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Companions of fields of rational and real algebraic numbers

Abstract. Companions of the field of rational numbers and a real-closed algebraic expansion of the field of rational numbers are studied. The description of existentially closed companions of a real-closed algebraic expansion of a field of rational numbers refers to the field of study of classical algebraic structures. The general theory of companions and existentially closed companions, built on the basis of Fraisse's classes in the works of A.T. Nurtazin, is included in the classical field of existentially closed theories in model theory. The basic concept of a companion: two models of the same signature are called companions if for any finite submodel of one of them, there is an isomorphic finite submodel in the other. This approach, applied to specific classical structures and their theories, provides new tools for the study of these objects. The study of the companion class of rational and algebraic real number fields reveals companion fields containing transcendental and possibly algebraic elements with special properties of polynomials defining these elements.

Keywords: companion, field of rational numbers, real closed field, algebraic field extension, algebraic element, transcendental element.

Introduction

The theory of existential closure arose in the middle of the twentieth century in the works of one of the recognized classics of model theory Abraham Robinson [1], [2], as well as in the works [3] – [8]. Currently, it is one of the most significant and most developed areas of modern model theory. In previous studies, the most basic form of the concept of companion theory, widely known in the theory of existential closure, is introduced and studied. The criterion of the countable categoricity of this companion theory was found. Some properties of existentially closed and forcing companions have been studied [3] – [9]. Another promising approach to constructing the theory of existentially closed structures based on Fraisse's works [6] is developed in [9] – [16].

Naturally, the development of the general theory of existentially closed companions should be accompanied by the study of classical structures and theories. Historically, one of the classical mathematical objects is the field of rational numbers and the field of all algebraic real numbers. The work studies companion extensions of the named fields. This study is an example of studying a classical object through an approach developed by Nurtazin A.T. and based on Fraisse classes.

The aim and objectives of the study

The purpose of the work is to describe the companions of the fields of rational and real numbers. For this, for each of the named fields, companions of two types are described, namely, purely transcendental extensions and subsequent algebraic extensions of purely transcendental extensions.

Literature review and problem statement

Let: $\mathbb{Q}$ be the field of rational numbers; $\mathbb{R}$ is a real closed algebraic extension of the field $\mathbb{Q}$; $\mathbb{P}[\overline{x}]$, $\mathbb{P}(\overline{x})$ are, respectively, the ring of polynomials and the field of quotients over the field $\mathbb{P}$ of independent variables $\overline{x}=(x_1,\ldots,x_n)$. All used and not given definitions and designations are taken from the monograph [15]. The basic concept of a companion: two models of the same signature are called companions if for any finite submodel of one of them, there is an isomorphic finite submodel in the other. Consider some basic companions of the field $\mathbb{P}$. The class of companions of the field $\mathbb{P}$ is denoted by $C(\mathbb{P})$.

Obviously, purely algebraic field extensions are not its companions. In turn, the following theorem is devoted to the first basic companions of the field, which are purely transcendental extensions.

Materials and methods

The work uses classical algebraic methods for constructing transcendental and algebraic extensions of this field. To describe a simple algebraic extension $\mathbb{P}(\overline{x})[x]/f$ a purely transcendental extension $\mathbb{P}(\overline{x})$ of the field $\mathbb{P}$, as a companion of f, the set of zeros of the polynomial n is studied and a method for recognizing the companions of the original field $\mathbb{P}$ is indicated.

Results and Discussion

COMPANIONS

Field of quotients

THEOREM 1. Let $\mathbb{P}$ have a field of characteristic 0. Then, the field of quotients $\mathbb{P}(\overline{x})$ is a companion of the field $\mathbb{P}$ i.e. $\mathbb{P}(\overline{x})\in C(\mathbb{P})$.

Proof. Each finite submodel of the ring $\mathbb{P}$ is a submodel of the field $\mathbb{P}(\overline{x})$. Conversely, let there be a finite submodel $\mathfrak{F}$ of the field $\mathbb{P}(\overline{x})$, we can assume that $\mathfrak{F}$ is given by a finite system of equalities and inequalities ($\neq$), the right and left parts of which contain elements $\mathfrak{F}$ and the operations of addition and multiplication of the field $\mathbb{P}(\overline{x})$. We transform this system of equalities and inequalities into an equivalent system $\mathbb{P}(\overline{x})$ $\& S_i(\overline{x})=0 \,\&\, \& T_j(\overline{x})\neq 0$, where $S_i(\bar{x}) = \frac{f_i(\bar{x})}{g_i(\bar{x})}, T_j(\bar{x}) = \frac{u_j(\bar{x})}{v_j(\bar{x})}$ here $f_i(\bar{x}), g_i(\bar{x}), u_j(\bar{x}), v_j(\bar{x}) \in \mathbb{P}[\bar{x}], g_i(\bar{x}), v_j(\bar{x}) \neq 0$

