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A novel numerical implementation for solving problem for loaded depcag

Abstract. In this work, we propose a new computational approach is implemented to solve a two-point BVP for a loaded differential equation with piecewise constant argument of generalized type (DEPCAG) based on the Dzhumabaev parameterization method. In this method, as parameters, we take the values of the desired solution at the partition points which are chosen taking into account the specifics of the equation. The problem under consideration reduces to an equivalent parametric problem for ordinary differential equations. A solution to this problem, which in turn are found from a system of linear algebraic equations. The found values of the parameters are used to determine the values of the unknown function at the remaining points of the interval by solving auxiliary initial-value problems. We develop a novel numerical implementation for solving the considered a two-point BVP for loaded DEPCAG and provide two examples illustrating its application, where Mathcad software will be used for the calculation.
Keywords: piecewise-constant argument of generalized type, loadings, two-point boundary-value problem, Dzhumabaev parametrization method, numerical solution.

1 Introduction and preliminaries

The study of loaded differential equations is of interest both from the practical and theoretical points of view, in mathematical modeling and in general mathematics itself. In [1-9], various boundary-value problems for loaded differential equations are studied and solved by different methods.

Differential equations with piecewise constant argument of generalized type, introduced by M. Akhmet [10], arise in modeling of diverse phenomena and are widely used in applications such as neural networks, hybrid systems, dynamic systems with discontinuities, etc. The theory of such equations has been extensively developed; see, for instance, [10-15]. However, there still remain open questions regarding boundary-value problems for such equations on a finite interval.

The parametrization method proposed by professor D. Dzhumabaev [16, 17] is an effective method of qualitative investigate and numerical solving BVPs for a wide class of differential and integro-differential equations.

This paper is concerned with solving numerically a two-point BVP for a system of loaded DEPCAG by the modification of Dzhumabaev's parametrization method.

We consider the following a two-point BVP for the system of loaded DEPCAG:

$$\dot{x}(t) = A_0(t)x + A_1(t)x(\theta_1) + \\ +K(t)x\big(\gamma(t)\big) + f(t), t \in (0,T), \quad (1)$$

$$B_0x(0) + C_0x(T) = d, x \in R^n, d \in R^n, \quad (2)$$

where $(n \times n)$-matrices $A_0(t)$, $A_1(t)$, and $K(t)$ are continuous on $[0,T]$, the n-vector-function $f(t)$ is piecewise continuous on $[0,T]$ with the possible discontinuity of the first kind at the point $t = \theta_1$; B_0 and C_0 are constant $(n \times n)$-matrices. Here $0 = \theta_0 < \theta_1 < \theta_2 = T$, $\|x\| = \max_{i=\overline{1,n}} |x_i|$.

The argument $\gamma(t)$ is a step function defined as

$$\gamma(t) = \xi_0 \text{ if } t \in [\theta_0, \theta_1); \theta_0 < \xi_0 < \theta_1,$$

and

$\gamma(t) = \xi_1$ if $t \in [\theta_1, \theta_2)$; $\theta_1 < \xi_1 < \theta_2$.

We will call a function $x(t)$ a solution to problem (1), (2) if:

(i) it is continuous on $[0, T]$ and differentiable on $(0, \mathrm{T})$ with the possible exception of the points θ_0 and θ_1, at which the one-sided derivatives exist;

(ii) it satisfies (1) on each interval (θ_{i-1}, θ_i), $i = \overline{1,2}$; at the points θ_0 and θ_1, Eq. (1) is satisfied by the right-hand derivatives of $x(t)$;

(iii) it satisfies the boundary condition (2).

2 A numerical algorithm for solving problem (1), (2)

We divide the interval $[0, T]$ as follows: $[0, T) = \bigcup_{r=1}^{4}[t_{r-1}, t_r)$. Here $t_0 = \theta_0$, $t_1 = \xi_0$, $t_2 = \theta_1$, $t_3 = \xi_1$, and $t_4 = \theta_2 = T$.

