimensionalized by means of the radius R:

$$\hat{v} = \frac{v}{R}, \quad \hat{w} = \frac{w}{R}. \tag{1}$$

According to the Sanders-Koiter nonlinear shell theory [13], dimensionless mid-surface strain-displacement relations and changes in the curvature of the infinite thin-elastic circular cylindrical shell with no initial imperfections are written as

$$\begin{gathered} \varepsilon_\xi = \varepsilon_{\xi\varphi} = 0, \\ \varepsilon_\varphi = \frac{\partial v}{\partial \varphi} + w + \frac{1}{2}\left(\frac{\partial w}{\partial \varphi} - v\right)^2, \\ \kappa_\xi = \kappa_{\xi\varphi} = 0, \quad \kappa_\varphi = \frac{\partial v}{\partial \varphi} - \frac{\partial^2 w}{\partial \varphi^2}, \end{gathered} \tag{2}$$

where φ is polar angle. Hereinafter we omit the hat for the dimensionless displacements v and w.

Then, applying the Hamilton variation principle, we arrive at the following nonlinear partial differential equations, which describe the vibrations of an infinite thin circular cylindrical shell taking into account a weak nonlinearity (see [6, 7] for details):

$$\frac{\partial^2 v}{\partial \tau^2}-\left(1+\frac{\beta^2}{12}\right)\frac{\partial^2 v}{\partial \varphi^2}-\frac{\partial w}{\partial \varphi}+\frac{\beta^2}{12}\frac{\partial^3 w}{\partial \varphi^3}+\left(v-\frac{\partial w}{\partial \varphi}\right)\left(w+\frac{\partial^2 w}{\partial \varphi^2}\right)+\frac{1}{2}\left(v-\frac{\partial w}{\partial \varphi}\right)^3=0, \quad (3)$$

$$\frac{\partial^2 w}{\partial \tau^2}+w+\frac{\beta^2}{12}\frac{\partial^4 w}{\partial \varphi^4}+\frac{\partial v}{\partial \varphi}-\frac{\beta^2}{12}\frac{\partial^3 v}{\partial \varphi^3}+\frac{1}{2}\left(v^2-\left(\frac{\partial w}{\partial \varphi}\right)^2\right)+\left(w+\frac{\partial v}{\partial \varphi}\right)\left(\frac{\partial v}{\partial \varphi}-\frac{\partial^2 w}{\partial \varphi^2}\right)+$$
$$+\left(v-\frac{\partial w}{\partial \varphi}\right)\frac{\partial^2 v}{\partial \varphi^2}+\frac{1}{2}\frac{\partial}{\partial \varphi}\left(v-\frac{\partial w}{\partial \varphi}\right)^3=0, \quad (4)$$

where β is a small parameter corresponding to the relative thickness of the shell.

In accordance with the method of multiple scales [14], both fast τ_0 and slow τ_1 times are specified:

$$\tau_0=\beta\tau, \quad \tau_1=\beta^3\tau. \quad (5)$$

Expanding the displacements v and w into asymptotic series in terms of the small parameter β, the following set of equations at leading order is obtained:

$$\frac{\partial^2 v_0}{\partial \varphi^2}+\frac{\partial w_0}{\partial \varphi}=0,$$
$$\frac{\partial v_0}{\partial \varphi}+w_0=0. \quad (6)$$

The displacements v_0 and w_0 in Eqs. (6) are given by the expressions with the Fourier coefficients V_0 and W_0:

$$v_0=V_0(\tau_0,\tau_1)\sin n\varphi, \quad w_0=W_0(\tau_0,\tau_1)\cos n\varphi, \quad (7)$$

where n is the circumferential wave number.

On substituting relations (7) into Eqs. (6), we have the following inextensibility condition of the semi-membrane shell theory:

$$W_0+nV_0=0. \quad (8)$$

This theory is widely utilized in the analysis of long closed and open cylindrical shells, when the membrane theory is not applicable, for example, while analyzing the stress state of shells.

