IRSTI 29.15.03 https://doi.org/10.26577/ijmph.2022.v13.i2.06

A. Muratkhan[1], A. Orazymbet[1], M. Zhakipova[1], M. Assylbek[2], S. Toktarbay[1,2*]

[1]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[2]Kazakh National Women's Teacher Training University, Almaty, Kazakhstan
*e-mail: saken.yan@yandex.ru
(Received 8 September 2022; accepted 14 December 2022)

A shadows from the static black hole mimickers

Abstract. In this work, we study shadows from the naked singularity spacetime. The most analytical solutions of black hole shadows only investigated the case that the geodesic equations for photons can separate variables. We review the spherical null naked singularity metric and this spherically symmetric naked singularity spacetime metric is the solution of Einstein equations with an anisotropic fluid source which has no photon sphere. We also review a static, axially-symmetric singular solution of the vacuum Einstein's equations without an event horizon which is can be used to describe the exterior gravitational field of a mass distribution with quadrupole moment. Moreover, the corresponding spacetime is characterized by the presence of naked singularities. It is theoretically known that not only a black hole can cast shadow, but other compact objects such as naked singularities, gravastar or boson stars can also cast shadows. We present the analytical calculation of shadows for both naked singularities spacetime and compare with the shadow of Schwarzschild static black hole, we show that this can serve as a black hole mimicker.
Keywords: compact object, naked singularity, shadow.

Introduction

By the Event Horizon Telescope (EHT) collaboration have unveiled the first image of the supermassive black hole shadow at the centre of our own Milky Way galaxy [1]. The Event Horizon Telescope (EHT) has mapped the first image of a black hole at the centre of the more distant Messier 87 galaxy in 2019 [2]. However, the images of two black holes similar, even they from the two completely different types of galaxies and two very different black hole masses. These results allows us to tests and verify of gravity theories and corresponding black hole solutions near a regime of the gravitational field for which the validity of General Relativity (GR). Therefore, it is important to consider any theory or calculation that satisfies the observational results in order to understand the nature of the geometry in the vicinity of an astrophysical black hole candidate and to test the validity of black hole hypotheses.

Black hole mimickers are possible alternatives for black holes, they would look observationally almost like black holes but would have no horizon. The properties in the near-horizon region where gravity is strong can be quite different for both type of objects, but at infinity it could be difficult to discern black holes from their mimickers.

In [3] it was provide a review of the current state of the research of the black hole (BH) shadow, focusing on analytical studies (see [4–7]. A black hole captures all light falling onto it and it is not possible to obtain a direct image of them, an observer will see a dark spot in the sky where the BH is supposed to be located. Due to the strong bending of light rays by the Black Hole gravity, both the size and the shape of this spot are different from what we naively expect on the basis of Euclidean geometry from looking at a non-gravitating black ball. Also, the authors [3] tried to give a complete list that have historically been used to refer to the visual appearance of a black hole and related concepts and they noted that despite the different names and different physical formulation of the problem, all these concepts are strongly intertwined. The word 'shadow' in different languages has several meanings. In the case of the BH shadow, it can be understood as a dark silhouette of the BH against a bright background which, however, is strongly influenced by the gravitational bending of light.

In [8], the authors construct a space-time configuration that has a central naked singularity, but without photon sphere, and it can give both a shadow and a negative perihelion precession. Their results imply that if the presence of a shadow and positive perihelion 2 precession implies either a black hole or a naked singularity, the presence of a shadow and negative perihelion precession simultaneously would only imply a naked singularity.

This work is organized as follows. In Sec.II we review the metric of null naked singularity spacetime which is the solution of Einstein's field equations with an anisotropic fluid source, we calculate the shadows from this spacetime in section III and using the same procedure, in section IV we investigate the shadows in the axisymmetric spacetime. Finally, Sec. V contains a summary of our results.

Spherical symmetric null naked singularity

There has been a significant amount of work regarding the singular spacetimes and a lot of literature where timelike, lightlike geodesics around the black hole and naked singularity are investigated. Generally, shadow is considered to be formed due to the existence of a photon sphere outside the event horizon of a black hole.

