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Leptonic and radiative decays of mesons with beauty and charm in the relativistic hamiltonian approach

Abstract. On the basis of the investigation of the asymptotic behavior of the correlation functions of the corresponding field currents with the necessary quantum numbers the analytic method for the determination of the mass spectrum and decay constants of mesons consisting of *c* and *b* quarks with relativistic corrections is proposed. The dependence of the constituent mass of quarks on the current mass and on the orbital and radial quantum numbers is analytically derived. The mass and wave functions of the mesons are determined via the eigenvalues of nonrelativistic Hamiltonian in which the kinetic energy term is defined by the constituent mass of the bound state forming particles and the potential energy term is determined by the contributions of every possible type of Feynman diagrams with an exchange of gauge field. In the framework of our approach the mass splitting between the singlet and triplet states is determined, and the width of $E1$ transition rates in the $\bar{c}c$, $\bar{b}b$ and $\bar{b}c$ systems are calculated. The obtained results are satisfactorily agree with the experimental data.
Keywords: mass spectrum, correlation function, constituent mass, nonrelativistic Hamiltonian, radiative decay.

Introduction

The energy spectrum of the bound state can be determined with a good precision within the framework of nonrelativistic quantum mechanics (NRQM) when a good selection of the potential is made. However, the nonrelativistic Schrodinger equation (SE), which gives a mathematically correct description of the bound state, is no longer sufficient since for the description of modern experimental results, obtained in both atomic [1] and hadronic physics [2], it is necessary to take into account the relativistic correction. Nevertheless, the nonrelativistic SE is the reliable tool for the bound state energy research and its determination. In this case, real relativistic corrections are small, so the theoretical problem reduces to obtaining the relativistic corrections to the nonrelativistic interaction potential in the formalism of quantum field theory (QFT). This idea underlies the Breit potential [3] and the effective nonrelativistic quantum field theory of Caswell and Lepage [4]. Both these approaches use the scattering matrix as a source of required corrections. In the framework of quantum electrodynamics (QE) the authors of [5] studied the scattering matrix with appropriate Feynman diagrams by taking into account the renormalization and then taking the nonrelativistic limit, so they obtained the interaction potential with the relativistic corrections. Thus, the nonrelativistic QED or NRQED method for the determination of the energy spectrum by taking into account relativistic corrections was formulated. Subsequently, this method was improved in [6]. However, in these works, the relativistic corrections within the framework of the perturbation theory were taken into account mainly to the interaction potential, and the correction to the kinetic part of the interaction Hamiltonian was almost ignored. The relativistic correction to the kinetic part of the Hamiltonian in the usual quantum mechanical formalism is included only in

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the framework of the relativistic SE. It is known that the determination of the energy spectrum and wave functions of the bound state consisting of a few particles from the relativistic SE, from the point of view of mathematical calculations is almost impossible. Therefore, the inclusion of the relativistic corrections into the determination of the properties of the relativistic bound state as a potential and kinetic part of the interaction Hamiltonian is one of the most urgent problems of modern theoretical study. Our work is devoted to studying this problem.

In our approach [4], the mass of the bound state is determined by the asymptotic behavior of the correlation function of the corresponding currents with the necessary quantum numbers. The correlation function, which is expressed in terms of the quantum-field Green function is represented as a functional integral, which allows one to allocate the necessary asymptotic behavior, and the averaging over the external gauge field can be performed accurately. The resulting representation is similar to the Feynman functional path integral [7] in nonrelativistic quantum mechanics. In this case, the interaction potential is determined by the Feynman diagram, the resulting exchange of the gauge field, and the mass in the SE is the constituent differing from the mass of the initial state of the system, i.e. one kinetic part of the Hamiltonian is expressed in terms of the constituent mass of the constituent particles, and it differs from the initial mass state. Our results show that the difference between these masses for the light particles is essential, in particular, for the electron and for heavy particles such as an isotope of hydrogen it is not noticeable. Thus, thanks to the constituent mass of the constituent particles one can take into account relativistic corrections to the kinetic part of the interaction Hamiltonian.

The paper is organized as follows. In section 2, we describe in detail the determination of the mass and constituent mass bound state system. In section 3, the mass spectrum of mesons consisting of *c* and *b* quarks with the orbital and radial excitations is defined. The dependence of the constituent mass of the constituent particles on the mass of the initial state, as well as radial and orbital quantum numbers is determined. The obtained results are in satisfactory agreement with the available experimental data.

1. Determining the mass of the relativistic bound state

We now briefly discuss the details of our approach. Let us denote $J(x)=\Phi^{+}(x)\Phi(x)$ as the current of scalar charged particles. If we neglect the annihilation channel, then it is convenient to represent the considered correlators as the averaging over the gauge field $A_\alpha(x)$ of a product of the Green functions $G_m(x,y\,|\,A)$ of the scalar charged particles in the external gauge field:

$$\Pi(x-y)=\langle J(x)J(y)\rangle=\langle \Phi^{+}(x)\Phi(x)\Phi^{+}(y)\Phi(y)\rangle=\langle G_{m_1}(x,|\,A)G_{m_2}(y,x\,|\,A)\rangle_A . \quad (1)$$

The Green function $G_m(x,y\,|\,A)$ for the scalar particle in the external gauge field is determined from the equation

$$\left[(i\frac{\partial}{\partial x_\alpha}+\frac{g}{c\hbar}A_\alpha(x))^2+\frac{c^2m^2}{\hbar^2}\right]G_m(x,y\,|\,A)=\delta(x-y) \quad (2)$$

The solution of (2) can be represented as a functional integral in the following way (for details see [8]):

$$G_m(x,y\,|\,A)=\int_0^\infty\frac{ds}{(4s\pi)^2}\exp\left\{-sm^2-\frac{(x-y)^2}{4s}\right\}\int d\sigma_\beta\exp\left\{ig\int_0^1 d\xi\,\frac{\partial Z_\alpha(\xi)}{d\xi}A_\alpha(\xi)\right\} \quad (3)$$