The latter system is equivalent in $\mathbb{P}[\overline{x}]$ system of equations and one inequality $\&(f_i(\bar{x}) = 0)\&T(\bar{x}) = \prod g_i(\bar{x})\, u_j(\bar{x}) v_j(\bar{x}) \neq 0$, where $f_i(\overline{x}), g_i(\overline{x}), u_j(\overline{x}), v_j(\overline{x}) \in \mathbb{P}[\overline{x}], g_i(\overline{x}), v_j(\overline{x}) \neq 0$.

Let us prove the existence of a set $\overline{a}=(a_1,...,a_n)\in\mathbb{P}$, such that in $\mathbb{P}$ is fulfilled

$\&(f_i(\bar{a}) = 0)\&T(\bar{a}) \neq 0$ $(\mathbb{P}|=\&(f_i(\bar{a}) = 0)\&T(\bar{a}) \neq 0)$. For any choice of $\bar{a} = (a_1, \dots, a_n) \in \mathbb{P}$, the equality $\&f_i(\bar{a}) = 0$is obvious, since all the coefficients of the variables and the free terms in $\& f_i(\overline{x})$ are equal to zero. By induction on the number of variables n, we prove the existence of $\bar{a} = (a_1, \dots, a_n) \in \mathbb{P}$, which is fulfilled $\mathbb{P}\models T(\overline{a})\neq 0$. For $n=1$ we choose $a_1\in\mathbb{P}$ which is not a root $T(x_1)$.

Step of induction. Let $\bar{a}_{n-1} = (a_1, \dots, a_{n-1}) \in \mathbb{P}$ be such that the higher coefficient of the polynomial $T(\overline{x})$ considered as a polynomial of x_1 over the ring $\mathbb{P}[\overline{x}_{n-1}]$ is not zero. Then, the polynomial $T(\bar{a}_{n-1}, x_n) \neq 0$ and hence there is $a_n\in\mathbb{P}$, such that $\mathbb{P}| = T(\bar{a}) \neq 0$ is satisfied. So $\mathbb{P}|=\&(f_i(\bar{a}) = 0)\&T(\bar{a}) \neq 0$ is done.

The theorem has been proved.

CONSEQUENCE 1. Let $R_i(\overline{x}), S_j(\overline{x})\in\mathbb{P}(\overline{x})$, here $\mathbb{P}$ is formally a real field. The system $\& R_i(\overline{x})=0 \,\&\, \& S_j(\overline{x})\neq 0$ is equivalent in $\mathbb{P}(\overline{x})$ system of one equation and one inequality $P_1(\overline{x})=0 \,\&\, P_2(\overline{x})\neq 0$ where $P_1(\overline{x}), P_2(\overline{x})\in\mathbb{P}[\overline{x}]$.

Simple algebraic extensions

Let $A_f = \{ (\bar{a}, a) | f(\bar{a}, a) = 0, \bar{a}, a \in \mathbb{P}\}$be an annihilator f, where $f(\overline{x},x)\in\mathbb{P}(\overline{x})[x]$. When considering algebraic extensions of the field $\mathbb{P}(\overline{x})$ by means of an irreducible polynomial $f(\overline{x},x)\in\mathbb{P}(\overline{x})[x]$, we will assume that $f(\overline{x},x)\in\mathbb{P}[\overline{x},x]$ and has content 1, as a polynomial in x over the ring $\mathbb{P}[\overline{x}]$.

Then, the divisibility of any polynomial $g(\overline{x},x)\in\mathbb{P}(\overline{x})[x]$ by $f(\overline{x},x)$ is equivalent to the divisibility of a polynomial $g'[\overline{x},x]\in\mathbb{P}(\overline{x})[x]$ such that $g(\overline{x},x)=g'(\overline{x},x)\dfrac{p(\overline{x})}{q(\overline{x})}$, where $g'[\overline{x},x]$is a polynomial with content 1 over the ring $\mathbb{P}[\overline{x}]$, and $q(\overline{x})$ is the least common multiple of the denominators of the coefficients in $g(\overline{x},x)$.