Following the asymptotic procedure, nonlinear equations of the shell vibrations taking into account quadratic and cubic nonlinearities at first, second, third, and fourth asymptotic orders are obtained (see [7] for more details). Solving the asymptotic scheme at fourth order, we arrive at the expression for the leading order term W_0:

$$W_0=2D_{1n}\cos\left(\frac{\Omega_{1n}^2-c_n D_{1n}^2}{2\Omega_{0n}}\tau_1+\Omega_{0n}\tau_0+D_{2n}\right), \quad (9)$$

where D_{1n} and D_{2n} are arbitrary constants, Ω_{0n} and Ω_{1n} are eigenfrequencies and their linear correction, respectively, determined by

$$\Omega_{0n}^2=\frac{n^2\left(1-n^2\right)^2}{12\left(1+n^2\right)}, \quad (10)$$

$$\Omega_{1n}^2=-\frac{n^6\left(1-n^2\right)^2}{36\left(1+n^2\right)^3}, \quad (11)$$

and

$$c_n = \frac{\left(1-n^2\right)^4\left(1-20n^2+78n^4-50n^6-92n^8+45n^{10}-20n^{12}-32n^{14}\right)}{24n^2\left(1+n^2\right)^3\left(1+4n^2\right)\left(3-5n^2-4n^4\right)}. \quad (12)$$

Numerical Results and Discussion

This section is concerned with the numerical analysis of the infinite cylindrical shell vibrations with nonlinearity based on the relations presented in the previous sections and further visualization of elastic displacements of the shell as a limiting case of a nanotube.

The values for the radius and wall thickness of the cylindrical shell were chosen in accordance with the work [15]: $R = 5\cdot 10^{-9}$m, $h = 0.066\cdot 10^{-9}$m. Numerical calculation and graphic visualization of displacements of the cylindrical shell were performed in the Wolfram Mathematica package.

Figures 2 and 3 show the graphs of the cylindrical shell displacements at leading order relative to the Fourier coefficients V_0 and W_0, and Fig. 4 illustrates the displacements v_0 and w_0 given by expressions (7) for circumferential wave number $n = 2$ and constants $D_{1n} = 0.5$, $D_{2n} = 1$. The value for the polar angle φ is $\frac{\pi}{6}$.

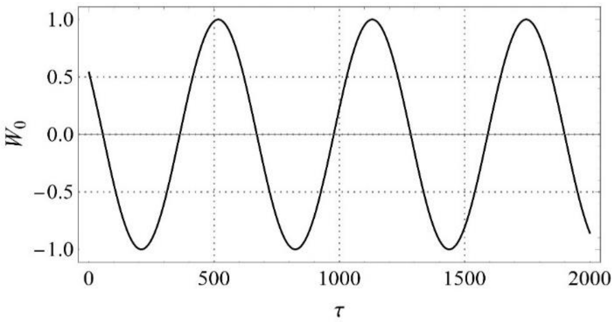

Figure 2 – Radial displacements W_0 of the circular cylindrical shell

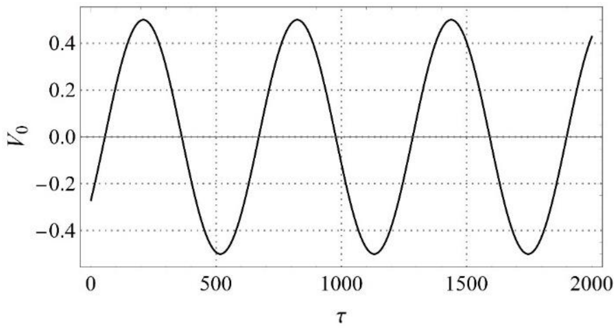

Figure 3 – Tangential displacements V_0 of the circular cylindrical shell

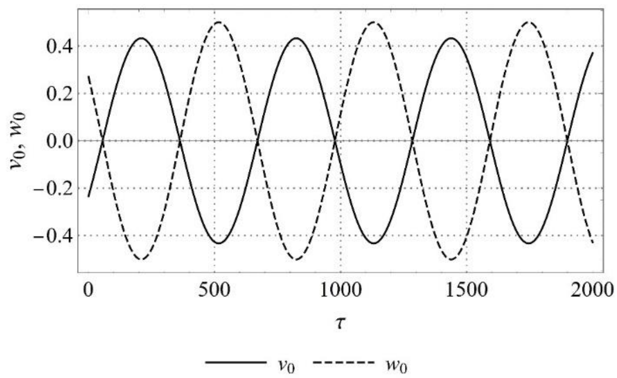

Figure 4 – Tangential and radial displacements of the cylindrical shell at leading order, $n = 2$

As can be seen from the obtained graphs, for the wave number $n = 2$, the maximum amplitude values of the shell tangential displacements v_0 slightly exceed the largest amplitudes of its radial displacements w_0. The oscillatory process of the circular cylindrical shell remains stable.

The results of numerical simulation when the value of the wave number increases are presented in

Figs. 5-10. As the constructed graphs show, an increase in the wave number from $n=2$ to $n=8$ results in a significant decrease in the period of the arising vibrations of the cylindrical shell.

The displacement field of the cylindrical shell depending on the change in the polar angle φ at different time moments is also studied (Figs. 11-13). The value for the circumferential wave number is chosen to be $n=2$.