Here the following notation is used:

$$Z_\alpha(\xi) = (x-y)_\alpha \xi + y_\alpha - 2\sqrt{s}B_\alpha(\xi); \quad d\sigma_\beta = N\delta B_\beta \exp\left\{-\frac{1}{2}\int_0^1 d\xi B^2(\xi)\right\}, \tag{4}$$

with the normalization

$$B_\beta(0) = B_\beta(1) = 0 \qquad \int d\sigma_\beta = 1,$$

where N is the normalization constant. In averaged over the external gauge field $A_\alpha(x)$ we limit ourselves to the lowest order, i.e. we take into account only the two-point Gaussian correlator:

$$\left\langle \exp\left(i\int dx A_\alpha(x) J_\alpha(x)\right)\right\rangle_A = \exp\left(-\frac{1}{2}\int\int dx dy J_\alpha(x) D_{\alpha\beta}(x-y) J_\beta(y)\right). \tag{5}$$

Here $J_\alpha(x)$ is the real current. The propagator of the gauge field has the following form:

$$D_{\alpha\beta}(x-y) = \left\langle A_\alpha(x) A_\beta(y)\right\rangle_A = \delta_{\alpha,\beta} D(x-y) + \frac{\partial^2}{\partial x_\alpha \partial y_\beta} D_d(x-y) \tag{6}$$

where

$$D(x) = \int \frac{dq}{(2\pi)^4}\frac{\exp(iqx)}{q^2}, \qquad D_d(x) = \int \frac{dq}{(2\pi)^4}\frac{\exp(iqx)}{q^2}\frac{d(q^2)}{q^2}. \tag{7}$$

So the external field exists only in a virtual state. The mass of the bound state is usually defined through the correlation function in the following way:

$$M = -\lim_{|x-y|\to\infty} \frac{\ln \Pi(x-y)}{|x-y|}. \tag{8}$$

Thus, if we know the correlation function, then we can determine the bound state mass.

From (8) one can see that for determination of the mass M one needs to calculate correlation function $\Pi(x)$ in the asymptotics $|x| \to \infty$. Substituting (3) into (1) and averaging over the external gauge field we obtain:

$$\Pi(x) = \int\int_0^\infty \frac{d\mu_1 d\mu_2}{(8\pi^2 x)^2} J(\mu_1,\mu_2) \exp\left\{-\frac{|x|}{2}\left(\frac{m_1^2}{\mu_1}+\mu_1\right) - \frac{|x|}{2}\left(\frac{m_2^2}{\mu_2}+\mu_2\right)\right\} \tag{9}$$

here

$$J(\mu_1,\mu_2) = N_1 N_2 \int\int \delta r_1 \delta r_2 \exp\left\{-\frac{1}{2}\int_0^x d\tau[\mu_1 \dot{r}_1^2(\tau) + \mu_2 \dot{r}_2^2(\tau)]\right\}\exp\{-W\}, \qquad W = W_{1,1} + W_{2,2} + 2W_{1,2}, \tag{10}$$

and following notation is used:

$$W_{i,j} = \frac{g^2}{2}(-1)^{i+j}\int_0^\infty\int_0^\infty d\tau_1 d\tau_2 Z'^{(i)}_\alpha(\tau_1) D_{\alpha\beta}(Z^{(i)}(\tau_1) - Z^{(j)}(\tau_2)) Z'^{(j)}_\beta(\tau_2) \tag{11}$$

Representation (10) is analogous to the quantum Green function in a from the Feynman functional integral, when two particles with masses μ_1 and μ_2 interacts via the nonlocal potential $W_{i,j}$. Therefore, we call masses m_1 and m_2 the current masses, and parameters μ_1 and μ_2 the constituent masses. Note that the functional integration in (10) is over the four dimensional vectors $r_1 = (\mathbf{r}_1, r_1^{(4)})$ and $r_2 = (\mathbf{r}_2, r_2^{(4)})$. The value of $W_{i,j}$ is determined by the contribution of various types of Feynman diagrams. There are two types of interaction: the first is the interaction of the constituent particles by the gauge field, the contribution of which is determined immediately as $W_{1,2}$; the second is the interaction of the constituent particles with each other, i.e. the self-energy diagrams, the contribution of which is determined as $W_{1,1}$ and $W_{2,2}$. In the nonrelativistic limit the value of $W_{1,2}$ correspond to a potential interaction $W_{1,1}$, $W_{2,2}$ that determine the contribution of the particle mass renormalization.

In the asymptotic $|x| \to \infty$ the integral (10) behaves as:

$$\lim_{|x|\to\infty} J(\mu_1, \mu_2) \Rightarrow \exp\{-xE(\mu_1, \mu_2)\}, \quad (12)$$

where the function $E(\mu_1, \mu_2)$ depends on the coupling constant g and the parameters μ_1, μ_2 and is independent of the masses m_1, m_2. If $|x| \to \infty$ then the integral (10) is evaluated by the saddle point method. And for the bound state mass we obtain:

$$M = \frac{1}{2}\min\left\{\frac{m_1^2}{\mu_1} + \mu_1 + \frac{m_2^2}{\mu_2} + \mu_2 + 2E(\mu_1, \mu_2)\right\}. \quad (13)$$

Thus, the problem reduced to calculation of the functional integral (10). However, this integral is not calculated in a general way and is defined in the framework approaches. Nowadays, the exact mathematical methods of evaluating this integral are absent. Therefore, it is necessary to involve different physical assumptions or approaches to somehow perform the integration over the fourth component $r_1^{(4)}, r_2^{(4)}$. The integration over the fourth component electively corresponds to the transition to the nonrelativistic limit. In other words, the interaction potential with the corrections related to the nonperturbative, relativistic and non-local character of the interaction is determined. In particular, if in the functional $W_{i,j}$ in (11) we neglect the dependence on $r_1^{(4)}, r_2^{(4)}$, then the system (10) reduced to the Feynman path integral for the motion of scalar particles with masses μ_1 and μ_2 in the NRQM [7] with the local potential. In this approximation, according to (10), the interaction Hamiltonian of the scalar particles with masses μ_1 and μ_2 can be represented as:

$$H = \frac{1}{2\mu_1}\mathbf{P}_1^2 + \frac{1}{2}\mathbf{P}_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2), \quad (14)$$

where $V(\mathbf{r}_1 - \mathbf{r}_2)$ is the interaction potential, which is expressed through $W_{i,j}$, then $E(\mu_1, \mu_2)$ is an eigenvalue of the interaction Hamiltonian (14), i.a.