Next, the field $\mathbb{P}$ is one of the fields $\mathbb{Q},\mathbb{R}$.

Consider a simple algebraic extension $\mathbb{P}(\overline{x})[x]/f$ of the field that is the companion of the field $\mathbb{P}$.

THEOREM 2. Let $f(\overline{x},x)$ be an irreducible polynomial over the field $\mathbb{P}(\overline{x})$. Then, the algebraic extension $\mathbb{P}(\overline{x})[x]/f$ of the field $\mathbb{P}(\overline{x})$ is a companion $\mathbb{P}$ if and only if the condition is met: if an arbitrary polynomial $g(\overline{x},x)$ is not divisible by $f(\overline{x},x)$, then there is a tuple $(\overline{a},a)\in\mathbb{P}$ such that $g(\overline{a},a)\in A_f\setminus A_g$ is satisfied.

Proof. Necessity. Let $\{c_1,...,c_k\}\in\mathbb{P}$ be all coefficients of polynomials $g(\overline{x},x)$ and $f(\overline{x},x)$. By the primitive element theorem, there exists an irreducible polynomial $p(y)\in\mathbb{Q}[y]$ and an element $c^*\in\mathbb{P}$ such that $c_i=q_i(c^*),q_i(y)\in\mathbb{Q}[y]$.

Let us replace each of the elements $c_1,...,c_k$ by $q_i(c^*)$ in the polynomials $g(\overline{x},x)$ and $f(\overline{x},x)$, and obtain $g^*(\overline{x},x)=g(\overline{x},x)$ and $f^*(\bar{x},x)=f(\bar{x},x)$,$g^*(\bar{x},x),f^*(\bar{x},x)\in\mathbb{Q}(\bar{x})[x]$.

We have that $g^*(\overline{x},x)$ is not divisible by $f^*(\bar{x},x)$ in $\mathbb{P}(\overline{x})[x]$ $(\mathbb{P}(\bar{x})[x]| = f^*(\bar{x},x)\boxed{|}g^*(\bar{x},x))\Leftrightarrow\mathbb{P}(\bar{x})[x]| = f^*(\bar{x},x)\boxed{|}g^*(\bar{x},x)$ $\Rightarrow$ there is $(\bar{a},a)\in\mathbb{P}$ such that $g^*(\bar{a},a)\in A_{f^*}\backslash A_{g^*}\Leftrightarrow g(\bar{a},a)\in A_f\backslash A_g$ is satisfied. The necessity has been proved.

Sufficiency. By virtue of consequence 1, we can assume that the existential sentence is true in $\mathbb{P}(\overline{x})[x]/f$, after substituting solutions in it is one equality and one inequality$\mathbb{P}(\bar{x})[x]/f\models h_1(\bar{x},y_f)=0\&h_2(\bar{x},y_f)\neq 0$, where $h_1(\bar{x},y),h_2(\bar{x},y)\in\mathbb{P}[\bar{x},y]$.

Then there is a tuple $(\overline{a},a)\in\mathbb{P}$ such that $h_1(\overline{a},a)\in A_f,h_2(\overline{a},a)\in A_f\setminus A_g$ is satisfied. Thus, $\mathbb{P}\models h_1(\bar{x},y_f)=0\&h_2(\bar{x},y_f)\neq 0$ is satisfied. Sufficiency, and with it the theorem has been proved.

Let $I(A_f)=\{g|g\in\mathbb{P}(\bar{x})[x],A_f\subseteq A_g\}$, where $f\in\mathbb{P}(\overline{x})[x]$.

PROPOSITION 1. Let $f(\overline{x},x)$ be an irreducible polynomial over a field $\mathbb{P}(\overline{x})$. Then the following two properties of the polynomial $f(\bar{x},x)$ are equivalent:

a) If an arbitrary polynomial $g(\overline{x},x)$ is not divisible by $f(\overline{x},x)$ then there is a tuple $(\overline{a},a)\in\mathbb{P}$ that satisfies $g(\overline{a},a)\in A_f\setminus A_g$;

b) The equality $I(A_f)=(f)$ is fulfilled.

Proof. $a)\Rightarrow b)$. Definitely, $I(A_f)\supseteq(f)$. Let $g\in I(A_f)$ and $g\not\in(f)$, moreover, n, then by assumption $a)$ there is a set $(\overline{a},a)\in A_f$ such that $g(\overline{a},a)\neq 0$ is a contradiction. So, $I(A_f)\subseteq(f)$.