The line element representation for null naked singularity given by [9],

$$ds^2 = \frac{dt^2}{\left(1+\frac{M}{r}\right)^2} + \left(1+\frac{M}{r}\right)^2 dr^2 + \\ +r^2(d\theta^2 + sin^2\theta\, d\phi^2), \tag{1}$$

where M is the Arnowitt-Deser-Misner (ADM) mass of the above spacetime. The expression

of the Kretschmann scalar and Ricci scalar for this spacetime are:

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{4M^2((M-2r)^2r^4 + 4(M+r)^2r^4 + (M+r)^4(M+2r)^2)}{r^4(M+r)^8}, \tag{2}$$

$$R = \frac{2M^3(M+4r)}{r^2(M+r)^4}. \tag{3}$$

From the above expressions of the Kretschmann scalar and Ricci scalar it can be seen that the spacetime has a strong curvature singularity at the center $r = 0$. No null surfaces such as an event horizon exist around the singularity in this spacetime.

This metric this is the solution of Einstein's field equations with an anisotropic fluid source The energy-momentum tensor for anisotropic fluid given as

$$T_{ab} = diag(\rho, p_r, p_\theta, p_\varphi). \tag{4}$$

The solutions of EFE for the energy density and pressures as:

$$\rho = -T^t{}_t = \frac{M^2(M+3r)}{r^2(M+r)^3} \tag{5}$$

$$p_r = T^r{}_r = -\frac{M^2(M+3r)}{r^2(M+r)^3} \tag{6}$$

$$p_\theta = p_\varphi = T^\theta{}_\theta = T^\varphi{}_\varphi = \frac{3M^2}{r^4}\left(1+\frac{M}{r}\right)^{-4} \tag{7}$$

and it is also shown that this metric satisfies all energy conditions, i.e. strong, weak and null energy conditions [6]; The anisotropy in the pressures is:

$$p_r - p_\theta = -\frac{M^2(M^2+4Mr+6r^2)^2}{r^2(M+r)^4} \tag{8}$$

The equation of state (α) for an anisotropic fluid can be written as:

$$\alpha = \frac{p_r + p_\theta + p_\varphi}{3\rho} \tag{9}$$

from the equations eq.(5- 7) the equation of state for this spacetime as

$$\alpha = \frac{2}{\left(3+\frac{M}{r}\right)\left(1+\frac{M}{r}\right)} - \frac{1}{3} \tag{10}$$

where if r tends to zero, equation of state becomes $-1/3$; if r tends to infinity, equation of state becomes $+1/3$.

From the above equation (1) the expansion of component of metric tensor can be written as

$$g_{tt} \approx -\left[1 - \frac{2M}{r} + 3\left(\frac{M}{r}\right)^2 - \dots\right], \quad (11)$$

it is clear that in the large r limit, this metric is symptotically flat. Even though the metric resembles the Schwarzschild metric at a large distance, near the singularity, the causal structure of this spacetime becomes different from the causal structure of Schwarzschild spacetime.

Shadows of null naked singularity

Even though the metric resembles the Schwarzschild metric at a large distance, near the singularity, the causal structure of this spacetime becomes different from the causal structure of Schwarzschild spacetime. The most analytical solutions of black hole shadows only investigated the case that the geodesic equations for photons can separate variables. For example, In Kerr black hole space-time for the null geodesics has a third motion of constant, namely the Carter constant which is can be found by the calculation of HamiltonJacobi equation, except for the energy E and the z-component of the angular momentum L_z and the photon motion system is integrable.[11].

Let's rewrite the line element 1 in this form:

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + +r^2(d\theta^2 + sin^2\,\theta\, d\phi^2), \quad (12)$$

where the functions

$$A(r) = \frac{1}{\left(1 + \frac{M}{r}\right)^2}, \quad (13)$$

The Hamilton of a photon is given by

$$H = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu, \quad (14)$$

The photon motions can be obtained from the Hamiltom equation

$$\dot{P}_\mu = -\frac{\partial H}{\partial x^\mu}, \dot{x}^\mu = \frac{\partial H}{\partial P_\mu}, \quad (15)$$

where $P_\mu = \frac{dx^\mu}{d\lambda}$ is the four-momentum of the photon, and λ is the affine parameter.