$$H\Psi(\mathbf{r}_1, \mathbf{r}_2) = E(\mu_1, \mu_2)\Psi(\mathbf{r}_1, \mathbf{r}_2). \quad (15)$$

By minimizing (13), we get the equation for μ_j:

$$\mu_j - \frac{m_j^2}{\mu_j} + 2\mu_j \frac{dE(\mu_1, \mu_2)}{d\mu_j} = 0; \qquad j = 1, 2. \quad (16)$$

The parameters μ_1, μ_2 have the dimension of mass. In further calculations we introduce a new parameter

$$\frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{\mu_2}. \quad (17)$$

Then equation (13) takes the form

$$M = \mu_1 + \mu_2 + \mu\frac{dE}{d\mu} + E(\mu); \quad E(\mu_1, \mu,) = E(\mu), \quad (18)$$

where

$$\mu_1 = \sqrt{m_1^2 - 2\mu^2 \frac{dE}{d\mu}};$$
$$\mu_2 = \sqrt{m_2^2 - 2\mu^2 \frac{dE}{d\mu}}. \quad (19)$$

In our approach, the energy spectrum and the wave function of the bound state are determined by the SE with the constituent masses μ_1 and μ_2.

Now we apply our research to determine the mass and energy spectrum, as well as to determine the decay width of mesons consisting of b and c quarks with orbital and radial excitations.

2.Meson spectra with orbital excitation

2.1 The interaction Hamiltonian

In this section, the mass spectrum of the charmonium, bottom and B_c mesons with spin-spin and spin-orbit interactions is determined from the SE with the constituent mass. The total interaction Hamiltonian of quarks is represented as:

$$H = H_c + H_{spin}, \qquad (20)$$

where H_c is the central part

$$H_c = \frac{1}{2\mu}\vec{P}^2 + \sigma \cdot r - \frac{4}{3}\frac{\alpha_s}{r}. \qquad (21)$$

The second part of the Hamiltonian is defined in the standard from (for details see [9,10]):

$$H_{spin} = H_{SS} + H_{LS} + H_{TT}, \qquad (22)$$

here H_{SS} is the spin-spin interaction Hamiltonian:

$$H_{SS} = \frac{2}{3\mu_1\mu_2}(\vec{S}_1\vec{S}_2)\Delta V_V = \frac{32\pi\alpha_s(\vec{S}_1\vec{S}_2)}{9\mu_1\mu_2}\cdot\delta(\vec{r}), \qquad (23)$$

and H_{LS} is the spin-orbital interaction Hamiltonian:

$$H_{LS} = \frac{1}{4\mu_1^2\mu_2^2}\frac{1}{r}\{[((\mu_1+\mu_2)^2 + 2\mu_1\mu_2)(\mathbf{L}\cdot\mathbf{S}_+) + (\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)]\frac{\partial}{\partial r}V_V - [(\mu_1^2+\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_+) + (\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)]\frac{\partial}{\partial r}V_s\}, \qquad (24)$$

and at last, the tensor interaction Hamiltonian is

$$H_{TT} = \frac{1}{12\mu_1\mu_2}[\frac{1}{r}\frac{\partial}{\partial r}V_V - \frac{\partial^2}{\partial r^2}V_V]S_{12}. \qquad (25)$$

here V_V is the vector potential corresponding to the one-gluon exchange:

$$V_v = -\frac{4\alpha_s}{3}\frac{1}{r} \qquad (26)$$

and V_S is the confinement potential

$$V_s = r\sigma \qquad (27)$$

and also the following notation is used,

$$\mathbf{S}_+ = \mathbf{S}_1 + \mathbf{S}_2; \quad \mathbf{S}_- = \mathbf{S}_1 - \mathbf{S}_2;$$
$$S_{12} = \frac{4}{(2\ell+3)(2\ell-1)}\left[\mathbf{L}^2\mathbf{S}^2 - \frac{3}{2}(\mathbf{LS}) - 3(\mathbf{LS})^2\right]. \qquad (28)$$

Using expressions (20)-(28) for the interaction Hamiltonian we calculate the mass spectrum of the mesons.

2.2 Determination of the quarkonium energy spectra

Now using the explicit form of the total Hamiltonian let us start to determine the quarkonium energy spectrum. According to (14,15) all from the Schrodinger equation

$$H\Psi = E\Psi. \qquad (29)$$

we will determine the energy spectrum and the wave function (WF). We will apply the oscillator-representation (OR) method [11] for the determination of eigenvalues and the wave functions (WF). According to the OR, let us change the variables in this following way (see for details in. [11]):

$$r = q2\rho,\ \Psi \Rightarrow \Psi(q2) = q2\rho\ell\Phi(q2) \qquad (30)$$

Using the atomic system of units ($\hbar = c = 1$), considering (20)-(28) and after some standard simplifications we obtain from eq. (24) for the modified SE:

$$\left\{-\frac{1}{2}\left(\frac{\partial^2}{\partial q^2}+\frac{d-1}{q}\frac{\partial}{\partial q}\right)-4\rho^2\mu E q^{2(2\rho-1)}+4\rho^2\mu\sigma q^{2(3\rho-1)}-\frac{16\rho^2\mu\alpha_s}{3}\cdot q^{2(\rho-1)}\right.$$
$$+\frac{64\alpha_s\mu\rho^2}{9\pi\mu_1\mu_2}\cdot(\vec{S}_1\vec{S}_2)\lim_{\Lambda\to\infty}\int_0^\Lambda dtq^{2(\rho-1)}t\sin(tq^{2\rho})-\frac{\sigma\rho^2\mu}{\mu_1^2\mu_2^2}q^{2(\rho-1)}$$
$$\times[((\mu_1^2+\mu_2^2))(\mathbf{L}\cdot\mathbf{S}_+)+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)]+\frac{4\mu\rho^2\alpha_s}{3\mu_1\mu_2}\cdot\frac{S_{12}}{q^{2(\rho+1)}} \tag{31}$$
$$\left.+\frac{4\mu\rho^2\alpha_s}{3\mu_1^2\mu_2^2q^{2(\rho+1)}}[((\mu_1+\mu_2)^2+2\mu_1\mu_2)(\mathbf{L}\cdot\mathbf{S}_+)+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)]\right\}\Phi(q^2)=0$$