$b)\Rightarrow a)$. Let $g(\overline{x},x)$ not be divisible by $f(\overline{x},x)$. According to the condition $I(A_f)=(f)$. From here $g(\overline{x},x)\not\in I(A_f)$ and hence there is a tuple $(\overline{a},a)\in\mathbb{P}$, which is done $g(\overline{a},a)\in A_f\setminus A_g$. The proposition has been proved.

Here is one necessary property of the companion $\mathbb{P}(\overline{x})[x]/f$ of the field $\mathbb{P}$.

PROPOSITION 2. Let $f(\overline{x},x)$ be an irreducible polynomial over a field $\mathbb{P}(\overline{x})$. Then, if the algebraic extension $\mathbb{P}(\overline{x})[x]/f$ of the field $\mathbb{P}(\overline{x})$ is a companion of $\mathbb{P}$,then each projection of the annihilator A_f is an infinite set.

Proof. Assume the opposite, and let the projection A_f be, for example, finite in the variable x_1, i.e.$A_f^1=\{(c_1,\dots,c_k)|\exists\bar{b}_1,\dots,\bar{b}_k)|(c_1,\bar{b}_1),\dots,(c_k,\bar{b}_k)\in A_f,\bar{b}_i=(b_{1i},\dots,b_{ni}),b_{ji},c_i\in\mathbb{P}\}$Then it is fulfilled: $\frac{\mathbb{P}(\bar{x})[x]}{f}|=\exists u_1,\dots,u_n u(f(u_1,u_2,\dots,u_n,u)=0\underset{i}{\&\&}u_1\neq c_i)$ but the same sentence is false in $\mathbb{P}$. Contradiction. In the case when A_f is empty, the same proposition is true in $\mathbb{P}(\overline{x})[x]/f$ but false in $\mathbb{P}$. The proposition is proved.

In the case of a simple algebraic extension $\mathbb{P}(x_1)[x]/f$ of the field $\mathbb{P}(x_1)$, Proposition 2 is inverted.

PROPOSITION 3. Let $f(x_1,x)$ be an irreducible polynomial over a field $\mathbb{P}(x_1)$. Then, an algebraic extension $\mathbb{P}(x_1)[x]/f$ from the field $\mathbb{P}(x_1)$ is a companion of $\mathbb{P}$, if and only if each projection of the annihilator A_f onto each of the coordinate axes Ox_1, Ox is an infinite set.

Proof. The necessity was proved in Proposition 2.

Sufficiency. By Proposition 1 and Theorem 2, it suffices to prove the equality $I(A_f)=(f)$. It's obvious that $I(A_f) \supseteq (f)$. Let us prove the inclusion $I(A_f) \subseteq (f)$

Suppose, $I(A_f) \not\subseteq (f)$, and $g(x_1,x) \in I(A_f) \setminus (f)$. Let $d(x_1,x)$ be the greatest common divisor of polynomials $g(x_1,x)$ and $f(x_1,x)$ over a field $\mathbb{P}(x_1)$. Note that due to the fact $f(x_1,x)$ that the irreducible polynomial $d(x_1,x)$ is an element of the field $\mathbb{P}(x_1)$,we denote it by $d(x_1)$. There are polynomials $p(x_1,x), q(x_1,x) \in \mathbb{P}(x_1)[x]$, such that the equality $p(x_1,x)g(x_1,x)+q(x_1,x)f(x_1,x)=d(x_1)$ is satisfied in the ring $\mathbb{P}(x_1)[x]$. Since $A_g \supseteq A_f$, and by condition, the projection of A_f onto the coordinate axis Ox_1 is an infinite set, it follows from the last representation of the polynomial $d(x_1)$ in the variable x_1 that it has an infinite set of zeros. Contradiction, sufficiency and with it the proposition have been proved.

Let us give a criterion for the mismatch of the ideals $I(A_f)$ and (f).

PROPOSITION 4. Let $f \in \mathbb{P}(\overline{x})[x]$ be an irreducible polynomial over a field $\mathbb{P}(\overline{x})$. The ideals $I(A_f)$ and (f) do not match with $I(A_f) \neq (f)$ if and only if there exists a polynomial $c(\overline{x}) \in \mathbb{P}[\overline{x}]$ such that, to the Cartesian power $\mathbb{P}^{n+1}$ $c(\overline{x}) \in I(A_f)$ is satisfied, i.e. the cylindrical surface A_c in affine space $\mathbb{P}^{n+1}$ contains A_f.