Let $C([0, T], \theta, R^{4n})$ denote the space of function quadruples $x[t] = (x_1(t), x_2(t), x_3(t), x_4(t))$, whose components $x_r: [t_{r-1}, t_r) \to R^n$ are continuous on $[t_{r-1}, t_r)$ and have finite limits $\lim_{t \to t_r - 0} x_r(t)$ for all $r = \overline{1,4}$. The space is equipped with the norm $\|x[\cdot]\|_2 = \max_{r=\overline{1,4}} \sup_{t \in [t_{r-1}, t_r)} \|x_r(t)\|$.

The restrictions of $x(t)$ to the partition subintervals, denoted by $x_r(t)$ ($x_r(t) = x(t)$ for $t \in [t_{r-1}, t_r)$, $r = \overline{1,4}$), satisfy the following multipoint boundary-value problem

$$\frac{dx_1}{dt} = A_0(t)x_1 + K(t)x_2(t_1) + \\ +A_1(t)x_3(t_2) + f_1(t),\ t \in [t_0, t_1), \tag{3}$$

$$\frac{dx_2}{dt} = A_0(t)x_2 + K(t)x_2(t_1) + \\ +A_1(t)x_3(t_2) + f_1(t),\ t \in [t_1, t_2), \tag{4}$$

$$\frac{dx_3}{dt} = A_0(t)x_3 + K(t)x_4(t_3) + \\ +A_1(t)x_3(t_2) + f_2(t),\ t \in [t_2, t_3), \tag{5}$$

$$\frac{dx_4}{dt} = A_0(t)x_4 + K(t)x_4(t_3) + \\ +A_1(t)x_3(t_2) + f_2(t),\ t \in [t_3, T), \tag{6}$$

$$B_0 x_1(0) + C_0 \lim_{t \to T-0} x_4(t) = d, \tag{7}$$

$$\lim_{t \to t_p - 0} x_p(t) = x_{p+1}(t_p),\ p = \overline{1,3}. \tag{8}$$

Here $f_1(t) = f(t)$ if $t \in [t_0, t_2)$ and $f_2(t) = f(t)$ if $t \in [t_2, t_4)$.

Applying the substitution $x_r(t) = w_r(t) + \mu_r$ on each r-th subinterval, with $\mu_r = x_r(t_{r-1})$, $r = \overline{1,4}$, we pass to the boundary-value problem with parameters μ_r:

$$\frac{dw_1}{dt} = A_0(t)(w_1 + \mu_1) + K(t)\mu_2 + \\ +A_1(t)\mu_3 + f_1(t),\ t \in [t_0, t_1), \tag{9}$$

$$w_1(t_0) = 0, \tag{10}$$

$$\frac{dw_2}{dt} = A_0(t)(w_2 + \mu_2) + K(t)\mu_2 + \\ +A_1(t)\mu_3 + f_1(t),\ t \in [t_1, t_2), \tag{11}$$

$$w_2(t_1) = 0, \tag{12}$$

$$\frac{dw_3}{dt} = A_0(t)(w_3 + \mu_3) + K(t)\mu_4 + \\ +A_1(t)\mu_3 + f_2(t),\ t \in [t_2, t_3), \tag{13}$$

$$w_3(t_2) = 0, \tag{14}$$

$$\frac{dw_4}{dt} = A_0(t)(w_4 + \mu_4) + K(t)\mu_4 + \\ +A_1(t)\mu_3 + f_2(t),\ t \in [t_3, T), \tag{15}$$

$$w_4(t_3) = 0, \tag{16}$$

$$B_0\mu_1 + C_0\mu_4 + C_0 \lim_{t \to T-0} w_4(t) = d, \tag{17}$$

$$\mu_p + \lim_{t \to t_p - 0} w_p(t) = \mu_{p+1},\ p = \overline{1,3}. \tag{18}$$

A pair $(w^*[t], \mu^*)$, whose components are $w^*[t] = (w_1^*(t), w_2^*(t), w_3^*(t), w_4^*(t)) \in C([0, T], \theta, R^{4n})$ and $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*) \in R^{4n}$, is called a solution to problem (9)-(18) if the functions $w_r^*(t)$, $r = \overline{1,4}$, are continuously differentiable on $[t_{r-1}, t_r)$ and satisfy equations (9), (11), (13), (15) with respective initial conditions and additional conditions (17), (18) with $\mu_j = \mu_j^*$, $j = \overline{1,4}$.