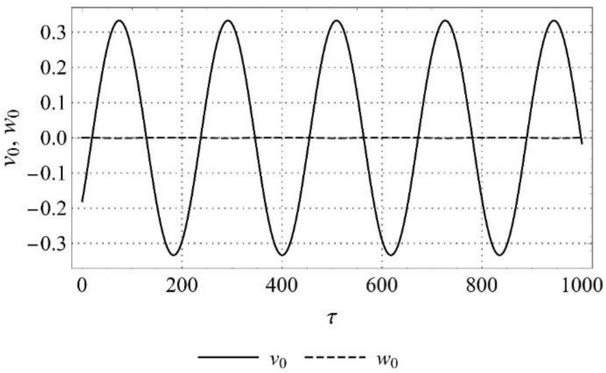

Figure 5 – Tangential and radial displacements of the shell, $n=3$

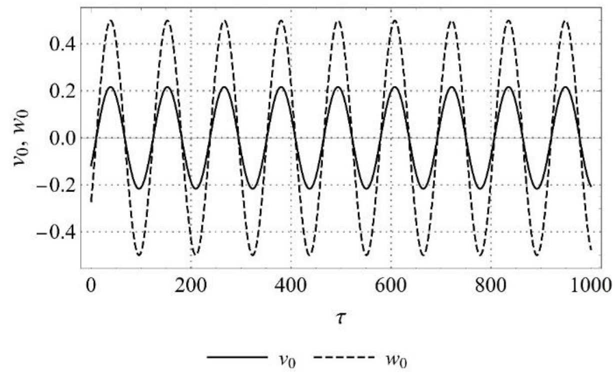

Figure 6 – Tangential and radial displacements of the shell, $n=4$

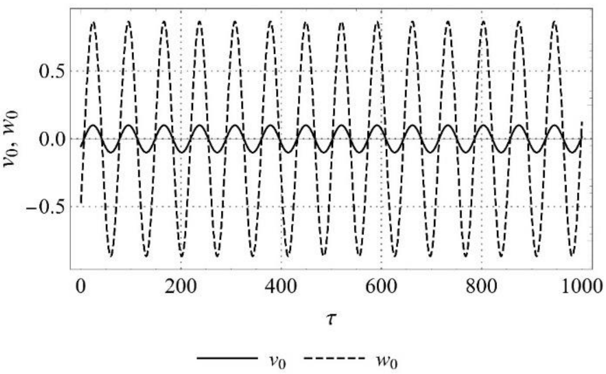

Figure 7 – Tangential and radial displacements of the shell, $n=5$

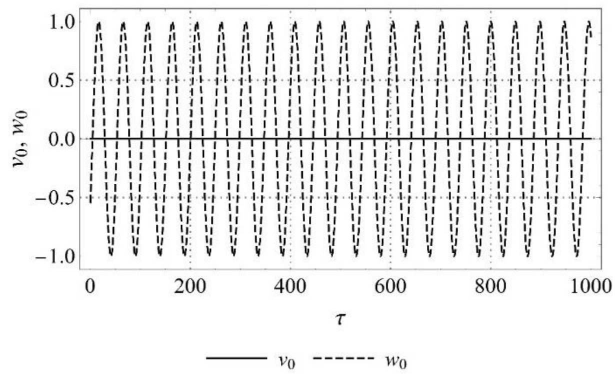

Figure 8 – Tangential and radial displacements of the shell, $n=6$

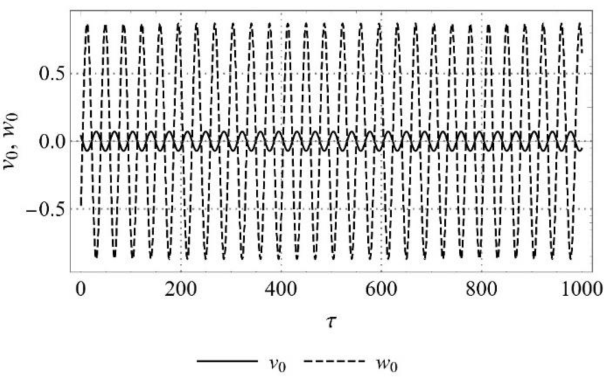

Figure 9 – Tangential and radial displacements of the shell, $n=7$

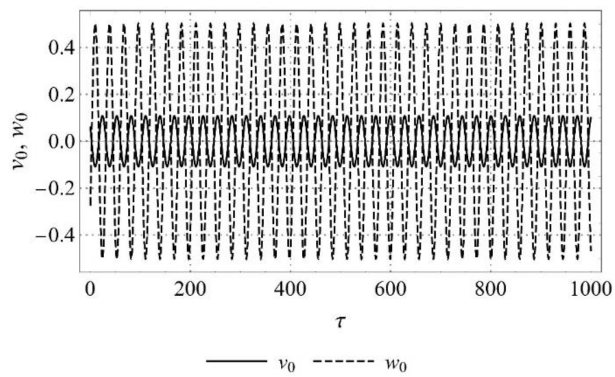

Figure 10 – Tangential and radial displacements of the shell, $n=8$

For $n = 3$, there is a sharp decrease in the amplitude of the shell radial displacements w_0 compared to its tangential displacements v_0 for considered values of the system parameters (Fig. 5), while for the values of the circumferential wave number $n = \overline{4,8}$, the amplitude of the shell radial displacements predictably exceeds the amplitude of its tangential displacements (Figs. 6-10), which directly follows from the basic hypothesis of the semi-membrane shell theory (8). It is also worth noting that the maximum amplitude values of the tangential displacements and the minimum amplitude values of the radial displacements of the cylindrical shell are observed at $n = 6$.