where d is the dimension of the auxiliary space

$$d=2+2\rho+4\rho\ell \tag{32}$$

As a result of the change of variables, we get the modified SE in the d-dimensional auxiliary space R^d. From (12) and (13) it follows that the orbital quantum number ℓ has entered into the dimension d of the space. This technique allows us to determine all characteristics we are interested in, the spectrum and WF by solving the modified SE only for the ground state in the d-dimensional auxiliary R^d space.

The wave function $\Psi_m(q^2)$ of the ground state in R^d depends only on the q^2 variable. Thus, the operator

$$\frac{\partial^2}{\partial q^2}+\frac{d-1}{q}\frac{\partial}{\partial q}\equiv\Delta_q \tag{33}$$

can be identified with the Laplacian in the auxiliary R^d space which acts on the ground-state wave function depending only on the radius q. Proceeding from the modified SE:

$$H\Phi(q)=\varepsilon(E)\Phi(q) \tag{34}$$

according to (31),we obtain that the energy spectrum $\varepsilon(E)$ is equal to zero in R^d.

$$\varepsilon(E)=0 \tag{35}$$

We will consider this equation as the condition for determination of the energy spectrum E of the initial Hamiltonian. Following the OR method, let us represent the canonical variables in terms of the creation (a+) and annihilation (a) operators in the Rd space;

$$q_j=\frac{a_j+a_j^+}{\sqrt{2\omega}},\quad P_j=\sqrt{\frac{\omega}{2}}\cdot\frac{a_j-a_j^+}{i}, \tag{36}$$
$$j=1,...,d,\quad [a_i,a_j^+]=\delta_{i,j}$$

where ω is the oscillator frequency which has been unknown yet. Substituting (36) into (34) and carrying out ordering by the creation and annihilation operators we obtain the interaction Hamiltonian:

$$H=H_0+\varepsilon_0(E)+H_I, \tag{37}$$

Here H_0 is the Hamiltonian of free oscillators

$$H_0=\omega(a_j^+a_j) \tag{38}$$

and ε_0 is the energy of the ground state in the zeroth approximation

$$\varepsilon_0(E)=\frac{d\omega}{4}-\frac{4\rho^2E\mu}{\omega^{2\rho-1}}\frac{\Gamma(d/2+2\rho-1)}{\Gamma(d/2)}-\frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}}\frac{\Gamma(d/2+\rho-1)}{\Gamma(d/2)}+$$
$$+\frac{4\rho^2\sigma\mu}{\omega^{3\rho-1}}\frac{\Gamma(d/2+3\rho-1)}{\Gamma(d/2)}+\frac{32\alpha_s\mu\rho}{9\mu_1\mu_2}\frac{(\vec{S}_1\vec{S}_2)\omega^{d/2}}{\Gamma(d/2)}\delta_{\ell,0}- \tag{39}$$
$$-\frac{\rho^2\sigma\mu}{M_1^2\omega^{\rho-1}}\frac{\Gamma(d/2+\rho-1)}{\Gamma(d/2)}+\frac{4\alpha_s\mu\rho^2S_{12}}{3\mu_1\mu_2}\frac{\omega^{\rho+1}\Gamma(d/2-\rho-1)}{\Gamma(d/2)}+\frac{4\alpha_s\mu\rho^2}{3M_2^2}\frac{\omega^{\rho+1}\Gamma(d/2-\rho-1)}{\Gamma(d/2)}.$$

Here the following notation is used:

$$\frac{1}{M_1^2}=\frac{1}{\mu_1^2\mu_2^2}\left[(\mu_1^2+\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_+)+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\ ;$$
$$\tag{40}$$
$$\frac{1}{M_2^2}=\frac{1}{\mu_1^2\mu_2^2}\left[((\mu_1+\mu_2)^2+2\mu_1\mu_2)(\mathbf{L}\cdot\mathbf{S}_+))+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\ .$$

The interaction Hamiltonian H_I can be represented also in the normal form of the creation a^+ and annihilation a operators and it does not contain the quadratic terms of the canonical variables

$$H_I=\int_0^\infty dx\int\left(\frac{d\eta}{\sqrt{\pi}}\right)^d\exp\{-\eta^2(1+x)\}:e_2^{-i\sqrt{x\omega}(q\eta)}:$$
$$\left[-\frac{4\rho^2\mu}{\omega^{2\rho-1}}\frac{Ex^{-2\rho}}{\Gamma(1-2\rho)}+\frac{4\rho^2\mu}{\omega^{3\rho-1}}\frac{\sigma x^{-3\rho}}{\Gamma(1-3\rho)}-\frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}}\frac{x^{-\rho}}{\Gamma(1-\rho)}-\right.$$
$$-\frac{\sigma\rho^2\mu}{M_1^2\omega^{\rho-1}}\frac{x^{-\rho}}{\Gamma(1-\rho)}+\frac{4\rho^2\mu\alpha_sS_{12}}{3\mu_1\mu_2}\frac{\omega^{\rho+1}x^{\rho}}{\Gamma(1+\rho)}+\frac{4\rho^2\mu\alpha_s}{3M_2^2}\frac{\omega^{\rho+1}x^{\rho}}{\Gamma(1+\rho)}+ \tag{41}$$
$$\left.+\frac{16\rho^2\mu\alpha_s(\vec{S}_1\vec{S}_2)}{9\pi\mu_1\mu_2}\lim_{\Lambda\to\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\frac{\Lambda^{2j+3}}{2j+3}\frac{x^{-2\rho-2\rho j}}{\omega^{2\rho+2\rho j-1}\Gamma(1-2\rho-2\rho j)}\right].$$