Let us show the equivalence between problems (1), (2) and (9)-(18). If a function $x^*(t)$ solves problem (1), (2), then the pair $(w^*[t], \mu^*)$, where $w^*[t] = (x^*(t) - x^*(t_0), x^*(t) - x^*(t_1), x^*(t) - x^*(t_2), x^*(t) - x^*(t_3))$ and $\mu^* = (x^*(t_0), x^*(t_1), x^*(t_2), x^*(t_3))$, is a solution of problem (9)-(15). Conversely, if a pair $(\tilde{w}[t], \tilde{\mu})$ with elements $\tilde{w}[t] = (\tilde{w}_1(t), \tilde{w}_2(t), \tilde{w}_3(t), \tilde{w}_4(t)) \in C([0, T], \theta, R^{4n})$ and $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4) \in R^{4n}$, is a solution of (9)-(18), then the function $\tilde{x}(t)$ defined as $\tilde{x}(t) = \widetilde{w}_r(t) + \tilde{\mu}_r, t \in [t_{r-1}, t_r), \mathrm{r} = \overline{1,4},$ and $\tilde{x}(T) = \lim\limits_{t \to T-0} \widetilde{w}_{m+1}(t) + \tilde{\mu}_{m+1}$, will be a solution of the original problem (1), (2).

Let $X_r(t)$ be a fundamental matrix of the equation $\frac{dx}{dt} = A_0(t)x$ on $[t_{r-1}, t_r]$, $\mathrm{r} = \overline{1,4}$. Then we can represent the solutions of initial-value problems (9)-(16) in the following form:

$$w_1(t) = X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)A_0(\tau)d\tau\,\mu_1 + X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)K(\tau)d\tau\,\mu_2 + $$
$$+X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)A_1(\tau)d\tau\,\mu_3 + X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)f_1(\tau)d\tau\,, t \in [t_0, t_1), \quad (19)$$

$$w_2(t) = X_2(t)\int_{t_1}^{t} X_2^{-1}(\tau)A_0(\tau)d\tau\,\mu_2 + X_2(t)\int_{t_1}^{t} X_2^{-1}(\tau)K(\tau)d\tau\,\mu_2 + $$
$$+X_2(t)\int_{t_1}^{t} X_2^{-1}(\tau)A_1(\tau)d\tau\,\mu_3 + X_2(t)\int_{t_1}^{t} X_2^{-1}(\tau)f_1(\tau)d\tau\,, t \in [t_1, t_2), \quad (20)$$

$$w_3(t) = X_3(t)\int_{t_2}^{t} X_3^{-1}(\tau)A_0(\tau)d\tau\,\mu_3 + X_3(t)\int_{t_2}^{t} X_3^{-1}(\tau)K(\tau)d\tau\,\mu_4 + $$
$$+X_3(t)\int_{t_2}^{t} X_3^{-1}(\tau)A_1(\tau)d\tau\,\mu_3 + X_3(t)\int_{t_2}^{t} X_3^{-1}(\tau)f_2(\tau)d\tau\,, t \in [t_2, t_3), \quad (21)$$

$$w_4(t) = X_4(t)\int_{t_3}^{t} X_4^{-1}(\tau)A_0(\tau)d\tau\,\mu_4 + X_4(t)\int_{t_3}^{t} X_4^{-1}(\tau)K(\tau)d\tau\,\mu_4 + $$
$$+X_4(t)\int_{t_3}^{t} X_4^{-1}(\tau)A_1(\tau)d\tau\,\mu_3 + X_4(t)\int_{t_3}^{t} X_4^{-1}(\tau)f_2(\tau)d\tau\,, t \in [t_3, T). \quad (22)$$