Proof. Necessity. Let be $I(A_f) \neq (f)$. Since $I(A_f) \supseteq (f)$ is always satisfied, then $I(A_f) \not\subseteq (f)$ takes place. Then let be $g(\overline{x},x) \in I(A_f) \setminus (f)$.

Since $f(\overline{x},x)$ the field $\mathbb{P}(\overline{x})$, is irreducible, the greatest common divisor c of polynomials f and g can be represented as $u(\bar{x},x)f(\bar{x},x) + v(\bar{x},x)g(\bar{x},x) = c(\bar{x})$, where$u(\bar{x},x), v(\bar{x},x) \in \mathbb{P}[\bar{x},x], c(\bar{x}) \in \mathbb{P}[\bar{x}]$. From the last relation we deduce that $c(\overline{x}) \in I(A_f)$. The necessity has been proved.

Sufficiency. Let be $c(\overline{x}) \in I(A_f)$ and $I(A_c) \supseteq I(A_f)$. We have a polynomial $c(\overline{x})$ as a polynomial of zero degree in x is not divisible by a polynomial $f(\overline{x},x)$ of degree not less than the first in the same variable, therefore $c(\overline{x}) \in I(A_c) \setminus I(A_f)$.

Sufficiency has been proved.

Consider an example of the mismatch of ideals $I(A_f)$ and (f).

EXAMPLE 1. An example of a simple algebraic extension of the non-companion field of rational numbers. Consider in an affine space $\mathbb{Q}^3$ the curve S, given by the intersection of the cylinder $x^2 + y^2 - 1 = 0$ and the plane$x + y + z = 0$.

The curve S over the field $\mathbb{Q}$ can be equivalently given as the annihilator of $A_f = \{ (a,b,c) | f(a,b,c) = 0, a,b,c \in \mathbb{Q}\}$, of the polynomial $f(x,y,z) = (x^2+y^2-1)^2 + (x+y+z)^2$ over the field $\mathbb{Q}$. Let us write the polynomial $f(x,y,z)$ in powers of the variable $z : f(x,y,z) = z^2 + 2(x+y)z + x^4 + y^4 + 2x^2y^2 - x^2 - y^2 + 2xy + 1$. Let us prove that $f(x,y,z)$ is irreducible. Suppose that $f(x,y,z) = (z - p(x,y))(z - q(x,y))$. Obviously $p(x,y) \neq q(x,y)$, , then for a pair $(a,b) \in \mathbb{Q}$ such that $p(a,b) \neq q(a,b)$ is fulfilled by $f(a,b,p(a,b)) = 0$ and $f(a,b,q(a,b)) = 0$. Since from the equation $x+y+z=0$, the value of $c=-a-b$ is uniquely determined, then $p(a,b)=q(a,b)$. A contradiction, thus $f(x,y,z)$ is irreducible.

Thus, the cylindrical surface A_g where $g = x^2 + y^2 - 1$ in the affine space $\mathbb{Q}^3$ contains A_f, therefore, $I(A_f) \neq (f)$ and by Proposition 1 and Theorem 2, we obtain a simple extension $\mathbb{Q}(x,y)[z]/f$ of the field $\mathbb{Q}(x,y)$ that is not a companion of the field $\mathbb{Q}$.

An immediate consequence of Proposition 1 and Theorem 2 is the following description, in terms of ideals, of a simple algebraic extension $\mathbb{P}(\bar{x})[x]/f$ of the field $\mathbb{P}(\bar{x})$, that is a companion of the field $\mathbb{P}$.

THEOREM 3. Let $f(\bar{x},x)$ be an irreducible polynomial over a field $\mathbb{P}(\bar{x})$. An algebraic extension $\mathbb{P}(\bar{x})[x]/f$ of the field $\mathbb{P}(\bar{x})$ is a companion $\mathbb{P}$ if and only if the ideal $I(A_f)$ is the same as the ideal (f).