Here $:\star:$ is a symbol of normal ordering, and we also use the notation

$$e_2^{-x}=e^{-x}-1+x-\frac{1}{2}x^2$$

In quantum field theory, after the representation of the canonical variables in terms of the creation and annihilation operators and after transformation of the interaction Hamiltonian into the normal form, the requirement of the absence of the second-order field operators is equivalent, in essence, to the renormalization of the coupling constant and the wave function [12]. Moreover, such a procedure permits one to take the main contribution into consideration in terms of the mass renormalization and in terms of the vacuum energy. In other words, all quadratic terms are completely included in the free oscillator Hamiltonian. This requirement allows us to formulate the following condition, according to the OR

$$\frac{\partial\varepsilon_0(E)}{\partial\omega}=0 \tag{42}$$

for the purpose of finding the oscillator frequency ω which defines the main quantum contribution. Taking into account (39) and from equations (35), (42) we can calculate the energy spectrum E of the initial system. So let us restrict ourselves only to the consideration of the zeroth-order approximation.

2.3 Determination of the mesons mass spectrum for the ground state.

In this section, we will determine the mass spectrum and wave functions of mesons consisting of b and c quarks. Consider the ground state, i.e, define the properties of $\eta_c, J/\psi, \eta_b, \Upsilon$ and B_c mesons taking into account the spin-spin interaction. From (20) for the ground state:

$$\varepsilon_0(E) = \frac{d\omega}{4} - \frac{4\rho^2 E_\mu}{\omega^{2\rho-1}}\frac{\Gamma(3\rho)}{\Gamma(1+\rho)} - \frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}}\frac{\Gamma(2\rho)}{\Gamma(1+\rho)} + \frac{4\sigma\mu\rho^2}{\omega^{3\rho-1}}\frac{\Gamma(4\rho)}{\Gamma(1+\rho)} + \frac{16\alpha_s\mu\rho\omega^{\rho+1}}{3\mu_1\mu_2}\frac{[s(s+1)-3/2]}{\Gamma(1+\rho)}, \tag{43}$$

where s is spin of the mesons. Taking into account and from equations we obtain for the oscillator frequency:

$$\omega^{3\rho} - \frac{16\alpha_s\rho^2\omega^{2\rho}\mu}{3}\frac{\Gamma(2\rho)}{\Gamma(2+\rho)} - \frac{4\rho^2\mu\sigma\Gamma(4\rho)}{\Gamma(2+\rho)} + \frac{16\alpha_s\rho\mu}{3\mu_1\mu_2}\frac{\omega^{4\rho}[s(s+1)-3/2]}{\Gamma(1+\rho)} = 0. \tag{44}$$

and for the energy of the ground:

$$E = \min_\rho\left\{\frac{\omega^{2\rho}\Gamma(2+\rho)}{8\rho^2\mu\Gamma(3\rho)} - \frac{4\alpha_s\omega^\rho\Gamma(2\rho)}{3\Gamma(3\rho)} + \frac{\sigma\Gamma(4\rho)}{\omega^\rho\Gamma(3\rho)} + \frac{4\alpha_s[s(s+1)-3/2]\omega^{3\rho}}{9\rho\mu_1\mu_2\Gamma(3\rho)}\right\} \tag{45}$$

According to (35), the mass of singlet triplet states

$$\begin{aligned}\mu_1 - \frac{m_1^2}{\mu_1} + 2\mu_1\frac{dE}{d\mu_1} = 0\\ \mu_2 - \frac{m_2^2}{\mu_2} + 2\mu_2\frac{dE}{d\mu_2} = 0\end{aligned} \tag{46}$$

Here m_1 and m_2 are the current masses of the quarks. As follow, the experimentally fitted value of the current masses of quarks is reader as

$$\begin{aligned}m_c &= 1.275 \pm 0.025\ GeV;\\ m_b(1S) &= 4.65 \pm 0.03\,GeV.\end{aligned} \tag{47}$$

The value of the running coupling constant of the quark-gluon interactions is determined as follows:

$$\beta_0 = 11 - \frac{2}{3}n_f,\ \mu_{12} = \frac{2\mu_1\mu_2}{\mu_1+\mu_2},\ n_f \tag{48}$$

where n_f is the flavor quantum number and Λ = 0.169 GeV is the scale of confinement for heavy quarks. Then the mass of mesons consisting of these quarks is defined as:

$$M = \frac{1}{2}\left(\mu_1 + \frac{m_1^2}{\mu_1} + \mu_2 + \frac{m_2^2}{\mu_2}\right) + E \tag{49}$$

The results of numerical calculations are introduced in Table 1. According to (47), for the current quark masses use the values $m_c = 1.275$, $m_b = 4.62$ GeV. The oscillator frequency and the constituent masses are defined from the quark equation presented in (44) and (47), respectively. From Table 1 we can see that the constituent quark masses are greater than the current masses. According (49), with changing of the constituent quark masses a running coupling constant of the quark-gluon interaction changes, the values are also given in Table 1. It can be see that our results for the meson masses are in good agreement with the experimental data. The values of WF at the coordinate $\Psi(0)$ are also given in Table 1. For the ground state we have

$$|\Psi_n(0)|^2 = \frac{1}{4\pi}\frac{\omega^{3\rho}}{\rho\Gamma(3\rho)} \tag{50}$$

Using $|\Psi_n(0)|^2$ let us determine the leptonic decay constant of the vector and pseudo-scalar mesons:

$$f_p^{NR} = f_\upsilon^{NR} = \sqrt{\frac{12}{M_{p,\upsilon}}} \mid \Psi_{p,\upsilon}(0) \mid , \qquad (51)$$

where $M_{p,v}$ is the mass of the vector and the pseudo-scalar mesons. The leptonic decay width of the vector mesons is determined as follows:

$$\Gamma(V \to cc) = \frac{16\pi\alpha_{em}^2 c_Q^2}{M_V^2} \mid \Psi(0) \mid^2 (1 - \frac{16\alpha_s}{3\pi}) \qquad (52)$$

where $\alpha_{em} = 1/137$ is the electromagnetic coupling constant; e_Q is the quark current, and M_V is the vector meson mass. The numerical results for the ground state are in Table 1.