If we substitute the limit values for $w_r(t)$, $\mathrm{r} = \overline{1,4}$, present in conditions (17) and (18), by their corresponding expressions found from (19)-(22), we arrive at the system of linear algebraic equations in parameters μ_r, $r = \overline{1,4}$:

$$B_0\mu_1 + C_0\mu_4 + C_0a_4(A_0, T)\mu_4 + C_0a_4(K, T)\mu_4 + $$
$$+C_0a_4(A_1, T)\mu_3 = d - C_0a_4(f_2, T), \quad (23)$$

$$\mu_1 + a_1(A_0, t_1)\mu_1 + a_1(K, t_1)\mu_2 + a_1(A_1, t_1)\mu_3 -$$
$$-\mu_2 = -a_1(f_1, t_1), \quad (24)$$

$$\mu_2 + a_2(A_0, t_2)\mu_2 + a_2(K, t_2)\mu_2 + a_2(A_1, t_2)\mu_3 -$$
$$-\mu_3 = -a_2(f_1, t_2), \quad (25)$$

$$\mu_3 + a_3(A_0, t_3)\mu_3 + a_3(K, t_3)\mu_4 + a_3(A_1, t_3)\mu_3 -$$
$$-\mu_4 = -a_3(f_2, t_3). \quad (26)$$

Here by $a_r(P, t)$ we denote the unique solutions of the auxiliary initial-value problems

$$\frac{dz}{dt} = A_0(t)z + P(t), t \in [t_{r-1}, t_r),$$

$$z(t_{r-1}) = 0, \mathrm{r} = \overline{1,4}.$$

Let us rewrite system (23)-(26) in the matrix form:

$$\mathrm{Q}(\theta_1)\mu = -F(\theta_1), \mu \in R^{4n}\,, \quad (27)$$

where

$$F\ (\theta\ _1) = (-d$$
$$+ C_0a_4(f_2, T), a_1(f_1, t_1), a_2(f_1, t_2), a_3(f_2, t_3))$$

and

$$Q(\theta_1) =$$
$$= \begin{pmatrix} B_0 & O & C_0 a_4(A_1, T) & C_0[I + a_4(A_0, T) + a_4(K, T)] \\ I + a_1(A_0, t_1) & a_1(K, t_1) - I & a_1(A_1, t_1) & O \\ O & I + a_2(A_0, t_2) + a_2(K, t_2) & a_2(A_1, t_2) - I & O \\ O & O & I + a_3(A_0, t_3) + a_3(A_1, t_3) & a_3(K, t_3) - I \end{pmatrix},$$

here I and O are the identity matrix and the zero matrix, respectively, both of dimension n.

It may be verified without difficulty that the solvability of problem (1), (2) and that of system (27) are equivalent. The solution of system (27) is a vector $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*) \in R^{4n}$, whose components are $\mu_r^* = x^*(t_{r-1}), r = \overline{1,4}$. To find the values of the solution to problem (1),(2) at the remaining points of $[0, T]$, we plug the values μ_r^* into equations (3)-(6) and solve them as ordinary differential equations subject to the initial conditions $x^*(t_{r-1}) = \mu_r^*$.

Based on the above findings, we develop the following numerical algorithm for solving the boundary-value problem (1),(2).

1. Divide intervals $[t_{r-1}, t_r]$, $r = \overline{1,4}$, into M_r parts. Find the approximate values of the coefficients and the right-hand side of (27) by solving the following matrix and vector initial-value problems:

$$\frac{dz}{dt} = A_0(t)z + f_1(t), z(t_0) = 0, t \in [t_0, t_1);$$

$$\frac{dz}{dt} = A_0(t)z + f_1(t), z(t_1) = 0, t \in [t_1, t_2);$$

$$\frac{dz}{dt} = A_0(t)z + f_2(t), z(t_2) = 0, t \in [t_2, t_3);$$

$$\frac{dz}{dt} = A_0(t)z + f_2(t), z(t_3) = 0, t \in [t_3, t_4);$$

$$\frac{dz}{dt} = A_0(t)z + A_0(t), z(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r), r = \overline{1,4};$$

$$\frac{dz}{dt} = A_0(t)z + A_1(t), z(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r), r = \overline{1,4};$$

$$\frac{dz}{dt} = A_0(t)z + K(t), z(t_{r-1}) = 0, t \in [t_{r-1}, t_r), \quad r = \overline{1,4}.$$

2. Construct the linear in parameters

$$Q_*(\theta_1)\mu^* = -F_*(\theta_1), \mu^* \in R^{4n}. \tag{28}$$

Solve system (28) to find μ^*. As noted above, the components of $\mu^*=(\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*)$ are $\mu_r^* = x^*(t_{r-1}), r = \overline{1,4}$.