Algebraic extensions

Let $\mathbb{P}(\bar{x})[y]/f$ and $\mathbb{P}(\bar{x})[y]/f/g$ (here $\mathbb{P}(\bar{x})[y]/f/g = (\mathbb{P}(\bar{x})[y]/f)[z]/g$ be simple algebraic extensions of the fields $\mathbb{P}(\bar{x})$ and $\mathbb{P}(\bar{x})[y]/f$ respectively, and be irreducible polynomials over the fields $\mathbb{P}(\bar{x})$ and $\mathbb{P}(\bar{x})[y]/f$ respectively. Let be $I(A_{fg}) = \{\, h | h \in \mathbb{P}(\bar{x})[y,z], A_{fg} \subseteq A_h\}$, $A_{fg} = \{(\bar{a},b,c) \in \mathbb{P} | (\bar{a},b) \in A_f, (\bar{a},b,c) \in A_g\}$

It is obvious that $I(A_{fg})$ is an ideal in the ring $\mathbb{P}(\bar{x})[y,z]$, generated by polynomials $f(\bar{x},y)$ и $g(\bar{x},y,z)$ i.e. $I(A_{fg}) = (f,g)$.

Consider now an algebraic extension of a simple algebraic extension of the field $\mathbb{P}(\bar{x})$.

THEOREM 4. The algebraic extension $\mathbb{Q}(\bar{x})[y]/f/g$ of the field $\mathbb{P}(\bar{x})[y]/f$ is a companion $\mathbb{P}$ if and only if, for any polynomial $h(\bar{x},y,z) \in \mathbb{P}(\bar{x})[y,z]$ such that $h(\bar{x},y,z) \not\in (f,g)$ there is a tuple $(\bar{a},b,c) \in \mathbb{P}$, such that $(\bar{a},b,c) \in A_{fg} \setminus A_h$.

Proof. Necessity. In the algebraic extension $\mathbb{P}(\bar{x})[y]/f/g$ of the field $\mathbb{P}(\bar{x})$, the system $f(\bar{x},y_f)=0 \,\&\, g(\bar{x},y_f,z_g)=0 \& h(\bar{x},y_f,z_g) \neq 0$ is satisfied; therefore, there is a tuple $(\bar{a},b,c) \in \mathbb{P}$ $(\bar{a},b,c) \in A_{fg} \setminus A_h$.

Sufficiency. Let $\mathbb{P}(\bar{x})[y]/f/g \models \exists u \ldots \exists v \varphi(\bar{x},y_f,z_g,u,\ldots,v)$ be fulfilled. Since $u,\ldots,v \in \mathbb{P}(\bar{x})[y]/f/g$ we assume that this formula is equivalent in the algebraic extension $\mathbb{P}(\bar{x})[y]/f/g$ of the field $\mathbb{P}(\bar{x})$ to the system $h_1(\bar{x},y_f,z_g)=0 \& h_2(\bar{x},y_f,z_g) \neq 0$. Then, there is a tuple $(\bar{a},b,c) \in \mathbb{P}$ such that $(\bar{a},b,c) \in A_{fg} \setminus A_h$. The necessity, and with it the theorem, has been proved.

THEOREM 5. An algebraic extension $\mathbb{Q}(\bar{x})[y]/f/g$ of a field $\mathbb{P}(\bar{x})[y]/f$ is a companion of $\mathbb{P}$ if and only if the ideal $I(A_{fg})$ is the same as the ideal (f,g).

Proof. Necessity. Let be $h(\bar{x},y,z) \not\in (f,g)$. Then, in the algebraic extension $\mathbb{P}(\bar{x})[y]/f/g$ of the field $\mathbb{P}(\bar{x})$ the system $f(\bar{x},y_f) = 0 \& g(\bar{x},y_f,z_g)=0 \& h(\bar{x},y_f,z_g) \neq 0$ is satisfied. Hence, in the companion $\mathbb{P}$ there is a tuple $(\bar{a},b,c) \in \mathbb{P}$, so that $(\bar{a},b,c) \in A_{fg} \setminus A_h$ and $h(\bar{a},b,c) \notin (f,g)$ hence. If $h(\bar{x},y,z) \in (f,g)$, then obviously $h(\bar{x},y,z) \in I(A_{fg})$. The necessity has been proved.

Sufficiency. Let $\mathbb{P}(\bar{x})[y]/f/g \models \exists u \ldots \exists v \varphi(\bar{x},y_f,z_g,u,\ldots,v)$ be fulfilled. Since $u,\ldots,v \in \mathbb{P}(\bar{x})[y]/f/g$ we will assume that this formula is equivalent in the algebraic extension $\mathbb{P}(\bar{x})[y]/f/g$ of the field $\mathbb{P}(\bar{x})$ to the system $h_1(\bar{x},y_f,z_g)=0 \& h_2(\bar{x},y_f,z_g) \neq 0$. By definition, it follows from $h_2(\bar{x},y_f,z_g) \neq 0$ that $h_2(\bar{x},y,z) \notin I(A_{fg})$. Then it follows from the equality $I(A_{fg}) = (f,g)$, that there is a tuple $(\bar{a},b,c) \in \mathbb{P}$, such that $(\bar{a},b,c) \in A_{fg} \setminus A_h$ and satisfies $\mathbb{P} | = h_1(\bar{a},b,c)=0 \& h_2(\bar{a},b,c) \neq 0$. Sufficiency, and with it the theorem has been proved.