2.4 Determination of the mass spectrum of mesons with the orbital excitation

In this section, we will calculate the energy and mass spectrum of the mesons consisting of *c* and *b* quarks with orbital excitation. From (39) it is seen that when $l \neq 0$ then the spin interactions are defined by only the spin-orbit interaction. In this case, the interaction Hamiltonian H_I does not contribute. First of all, let us consider the case *S = 0*. After some calculations, and taking into account (39) from (42) we obtain en equation for determining the oscillator frequency:

$$\omega^{3\rho} - \omega 2\rho \frac{16\alpha_s \rho^2 \mu}{3} \frac{\Gamma(2\rho + 2\rho l)}{\Gamma(2 + \rho + 2\rho l)} + \frac{4\rho^2 \mu\sigma\Gamma(4\rho + 2\rho l)}{\Gamma(2 + \rho + 2\rho l)} = 0, \qquad (53)$$

Table 1 – Mass spectrum of mesons consisting of b and c quarks for the ground state. Experimental data from Ref [2].

		$\bar{c}c$	$\bar{b}b$	$\bar{b}c$
S=0	$m_c GeV$	1.275	-	1.275
	$m_b GeV$	-	4.62	4.62
	α_s	0.30366	0.194679	0.248935
	$\sigma\ GeV^2$	0.195	0.153	0.195
	$E\ GeV$	0.413530	0.157253	0.363173
	ρ	0.526448	0.651103	0.46495
	$\omega^\rho GeV$	0.652	1.164	0.648335
	$\mu_c GeV$	1.42862	–	1.51306
	$\mu_b GeV$	–	4.73493	4.68082
	$M_{our} MeV$	2980.05	9400.04	6.2773
	$M_{\exp} MeV$	2980.3 ± 1.2	–	6277 ± 4
	$\mid \Psi(0) \mid^2\ GeV^2$	0.047003	0.196457	0.525517
	$f^\eta\ GeV$	0.435053	0.500795	0.316955

S=1	α_s	0.2990855	0.194459	0.247683
	E	0.519023	0.216613	0.412532
	ρ	1.03926	1.24871	1.11493
	$\omega^{\rho}\,GeV$	1.4311	3.4511	2.0512
	$\mu_c\,GeV$	1.47617	–	1.53652
	$\mu_b\,GeV$	–	4.7581	4.71302
	$M_{our}\,MeV$	3.09644	9.4603	6.33071
	$M_{\exp}\,MeV$	3096.916 ± 0.11	9460.3 ± 0.26	–
	$\lvert\Psi(0)\rvert^2\;GeV^2$	0.1004	0.5973	0.219078
	$f^{n}\;GeV$	0.62372	0.8704	0.644412
	$\Gamma_{our}\,keV$	6.135	1.330	–
	$\Gamma_{\exp}\,keV$	5.55 ± 0.14	1.340 ± 0.018	–

and for the energy:

$$E = \min_{\rho}\left\{\frac{\omega^{2\rho}\Gamma(2+\rho+2\rho l)}{8\rho^2\mu\Gamma(3\rho+2\rho l)} - \frac{\alpha\omega^{\rho}\Gamma(2\rho+2\rho l)}{3\Gamma(3\rho+2\rho l)} + \frac{\sigma\Gamma(4\rho+2\rho l)}{\omega^{\rho}\Gamma(3\rho+2\rho l)}\right\}. \tag{54}$$

Taking into account (53), (54) and (46) from (49) we determine the meson masses with orbital excitation. The results of numerical calculations introduced in Tables 2 and 3.

Now we will start to calculate the energy spectrum of meson spin triplet state $S = 1$ with orbital excitations. The equation that determines the oscillator frequency can be written as

$$\omega^{3\rho} - \omega^{2\rho}\frac{16\alpha_s\rho^2\mu}{3}\frac{\Gamma(2\rho+2\rho l)}{\Gamma(2+\rho+2\rho l)} + \frac{4\rho^2\mu\sigma\Gamma(4\rho+2\rho l)}{\Gamma(2+\rho+2\rho l)} - \omega^{2\rho}\frac{\sigma\rho^2\mu}{M_1^2}\frac{\Gamma(2\rho+2\rho l)}{\Gamma(2+\rho+2\rho l)}$$
$$+\frac{4\rho^2\alpha_s\mu}{\mu_1\mu_2}\frac{S_{12}\omega^{4\rho}\Gamma(2\rho l)}{\Gamma(2+\rho+2\rho l)} + \frac{\rho^2\mu\alpha_s\omega^{4\rho}}{M_2^2}\frac{\Gamma(2\rho l)}{\Gamma(2+\rho+2\rho l)} = 0 \tag{55}$$

and for the energy

$$E = \min_{\rho}\left\{\frac{\omega^{2\rho}\Gamma(2+\rho+2\rho l)}{8\rho^2\mu\Gamma(3\rho+2\rho l)} - \frac{\alpha\omega^{\rho}\Gamma(2\rho+2\rho l)}{3\Gamma(3\rho+2\rho l)} + \frac{\sigma\Gamma(4\rho+2\rho l)}{\omega^{\rho}\Gamma(3\rho+2\rho l)} - \frac{\sigma}{4M_1^2}\frac{\Gamma(2\rho+2\rho l)}{\Gamma(3\rho+2\rho l)}\right.$$
$$\left.+\frac{\alpha_s S_{12}}{3\mu_1\mu_2}\frac{\omega^{3\rho}\Gamma(2\rho l)}{\Gamma(3\rho+2\rho l)} + \frac{\alpha_s\omega^{3\rho}}{M_2^2}\frac{\Gamma(2\rho l)}{\Gamma(3\rho+2\rho l)}\right\} \tag{56}$$

The numerical results for the P and D states are shown in Tables 2 and 3, respectively.