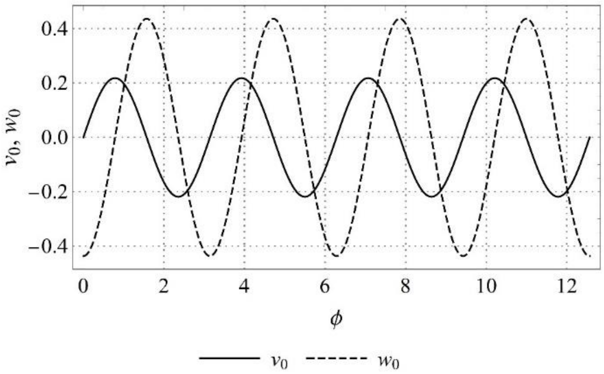

Figure 11 – Dependence of the shell tangential and radial displacements on the change in the polar angle φ at the time moment $\tau = 100$

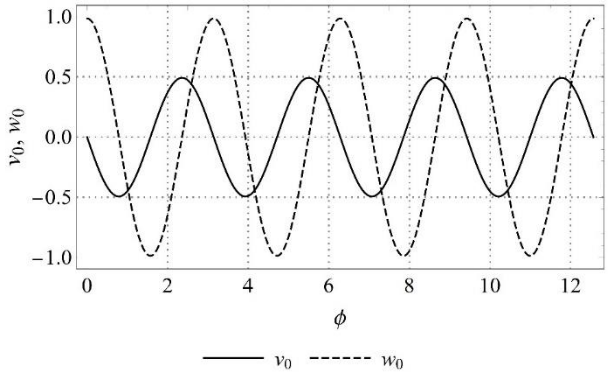

Figure 12 – Dependence of the shell tangential and radial displacements on the change in the polar angle φ at the time moment $\tau = 500$

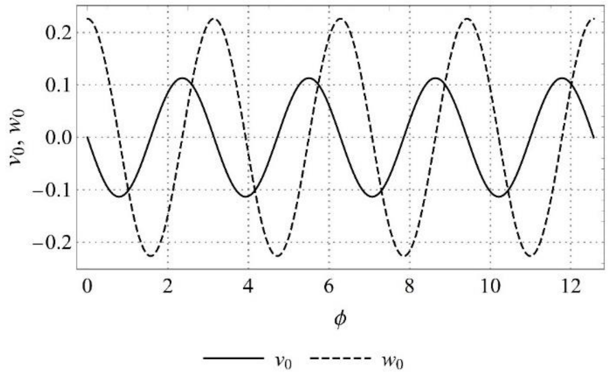

Figure 13 – Dependence of the shell tangential and radial displacements on the change in the polar angle φ at the time moment $\tau = 1000$

As follows from Figs. 11-13, the amplitude of the radial displacements w_0 is approximately two times higher than that of tangential displacements v_0 throughout the entire oscillatory process when the polar angle changes.

The influence of the mid-surface radius of the thin circular cylindrical shell as a limiting case of a nanotube on its tangential and radial displacements at leading order of the asymptotic expansion (6) is shown in Figs. 14 and 15, respectively, for the values of the wave number $n = 2$ and polar angle $\varphi = \frac{\pi}{6}$. Figures 16 and 17 demonstrate the results of numerical simulation for displacements of the infinite thin cylindrical shell when the wall thickness changes.