3. Solve the following initial-value problems

$$\frac{dz}{dt} = A_0(t)z + K(t)\mu_2^* + A_1(t)\mu_3^* + +f_1(t), z(t_0) = \mu_1^*, t \in [t_0, t_1];$$

$$\frac{dz}{dt} = A_0(t)z + K(t)\mu_2^* + A_1(t)\mu_3^* + +f_1(t), z(t_1) = \mu_2^*, t \in [t_1, t_2];$$

$$\frac{dz}{dt} = A_0(t)z + K(t)\mu_4^* + A_1(t)\mu_3^* + +f_2(t), z(t_2) = \mu_3^*, t \in [t_2, t_3];$$

$$\frac{dz}{dt} = A_0(t)z + K(t)\mu_4^* + A_1(t)\mu_3^* + +f_2(t), z(t_3) = \mu_4^*, t \in [t_3, t_4]$$

and determine the values of the solution $z^*(t)$ at the remaining points of the partition subintervals.

3 Examples

Example 1. Let us consider the following problem

$$\frac{dx}{dt} = \begin{pmatrix} t^2 & -3 \\ 5t & t+1 \end{pmatrix} x + \begin{pmatrix} 4 & t^3 \\ t & -4 \end{pmatrix} x(\gamma(t)) + + \begin{pmatrix} 1 & t \\ 4 & t^2 - 3 \end{pmatrix} x(1) + f(t), t \in (0,2), \tag{29}$$

$$\begin{pmatrix} 1 & 8 \\ 4 & 0 \end{pmatrix} x(0) + \begin{pmatrix} 5 & -4 \\ 8 & 9 \end{pmatrix} x(1) = \begin{pmatrix} 46 \\ 187 \end{pmatrix}, \quad x \in R^2. \tag{30}$$

Here if $t \in (0, 1)$: $\gamma(t) = \xi_0 = \frac{1}{2}$,

$$f(t) = \begin{pmatrix} \frac{31t^3}{8} - 5t^4 + 3t^2 + 10t + 2 \\ 3t^2 - 26t^3 - t^4 + \frac{71t}{4} - \frac{21}{2} \end{pmatrix};$$

if $t \in (1,2)$: $\gamma(t) = \xi_1 = \frac{3}{2}$,

$$f(t) = \begin{pmatrix} \frac{5t^3}{8} - 5t^4 + 3t^2 + 10t - 38 \\ 3t^2 - 26t^3 - t^4 + \frac{31t}{4} + \frac{5}{2} \end{pmatrix}.$$

The exact solution of problem (29), (30) is $x^*(t) = \begin{pmatrix} 5t^2 - 3 \\ t^3 - 1 \end{pmatrix}$.

To solve problem (29), (30) numerically, we implement the proposed algorithm. The interval $[0,2]$ is partitioned into the subintervals $\left[0, \frac{1}{2}\right]$, $\left[\frac{1}{2}, 1\right]$, $\left[1, \frac{3}{2}\right]$, $\left[\frac{3}{2}, 2\right]$. We take the step size $h = 0.05$ to numerically solve the auxiliary initial-value problems on the partition subintervals (step 1 of the algorithm).

Consider the system (28), where

$$Q_*(1) = \begin{pmatrix} 1 & 8 & 0 & 0 & -14.682896 & -10.74399 & -59.087058 & -8.667082 \\ 4 & 0 & 0 & 0 & -8.707397 & 28.771313 & 55.683542 & 50.097028 \\ 0.685574 & -1.788367 & 0.64101 & 1.749837 & -1.306734 & 1.398719 & 0 & 0 \\ 0.70852 & 0.992797 & 1.075304 & -3.217662 & 2.455186 & -1.576427 & 0 & 0 \\ 0 & 0 & 0.662779 & 0.307998 & -2.46031 & 1.374989 & 0 & 0 \\ 0 & 0 & 4.666998 & -1.497835 & 2.255643 & -1.566625 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.095413 & -0.532168 & -0.827204 & 2.513766 \\ 0 & 0 & 0 & 0 & 4.801938 & -0.492894 & 4.366706 & -0.105952 \end{pmatrix}, \quad F_*(1) = \begin{pmatrix} -548.417452 \\ 548.953597 \\ -5.534698 \\ 2.725685 \\ -6.349978 \\ -2.345375 \\ -7.045047 \\ 45.377496 \end{pmatrix}$$