In additional, due to symmetries of the metric one can introduce two integrals of motion, corresponding to cyclic coordinates t and φ, i.e., the conserved quantities of energy and angular momentum, respectively.

$$-P_t = E, P_\varphi = L, \quad (16)$$

From the Hamiltonian 14 with the eq.16 we can reduce

$$P_r^2 = \frac{E^2}{A(r)^2} - \frac{1}{r^2 A(r)}\left(P_\theta + \frac{1}{\sin^2\theta} L^2\right). \quad (17)$$

Because of the spherical symmetry, we can choose the orbit of the photon in the equatorial plane, which means$\theta = \frac{\pi}{2}, P^\theta = 0$. Also, the orbit equation for lightlike geodesics is $dr/d\varphi$, then, using eq.16 and eq.17 the orbit equation becomes

$$\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{P_r}{P_\varphi}\right)^2 = \frac{1}{r^2 A(r)}\left(\frac{r^2}{A(r)}\frac{E^2}{L^2} - 1\right). \quad (18)$$

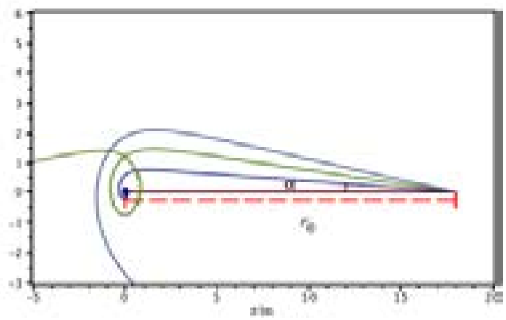

Figure 1 − Formation of a shadow in the case of a null naked singularity

We can see that eq.(18) is the same form as an energy conservation law in one-dimensional classical mechanics $(dr/d\varphi)^2 + V_{eff}(r) = 0$, where the effective potential depends on the impact parameter $b = L/E$.

According to (18), we can rewrite the effective potential for the metric (1)

$$V_{eff} = \frac{1}{r^2}\left(1+\frac{m}{r}\right)^2 - \frac{E^2}{L^2}\left(1+\frac{m}{r}\right)^4. \tag{19}$$

The unstable circular orbits of lightlike geodesics can be found when the equations for effective potential

$$V_{eff} = 0, V_{eff,r} = 0, V_{eff,rr} < 0, \tag{20}$$

From the above equation, one can determine the impact parameter b with a minimum radius of circular orbit

$$\frac{R^2}{A(R)} = b^2, \tag{21}$$

Let us also introduce the function

$$h(r)^2 = \frac{r^2}{A(r)}, \tag{22}$$

it is clear that the impact parameter and the function $h(r)$ are related by $b = h(R)$, then the equation 18 can be rewrite as a following

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{1}{r^2 A(r)}\left(\frac{h(r)^2}{h(R)^2} - 1\right). \tag{23}$$

Assume that a static observer at radius coordinate r_O sends light rays into the past.

Then, the angle α between such a light ray and the radial direction is can be calculated by

$$\cot\alpha = \frac{\sqrt{g_{rr}}}{\sqrt{g_{\varphi\varphi}}}\frac{dr}{d\varphi}\Big|_{r=r_0} = \frac{\frac{1}{A(r)^{1/2}}}{r}\frac{dr}{d\varphi}\Big|_{r=r_0}, \tag{24}$$

from the eq.(23) and eq.(24) we obtain

$$\cot^2\alpha = \frac{h\ (R)^2}{h\ (r_0)^2} - 1, \tag{25}$$

By elementary trigonometry, we get

$$\sin^2\alpha = \frac{h\ (R)^2}{h\ (r_0)^2}, \tag{26}$$

or

$$\sin^2\alpha = \frac{(R+m)^2}{(r_0+m)^2}. \tag{27}$$

From the condition in eq. (20), in the null naked singularity spacetime, the minimum turning point radius (r_{tp}) of the photon is $r_{tp} = R = 0$, then for an observer the angular size of the shadow is

$$\sin\alpha = \frac{m}{r_0+m}, \tag{28}$$

Then for an observer at a large distance the angular size

$$\alpha \approx \frac{m}{r_0}, \tag{29}$$

Synge calculated the shadow in the Schwarzschild spacetime as [8]

$$\sin^2\alpha_{Sch} = \frac{27m^2}{r_0^2}\left(1-\frac{2m}{r_0}\right). \tag{30}$$

For large distances we have:

$$\alpha_{Sch} \approx \frac{3\sqrt{3}m}{r_0}. \tag{31}$$

Shadow in *q*-metric

In spherical coordinates, the $q-$ metric can be written in a compact and simple form as

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + \\ +g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \tag{32}$$

where

$$g_{tt} = \left(1-\frac{2m}{r}\right)^{1+q},$$
$$g_{rr} = g_{tt}{}^{-1}\left(1+\frac{m^2\sin^2\theta}{r^2-2mr}\right)^{-q(2+q)},$$
$$g_{\theta\theta} = r^2 g_{rr}, g_{\phi\phi} = \left(1-\frac{2m}{r}\right)^{-q} r^2\sin^2\theta. \tag{33}$$