2.5 Determination of the mass spectrum of mesons with the radial excitation

In this section, we will determine the mass and energy spectrum of the mesons only with the radial excitation. In this case, the energy $\varepsilon 0(E)$ of the zeroth approximation in OR is determined by (43), and for the interaction Hamiltonian from (41), we have

$$H_I = \int^{\infty} dx \int (\frac{d\eta}{\sqrt{\pi}})^d e^{-\eta^2(1+x)} : e^{-2i\sqrt{x\omega}(q\eta)} :$$
$$\left\{ -\frac{4\rho^2\mu}{\omega 2\rho-1}\frac{Ex^{-2\rho}}{\Gamma(1-2\rho)} + \frac{4\rho^2\mu}{\omega^{3\rho-1}}\frac{\sigma x^{-2\rho}}{\Gamma(1-3\rho)} - \frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}}\frac{x^{-\rho}}{\Gamma(1-\rho)} + \frac{16\rho^2\mu\alpha_s(\vec{S}_1\vec{S}_2)}{9\pi\mu_1\mu_2}\lim_{\Lambda\to\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\frac{\Lambda^{2j+3}}{(2j+1)}\frac{x^{-2\rho-2\rho j}}{\omega^{2\rho+2\rho j-1}\Gamma(1-2\rho-2\rho j)} \right\} \quad (57)$$

In this case, the energy spectrum has the following form:

$$\varepsilon_n(E) = \varepsilon_0(E) + 2n_2\omega + \langle n_2 | H_I^c | n_2 \rangle + \langle n_2 | H_I^s | n_2 \rangle \quad (58)$$

where H_I^c is the central part and H_I^s is the spin part of the interaction Hamiltonian. After some simplifications for the energy spectrum with radial excitations we get

$$E_{n_2} = \frac{\omega^{2\rho}\Gamma(2+\rho)}{8\rho^2\mu\Gamma(3\rho)}\frac{1+\frac{n_2}{1+\rho}}{1+W_1} - \frac{\sigma\Gamma(4\rho)}{\omega^{\rho}\Gamma(3\rho)}\frac{1+W_3}{1+W_1} + \frac{4\alpha_s[S(S+1)-3/2]\omega^{3\rho}}{9\rho\mu_1\mu_3\Gamma(3\rho)}\frac{1+W_s}{1+W_1} \quad (59)$$

Table 2 – Mass spectrum of charmonium with the orbital exitations. Experimental data from Ref [2].

		$J=l-1$, $S=1$	$J=l+1$, $S=1$	$J=l$, $S=0$
l=1	α_s	0.3013	0.2981	0.2978
	E GeV	0.923955	0.960388	0.945799
	ρ	0.808694	0.613677	0.230383
	ω^{ρ} GeV	1.14386	0.618518	0.276542
	μ_c GeV	1.45188	1.48592	1.48936
	M_{h_c}	3.4955	3.54033	3.5266
	$\lvert\Psi(0)\rvert^2$ GeV^3	0.116538	0.0325412	0.00557
	f_h GeV	0.632513	0.332113	0.13763
l=2	α_s	0.2987	0.2944	0.2936
	E GeV	1.2229	1.22267	1.21638
	ρ	0.612313	0.595989	1.39076
	ω^{ρ} GeV	0.595536	0.560571	5.59744
	μ_c GeV	1.53846	1.5278	1.5371
	M_{h_c}	3.81728	3.8145	3.81107
	$\lvert\Psi(0)\rvert^2$ GeV^3	0.1165385	0.025338	1.34144
	f_h GeV	0.632505	0.282333	2.05519

Table 3 – Mass spectrum of bottomy with the orbital exitations. Experimental data from Ref [2].

		$J=l-1$ $S=1$	$J=l$ $S=1$	$J=l+1$ $S=1$	$J=l$ $S=0$
l=1	α_s	0.1944	0.1943	0.1943	0.1946
	$E\,GeV$	0.635856	0.6479	0.669121	0.657241
	ρ	0.628027	0.780369	0.312187	0.0915
	$\omega^\rho GeV$	0.985258	1.36504	0.49	0.273495
	$\mu_c GeV$	4.7567	4.76124	4.76007	4.76134
	M_ξ	9.87978	9.89209	9.91324	9.90144
	$\lvert\Psi(0)\rvert^2\ GeV^3$	0.0514891	0.09099	0.002881	0.03731
	$f_{bc}\ GeV$	0.250077	0.332242	0.186755	0.21496
l=2	α_s	0.1939	0.1939	0.1939	0.1939
	$E\,GeV$	0.906587	0.911645	0.916257	0.911824
	ρ	0.184697	0.177198	0.169101	0.0634526
	$\omega^\rho GeV$	0.327369	0.321223	0.315	0.2129
	$\mu_c GeV$	4.79492	4.79433	4.7926	4.7967
	M_ξ	10.153	10.158	10.1625	10.1583
	$\lvert\Psi(0)\rvert^2\ GeV^3$	0.000194	0.0089142	0.000156	0.0000124
	$f_{bc}\ GeV$	0.015147	0.102619	0.0136138	0.0038378

In this case, the oscillator frequency is determined from the following equation:

$$\omega^{3\rho}-\frac{16\alpha_s\mu\rho^2\omega^{2\rho}\Gamma(2\rho)}{3\Gamma(2+\rho)}\frac{(2\rho-1)\tilde{W}_2-(\rho-1)\tilde{W}_1}{\tilde{W}_1+(2\rho-1)(1+\frac{4n_2}{1+\rho})}+\frac{4\rho^2\mu\sigma\Gamma(4\rho)}{\Gamma(2+\rho)}\frac{(2\rho-1)\tilde{W}_3-(3\rho)\tilde{W}_1}{\tilde{W}_1+(2\rho-1)(1+\frac{n_2}{1+\rho})}$$
$$+\frac{16\alpha_s\mu\rho[S(S+1-3/2)]\omega^{4\rho}}{9\mu_1\mu_2\Gamma(2+\rho)}\frac{(2\rho-1)\tilde{W}_s+(1+\rho)W_1}{\tilde{W}_1+(2\rho-1)(1+\frac{n_2}{1+\rho})}, \quad (60)$$

where the following notation is used

$\tilde{W}_j=1+W_j$, j = 1, 2, 3 and s: using (59) from (37) and (38) we determine the meson mass and the constituent quarks mass, and the numerical results are shown in Table 3.