Solving the system (28), we find (step 2 of the algorithm):

$$\mu_1^* = \begin{pmatrix} -2.999981352 \\ -0.999992433 \end{pmatrix},$$

$$\mu_2^* = \begin{pmatrix} -1.750002872 \\ -0.874993786 \end{pmatrix},$$

$$\mu_3^* = \begin{pmatrix} 1.999989979 \\ -0.000015784 \end{pmatrix},$$

$$\mu_4^* = \begin{pmatrix} 8.249993363 \\ 2.374987821 \end{pmatrix}.$$

The results of all calculations are obtained using Mathcad software.

The proximity of the exact solution to the numerical solutions satisfies the estimate (step 3 of the algorithm).

If $h = 0.05$, then

$$\max_{k=0,40} \|x^*(t_k) - \tilde{x}(t_k)\| < 0.00002.$$

If $h = 0.025$, then

$$\max_{k=0,80} \|x^*(t_k) - \tilde{x}(t_k)\| < 0.000001.$$

If $h = 0.0125$, then

$$\max_{k=0,160} \|x^*(t_k) - \tilde{x}(t_k)\| < 0.00000009.$$

Example 2. Let us consider the following problem

$$\frac{dx}{dt} = \begin{pmatrix} t & 1 & t^2 \\ 2 & t & 2t^3 \\ t^2 & 0 & 4t \end{pmatrix} x + \begin{pmatrix} 1 & t & 2 \\ 0 & 5 & 5t \\ 6 & t+2 & 3t \end{pmatrix} x(\gamma(t)) +$$

$$+ \begin{pmatrix} 4t^2 & 6 & 0 \\ 5t & t-3 & 8 \\ 1 & 0 & t \end{pmatrix} x\left(\frac{1}{2}\right) + f(t), t \in (0,1), \quad (31)$$

$$\begin{pmatrix} 2 & 0 & 6 \\ 4 & 2 & 1 \\ 4 & 5 & -7 \end{pmatrix} x(0) + \begin{pmatrix} 1 & 5 & 11 \\ 0 & -4 & 2 \\ 6 & 8 & 9 \end{pmatrix} x(1) = \begin{pmatrix} -109 \\ -65 \\ 59 \end{pmatrix}, x \in R^3. \quad (32)$$

Here

$$\gamma(t) = \xi_0 = \frac{1}{4}, f(t) = \begin{pmatrix} \frac{27}{64}t - 5t^3 - t^4 + \frac{131}{8} \\ 18t^3 - 5t^4 - 2t^5 + 13t^2 + \frac{425}{16}t + \frac{5119}{64} \\ 3t^2 - 11t^3 + \frac{4671}{64}t + \frac{187}{32} \end{pmatrix}, t \in \left(0, \frac{1}{2}\right);$$

$$\gamma(t) = \xi_1 = \frac{3}{4}, f(t) = \begin{pmatrix} \frac{95}{8} - 5t^3 - \frac{167}{64}t - t^4 \\ 18t^3 - 5t^4 - 2t^5 + 13t^2 + \frac{385}{16}t + \frac{4149}{64} \\ 3t^2 - 11t^3 + \frac{4381}{64}t - \frac{679}{32} \end{pmatrix}, t \in \left(\frac{1}{2}, 1\right).$$

The exact solution of problem (31), (32) is $x^*(t) = \begin{pmatrix} 7t-3 \\ 5t^3+2t \\ t^2-9 \end{pmatrix}$.

To solve problem (31), (32) numerically, we implement the proposed algorithm. The interval $[0,1]$ is partitioned into the subintervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$, $\left[\frac{3}{4}, 1\right]$. We take the step size $h = 0.025$ to numerically solve the auxiliary initial-value problems on the partition subintervals (step 1 of the algorithm).