This metric is the simplest static vacuum solutions of Einstein's filed equations with quadrupole investigated in [13] and the geometric properties of the metric analyzed in detail. In the literature, this metric is known as the Zipoy–

Voorhees metric, δ - metric, γ - metric and q-metric [14–21]. Interior solutions of Einstein's field equations was found in [22] and the new generating method with the perfect fluid source presented in [23] which includes the multipole moments. Consider that the orbit of the photon in the equatorial plane. The first integral of timelike geodesic equation is

$$g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}=0. \tag{34}$$

Hence

$$-g_{tt}\dot{t}^2+g_{rr}\dot{r}^2+g_{\varphi\varphi}\dot{\varphi}^2=0. \tag{35}$$

We have used the expression for the energy E and the angular moment L which are constants of motion that associated with the Killing vector fields $\xi_t=\partial_t$ and $\xi_\varphi=\partial_\varphi$, respectively.

Consider the the boundary curve of the shadow corresponds to past-oriented light rays that asymptotically approach one of the unstable circular light orbits at radius r_{ph}. Therefore we have to consider the limit $R\rightarrow r_{ph}$ in (26) for getting the angular radius α_{sh} of the shadow

by the same procedure as in section III,

$$\sin^2\alpha_{sh}=\frac{h(r_{ph})^2}{h(r_0)^2}. \tag{36}$$

If we consider that the parameter $q\neq 0$, then the r_{ph} is

$$r_{ph}=(3+2q)m. \tag{37}$$

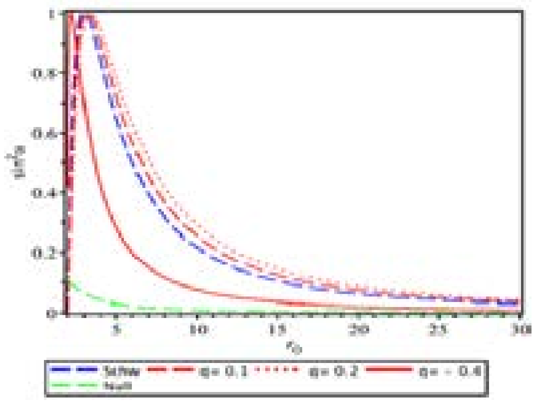

Figure 2 − Angular size of shadows in a different scenarios as a function of $\sin\alpha_{sh}^2$, for Schwarzschild (blue), q − metric (red) and for null naked singularity (green)

The critical value b_{cr} of the impact parameter is connected with r_{ph} by $b_{cr}=h(r_{ph})$

$$b_{cr}=m(2q+1)^{-q-\frac{1}{2}}(2q+3)^{3+\frac{3}{2}}, \tag{38}$$

the final calculation of angular radius of the shadow in the q–metric space-time is

$$\sin^2\alpha_{sh}=\frac{m^2}{r_0^2}(1+2q)^{-1-2q}\times$$
$$\times(3+2q)^{3+2q}\left(1-\frac{2m}{r_0}\right)^{1+2q}. \tag{39}$$

It is clear that when $q=0$ it is reduce to the radius of photon sphere in the Schwarzschild spacetime, i.e., $r_{ph}=3m$ and, after substitution into (36) we can find the angular radius α_{sh} of the shadow in the Schwarzschild spacetime.

In figure 2 shown the angular size of shadows in a different scenarios. For the large observer at r_0 they have not same angular size. The near the naked singularity apacetime, the size of shadows are quite different and the the size of shadow from the null naked singularity has a small angular size. At the large distance, Eq.(40) becomes

$$\alpha_{sh}=\frac{m}{r_0}(1+2q)^{-\frac{1}{2}-q}(3+2q)^{\frac{3}{2}+q}. \tag{40}$$

If q = 0, for large distances the shadow can be approximated and the expression reduce to the similar angular size of shadow in Schwarzschild spacetime as $\dfrac{3\sqrt{3}}{r_0}$.

Conclusions and remarks

In this work, we reviewed the null naked singularity solution of Einstein's filed equations that absence of photon sphere and calculated shadow size for a static observer. The angular size of the shadow in any spherically symmetric and static metric, for any position r_0 of a static observer, can be calculated in the simple manner.