Let us formulate a description of the companions of the field $\mathbb{P}$ in the general case.

Let $B = \{\beta_1, \beta_2, \ldots\}, X = \{x_1, x_2, \ldots\}$ be countable sets of independent variables, $\bar{\beta} = (\beta_1, \ldots, \beta_m)$, $\bar{x}_n = (x_1, \ldots, x_n)$, $\bar{f}_n = (f_1, \ldots, f_n)$, where $f_i(\bar{\beta}, \bar{x}_{i-1}^{\bar{f}_{i-1}}, x_i)$ is irreducible in the ring $\mathbb{P}(\bar{\beta})[\bar{x}_{i-1}^{\bar{f}_{i-1}}][x_i]$, where $\bar{x}_i^{\bar{f}_i} = (x_1^{f_1}, \ldots, x_i^{f_i})$, $x_i^{f_i}$ is the root of the polynomial $f_i(\bar{\beta}, \bar{x}_{i-1}^{\bar{f}_{i-1}}, x_i) \in \mathbb{P}(\bar{\beta})[\bar{x}_{i-1}^{\bar{f}_{i-1}}][x_i]$ over the field $f_i(\bar{\beta}, \bar{x}_{i-1}^{\bar{f}_{i-1}}, x_i) \in \mathbb{P}(\bar{\beta})[\bar{x}_{i-1}^{\bar{f}_i}$ $\bar{\beta})[\bar{x}_i^{\bar{f}_i}]$ is defined as follows. Let's put $\mathbb{P}(\bar{\beta})[\bar{x}_1^{\bar{f}_1}] = \mathbb{P}(\bar{\beta})[x_1]/f_1(\bar{\beta}, x_1)$, $\mathbb{P}(\bar{\beta})[\bar{x}_2^{\bar{f}_2}] = \mathbb{P}[\bar{\beta}][\bar{x}_1^{\bar{f}_1}][x_2]/f_2(\bar{\beta}, \bar{x}_1^{\bar{f}_1}, x_2)$. We define $\mathbb{P}(\bar{\beta})[\bar{x}_n^{\bar{f}_n}] = \mathbb{P}(\bar{\beta})[\bar{x}_{n-1}^{\bar{f}_{n-1}}][x_n]/$

$f_n(\bar{\beta}, \bar{x}_{n-1}^{\bar{f}_{n-1}}, x_n)$ by induction. Thus, in the sequence $\mathbb{P}(\bar{\beta})[\bar{x}_1^{\bar{f}_1}], \mathbb{P}(\bar{\beta})[\bar{x}_2^{\bar{f}_2}], \dots, \mathbb{P}(\bar{\beta})[\bar{x}_n^{\bar{f}_n}]$ – each subsequent field $\mathbb{P}(\bar{\beta})[\bar{x}_{i+1}^{\bar{f}_{i+1}}]$ is a simple algebraic extension of the previous field by means of an irreducible polynomial $f_{i+1}(\bar{\beta}, \bar{x}_i^{\bar{f}_i}, x_{i+1})$. We define the corresponding ideals $I(A_{\bar{f}_n}) = \{ h \in \mathbb{P}(\bar{\beta})[\bar{x}_n^{\bar{f}_n}] | A_{\bar{f}_n} \subseteq A_h\}, A_{\bar{f}_n} = \{(\bar{a}, \bar{b}_n) \in \mathbb{P}? | (\bar{a}, b_{,1}) \in A_{f_1}, \dots, (\bar{a}, b_1, \dots, b_n) \in A_{f_n}, \bar{b}_n = (b_1, \dots, b_n)\}$, where each $f_i \in \mathbb{P}(\bar{\beta})[\bar{x}_{i-1}^{\bar{f}_{i-1}}][x_i], i = 1, \dots, n$ is irreducible over the corresponding field $\mathbb{Q}(\bar{\beta})[\bar{x}_{i-1}^{\bar{f}_{i-1}}]$.

Let us give a general description of the companions of the field $\mathbb{P}$.