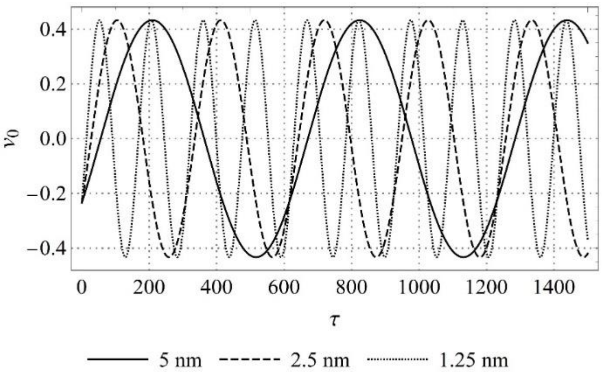

Figure 14 – The influence of the shell radius on its tangential displacements

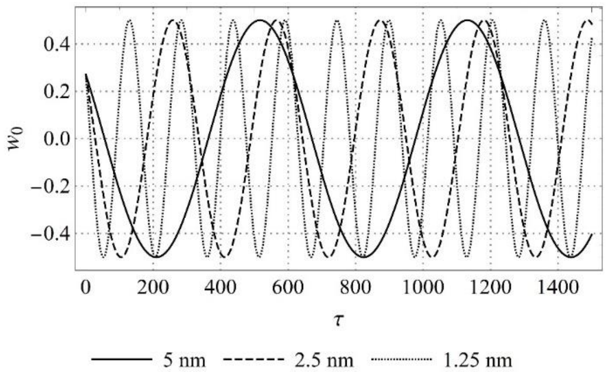

Figure 15 – The influence of the shell radius on its radial displacements

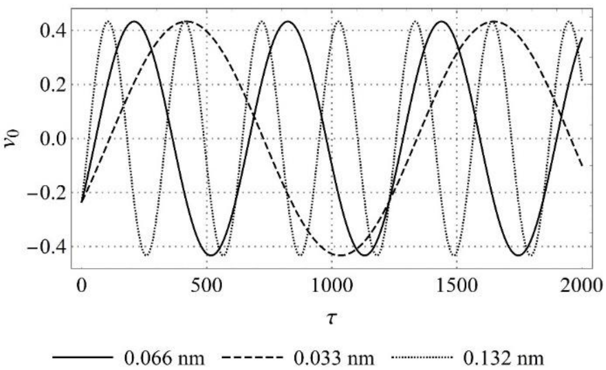

Figure 16 – The influence of the shell thickness on its tangential displacements

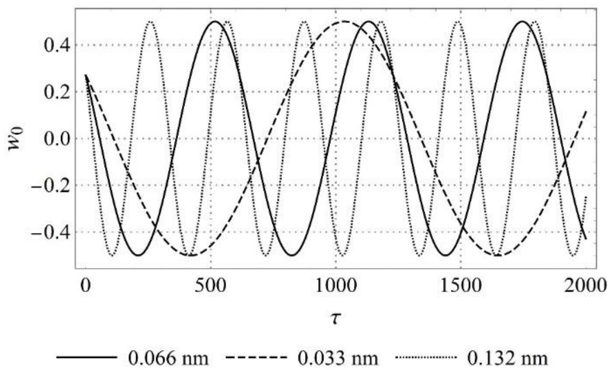

Figure 17 – The influence of the shell thickness on its radial displacements

The conducted analysis shows that the decrease of the shell radius (two and four times decrease in Figs. 14 and 15, respectively) does not affect the amplitude of the arising vibrations at nanoscale; however, a significant decrease in the period of the cylindrical shell vibrations is observed.

It follows from Figs. 16 and 17 that the twofold decrease and the corresponding increase of the shell wall thickness results only in the change of the period of the arising vibrations at leading order of asymptotic expansion; at the same time, there are no quantitative changes in their vibration amplitude.

Conclusion

The numerical analysis and visualization of tangential and radial displacements of an infinite circular cylindrical shell as a limiting case of a nanotube at leading order of asymptotic expansion, which allows giving a sufficiently accurate quantitative estimate of the oscillatory process, were carried out. A nonlinear mathematical model of vibrations of the cylindrical shell developed with the use of the fundamentals of Sanders-Koiter's nonlinear theory and the application of the Hamilton variation principle underlay the obtained asymptotic relations. The utilization of a fourth-order asymptotic scheme allowed determining the leading-order Fourier coefficients, which were further used for the numerical analysis of the thin shell vibrations.

The influence of the wave number, polar angle, radius, and wall thickness of the shell on the amplitude and period of the arising vibrations was studied. The conducted analysis showed that the change in the shell radius and the wall thickness did not affect the amplitude of the shell vibrations at nanoscale; however, a significant decrease in the period of the cylindrical shell vibrations was observed.

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