We calculated the size of shadows in Null naked singularity and static q-metric spacetime. The near the naked singularity apacetime, a size of shadows are quite different from the Schwarzschild spacetime, the null naked singularity spacetime has a shadow with small angular size. For the q-metric spacetime, the size of the shadow directly depends the value of quadrupole parameter. Near the naked singularity, the quantum gravity effects should be dominant, and therefore, such quantum gravity effects might be manifested or can be observed in the shadow cast by a naked singularity. This will require a detailed analysis of the various features encoded in such shadows.

For the large distance observer at r_0 the null naked singularity and q-metric spacetime asymptotically resembles the Schwarzschild spacetime. As a result, the null naked singularity and static q-metric spacetime can be thought of as a Schwarzschild black hole mimickers.

Reference

1. Akiyama, E.A. " Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way." *The Astrophysical Journal Letters* Volume 930, Number 2. (2022):L12.
2. Akiyama, E.A. " Event Horizon Telescope Results. V. Physical Origin of the Asymmetric Ring." *The Astrophysical Journal Letters* Volume 875, Number 1. (2019):L5.
3. Perlick, V., Tsupko, O.Y., "Light propagation in a plasma on an axially symmetric and stationary spacetime: Separability of the Hamilton–Jacobi equation and shadow." *Journal of Mathematical Physics* Volume 947, Number 1. (2022):L9.
4. Dey, D., Shaikh, R., Joshi, P.S., "Perihelion precession and shadows near black holes and naked singularities." *Physical Review D* (2020).
5. Joshi, A.B., Dey, D., Joshi, P.S., "Shadow of a naked singularity without photon sphere", *Physical Review D*. Volume 102, (2020):L3-1.
6. Madan, S., Bambhaniya, P., "Tidal force effects and periodic orbits in null naked singularity spacetime." *General Relativity and Quantum Cosmology* (2022).
7. Carter, B., "Global Structure of the Kerr Family of Gravitational Fields." *Physical Review Journals Archive* Volume 174, 1559 (1968):L5.
8. Synge, J.L., "The Escape of Photons from Gravitationally Intense Stars." *Monthly Notices of the Royal Astronomical Society* Volume 131, Pages 463–466, (1966):L3.
9. Toktarbay, S. , Quevedo, H., Abishev, M., Muratkhan, A., "Gravitational field of slightly deformed naked singularities." *European Physical Journal C* Volume 82, 382 (2022).
10. Zipoy, D.M., "Topology of some spheroidal metrics." *Journal of Mathematical Physics* Volume 7, 1137–1143 (1966).
11. Voorhees, B.H., "Static axially symmetric gravitational fields." *Phys. Rev. D* Volume 2, 2119–2122 (1970).
12. Malafarina, D., Magli, G., Herrera, L., in *General Relativity and Gravitational Physics*, *American Institute of Physics Conference Series*, Volume 751, ed. by Espositio, G., Lambiase G., Marmo, G., Scarpetta, G., Vilasi G. (2005), *American Institute of Physics Conference Series*, Volume 751, (1966).
13. Abishev, M., Boshkayev, K., Quevedo, H., Toktarbay, S., "Accretion disks around a mass with quadrupole." *Gravitation, Astrophysics and Cosmology,* (2016).
14. Boshkayev, K., Gasper´ E., ın, Guti´errez- Pin˜eres, A.C., Quevedo, H., Toktarbay, S., "Motion of test particles in the field of a naked singularity." *Physical Review D* Volume 93, (2016):L2.
15. Boshkayev, K., Quevedo, H., Toktarbay, S., Zhami, B., Abishev, M., "On the equivalence of approximate stationary axially symmetric solutions of the Einstein field equations." *Grav. Cosmol.* Volume **22**(4), (2016).
16. Benavides-Gallego, C.A., Abdujabbarov, A., Malafarina, D., Ahmedov, B., Bambi, C, "Charged particle motion and electromagnetic field in Unknown node type: font spacetime." *Physical Review D* Volume 99, (2019):L4-15.
17. Turimov, B., Ahmedov, B., Koloˇs, M., "Axially symmetric and static solutions of Einstein equations with self-gravitating scalar field." *Physical Review D* Volume 98, (2018):L8-15.
18. Stewart, B.W., Papadopoulos, D., Witten, L., Berezdivin, R., Herrera, L., "An interior solution for the gamma metric." *General Relativity and Gravitation* Volume 14(1), 97 (1982).
19. Quevedo, H., Toktarbay, S., "Generating static perfect-fluid solutions of Einstein's equations." *J.Math. Phys.* Volume 56, (2015).

Funded by Al-Farabi KazNU.