THEOREM 6. An algebraic extension $\mathbb{P}(\bar{\beta})[\bar{x}_n^{\bar{f}_n}] = \mathbb{P}(\bar{\beta})[\bar{x}_{n-1}^{\bar{f}_{n-1}}][x_n] / f_n(\bar{\beta}, \bar{x}_{n-1}^{\bar{f}_{n-1}}, x_n)$ of a field $\mathbb{P}(\bar{\beta})$ is a companion $\mathbb{P}$ if and only if the ideal $I(A_{\bar{f}_n})$ coincides with the ideal $(f_1, \dots, f_n)$.

The proof is a general reproduction of the proof of Theorem 5.

Discussion of results

The above results give a fairly complete description of the companions of the fields of rational and real numbers. Construction methods can be used for further studies of field companions and their classes.

Conclusion

The general theory of Fraisse's companion classes and their theories, developed by A.T. Nurtazin, constitutes a separate new area in model theory. This approach, applied to specific classical structures and their theories, provides new tools for the study of these objects. The study of the companion class of rational and algebraic real number fields reveals companion fields containing transcendental and possibly algebraic elements with special properties of polynomials defining these elements. The companions of each of the above-named fields are algebraic extensions of the fields of quotients of a certain set of independent variables over the corresponding field, using mutually agreed polynomials.

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References

1. Barwise J. Robinson A. (1970) Completing theories by forcing, Ann. Math. Logic, 2:119-142. DOI: 10.1016/0003-4843(70)90008-2 (in English).
2. Robinson A. (1951) On the Metamathematics of Algebra. Amsterdam, North-Holland, 540.
3. Belegradek O.V. (1974) Algebraically closed groups, Algebra i Logika, 13:239- 255. DOI: 10.1007/BF01463347 (in Russian).
4. Cohen P.J. (1966) Set Theory and the Continuum Hypothesis. NY, Benjamin, 467.
5. Ershov Yu.L., Palyutin E.A., Taimanov A.D. (1982) Model Theory, A Handbook of Mathematical Logic. Moscow, 492.
6. Fraisse R. (1955) Sur quelques classifications des systemes de relations, Publ.scient. de l'univ. d'Algers, 35-182 (in English).
7. Macintyre A. (1972) On algebraically closed groups, Ann. Math., 96:53-97. DOI: 10.2307/1970894 (in English).
8. Macintyre A. (1972) Omitting quantifier-free types in generic structures, Journal of Symbolic Logic, 37: 512 – 520. DOI: 10.2307/2272737 (in English).
9. Nurtazin A.T. (2016) Countable infinite existentially closed models of universally axiomatizable theories, Siberian Advances in Mathematics, 26: 99-125. DOI: 10.3103/S1055134416020036 (in English).
10. Nurtazin A.T. (2018) Properties of Existentially Closed Models [Svojstva jekzistencial'no zamknutyh modelej], Algebra i Logika, 57: 321–327 (in Russian).
11. Nurtazin A.T. (2018) Properties of exictentionally cloused companions, Algebra and Logic, 57: 211–221 (in English).

12. Nurtazin A.T. (2018) Forcing formulas in Frase structures and classes [Vynuzhdenie formul v strukturah i klassah Frjese], Algebra i Logic, 57: 567–586 (in Russian).

13. Nurtazin A.T. (2018) Forcing formulas in Fraise structures and classes, Algebra and Logic, 57: 368–380 (in English).

14. Nurtazin A.T. (2015) Countable Existentially Closed Models of Universally Axiomatizable Theories [Schetnye jekzistencial'no zamknutye modeli universal'no aksiomatiziruemyh teorij], Matematicheskie Trudy, 18: 48-97 (in Russian).

15. Nurtazin A.T. (2017) Introduction to model theory, quantifier elimination and existential closure [Vvedenie v teoriju modelej, jeliminacija kvantorov i jekzistencial'naja zamknutost'], Almaty: NTs GNTE, 87 (in Russian).

16. Nurtazin A.T. (2018) Any forcing type is implemented in some forcing model. Fourteenth intl. Asian school-seminar "Problems of optimization of complex systems", Issyk-Kul, Kyrgyzstan. 2018. . 109–113 (in Russian).

17. Khisamiev Z.G., Alimzhanova S.A. (2019) Companions of the ring of integers. Mater.scient.conf. IIVT MES RK "Modern problems of informatics and computing technologies", Almaty, Kazakhstan. 2019. P.361–365 (in Russian).

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