3 Determination of the width of the radiative decay

Now we proceed to the determination of the radiative decay width or $E1$ transition. The matrix elements of the $E1$ transition from the state ($n^{2s+1}J$, i,) to the state ($n^{'2s+1}J^{'}$, f,) is written as follows:

$$M(i\to f)_\mu=\delta_{s,s^{'}}k\sqrt{(2J+1)(2J'+1)(2l+1)(2l'+1)}\times\begin{pmatrix}J' & 1 & J\\ -M' & \mu & M\end{pmatrix}\begin{pmatrix}l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l & s & J\\ J\backslash & 1 & l'\end{pmatrix}e_Q I_{i,f}, \quad (61)$$

where the usual notation in brackets is: 3*j* is the symbol, e_Q is the quark charge, and $I_{i,f}$ is the radial matrix element of the $i \to f$ transition:

$$I_{i,f} = \int_0^\infty dr\, r^2\, \Psi^*_{n'l'}\, r\, \Psi_{nl}(r) \qquad (62)$$

where $\Psi_{i,f}$ is the radial wave function of the initial and the final state. Then the radiative decay width is determined as follows:

$$\Gamma(i \to f + \lambda) = \frac{4\alpha_{em} e_Q^2}{3}(2J'+1) S_{if}^E\, k^3 \,| I_{i,f} |^2 \qquad (63)$$

where *k* is the photon momentum and it is equal to

$$k = \frac{m_i^2 - m_f^2}{2m_i} \qquad (64)$$

and m_i, m_f is the mass of the initial and the final state. The statistical factor $S_{if}^E = S_{fi}^E$ is:

Table 4 – Mass spectrum of mesons consisting of b and c quarks with radial excitations. Experimental data from Ref [2].

		$\bar{c}c$	$\bar{b}b$	$\bar{b}c$
S=0	α_s	0.2745	0.19027	0.22974
	$E\, GeV$	0.939195	0.704855	0.79797
	ρ	0.504507	0.45040495	0.537141
	$\omega^\rho GeV$	0.61426	0.913661	0.732053
	$\mu_c GeV$	1.79312	–	2.01377
	$\mu_b GeV$	–	5.115	4.841
	$M_{our} MeV$	3.6389	9.99276	0.683353
	$M_{exp} MeV$	3638.9 ± 1.3	–	–
	$\|\Psi(0)\|^2\, GeV^2$	0.0409795	0.161118	0.0649499
	$f_\eta GeV$	0.367611	0.439865	0.33772
S=1	α_s	0.27479	0.18989	0.22996
	E	1.01391	0.728737	0.836904
	ρ	0.644051	0.452765	0.571577
	$\omega^\rho GeV$	0.629255	0.905397	0.73425
	$\mu_c GeV$	1.7888	–	2.03489
	$\mu_b GeV$	–	5.1501	4.896
	$M_{our} MeV$	3.71148	10.0233	6.88157
	$M_{exp} MeV$	3686.109 ± 0.012	10.02326 ± 0.0031	–
	$\|\Psi(0)\|^2\, GeV^3$	0.025839	0.15568	0.0604622
	$f_\eta\, GeV$	0.289038	0.43172	0.324705
	$\Gamma_{our} keV$	0.691849	0.260597	–
	$\Gamma_{exp} keV$	2.35 ± 0.04	0.612 ± 0.011	–

Table 5 – The radiative decay width results

Transition $i \to f$	k MeV	I_{if} GeV^{-1}	$\Gamma_{our}(i \to f)$ keV	$\Gamma_{exp}(i \to f)$ keV
$\chi_{c0} \to \gamma + J/\psi$	376.3	2.33	139.312	–
$\chi_{c1} \to \gamma + J/\psi$	416.06	1.73	310.3	295.84
$\chi_{c2} \to \gamma + J/\psi$	429.12	2.18	450.5	$\propto 500$

$1^3D_1 \to \gamma + 1^1P_0$	308.22	1.78	267.92	$\propto$ 299
$1^3D_1 \to \gamma + 1^1P_1$	266.90	3.274	146.9	$\propto$ 99
$1^3D_1 \to \gamma + 1^1P_2$	253.13	2.751	3.54	$\propto$ 3.88
$\chi_{c0} \to \gamma + \Upsilon$	410.57	1.422	16.81	–
$\chi_{c1} \to \gamma + \Upsilon$	422.366	1.57	66.9	–
$\chi_{c2} \to \gamma + \Upsilon$	442.592	0.6644	22.97	–
$1^3D_1 \to \gamma + 1^1P_0$	269.544	0.1526	0.33	–
$1^3D_1 \to \gamma + 1^1P_1$	257.56	0.135	0.06	–
$1^3D_1 \to \gamma + 1^1P_2$	236.929	0.4988	0.024	–

$$S^E_{if} = \max(l,l')\begin{Bmatrix} J & 1 & J' \\ l; & s & l \end{Bmatrix} \qquad (65)$$

Thus, it is necessary to calculate the transition of the matrix element which is presented in (61). The numerical results of the decay width are shown in Table 5.

Conclusions

Based on these results we conclude:

In our approach, constituent quark masses are not free parameters, are determined for each quarkonium separately and differ from the mass of a free state, i.e., from the valence quark masses. In this case, the constant α_s of the strong interaction differents from each other for meson. Free parameter in the our approach is the string tension σ and for quarkonium consisting of c quarks is σ = 19.5 *GeV* 2 and for bottomonium consisting of b is σ = 15.3 *GeV* 2.

The mass and wave functions of the mesons are determined via the eigenvalues of nonrelativistic Hamiltonian in which the kinetic energy term is defined by the constituent mass of the bound state forming and potential energy term is determined by the contributions of every possible type of Feynman diagrams with an exchange of gauge field. In this work, loop corrections to the potential interaction are not taken into account and this deserves further study.

In the framework of our approach the mass splitting between the singlet and triplet states is determined and the radiative decay widths of the $\bar{c}c$, $\bar{b}b$ and $\bar{b}c$ systems are calculated.

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