Solving the system (28), we find (step 2 of the algorithm):

$$\mu_1^* = \begin{pmatrix} -2.999999976 \\ -0.000000021 \\ -9.000000037 \end{pmatrix},$$

$$\mu_2^* = \begin{pmatrix} -1.249999988 \\ 0.578125004 \\ -8.937500018 \end{pmatrix},$$

$$\mu_3^* = \begin{pmatrix} 0.499999994 \\ 1.624999997 \\ -8.749999997 \end{pmatrix}$$

$$\mu_4^* = \begin{pmatrix} 2.249999991 \\ 3.609375057 \\ -8.437499952 \end{pmatrix}.$$

The results of calculations, obtained using Mathcad software, are presented in Table 1 (step 3 of the algorithm).

Table 1 – The proximity of the exact solution $x^*(t)$ to the numerical solutions $\tilde{x}(t)$ of the problem (31), (32)

k	**t_k**	$\lvert x_1^*(t_k) - \tilde{x}_1(t_k)\rvert$	$\lvert x_2^*(t_k) - \tilde{x}_2(t_k)\rvert$	$\lvert x_3^*(t_k) - \tilde{x}_3(t_k)\rvert$
0	0	0.2422E-7	0.2137E-7	0.3679E-7
1	0.025	0.2275E-7	0.1788E-7	0.3500E-7
2	0.05	0.2134E-7	0.1458E-7	0.3322E-7
3	0.075	0.2000E-7	0.1145E-7	0.3146E-7
4	0.1	0.1873E-7	0.0853E-7	0.2969E-7
5	0.125	0.1750E-7	0.0581E-7	0.2790E-7
6	0.15	0.1632E-7	0.0331E-7	0.2607E-7
7	0.175	0.1516E-7	0.0105E-7	0.2421E-7
8	0.2	0.1402E-7	0.0097E-7	0.2230E-7
9	0.225	0.1287E-7	0.0271E-7	0.2034E-7
10	0.25	0.1171E-7	0.0418E-7	0.1833E-7
11	0.275	0.1052E-7	0.0534E-7	0.1626E-7
12	0.3	0.0927E-7	0.0618E-7	0.1414E-7
13	0.325	0.0794E-7	0.0666E-7	0.1198E-7
14	0.35	0.0650E-7	0.0677E-7	0.0978E-7
15	0.375	0.0493E-7	0.0646E-7	0.0755E-7
16	0.4	0.0319E-7	0.0572E-7	0.0532E-7
17	0.425	0.0126E-7	0.0449E-7	0.0311E-7
18	0.45	0.0092E-7	0.0273E-7	0.0094E-7
19	0.475	0.0338E-7	0.0039E-7	0.0115E-7
20	0.5	0.0617E-7	0.0258E-7	0.0311E-7

Table continuation

k	t_k	$\lvert x_1^*(t_k) - \tilde{x}_1(t_k)\rvert$	$\lvert x_2^*(t_k) - \tilde{x}_2(t_k)\rvert$	$\lvert x_3^*(t_k) - \tilde{x}_3(t_k)\rvert$
21	0.525	0.0571E-7	0.0476E-7	0.0767E-7
22	0.55	0.0530E-7	0.1191E-7	0.1234E-7
23	0.575	0.0499E-7	0.1884E-7	0.0171E-7
24	0.6	0.0481E-7	0.2550E-7	0.2189E-7
25	0.625	0.0481E-7	0.3185E-7	0.2669E-7
26	0.65	0.0503E-7	0.3785E-7	0.3143E-7
27	0.675	0.0554E-7	0.4342E-7	0.3605E-7
28	0.7	0.0640E-7	0.4849E-7	0.4047E-7
29	0.725	0.0768E-7	0.5298E-7	0.4460E-7
30	0.75	0.0947E-7	0.5678E-7	0.4833E-7
31	0.775	0.1185E-7	0.5977E-7	0.5152E-7
32	0.8	0.1493E-7	0.6181E-7	0.5402E-7
33	0.825	0.1884E-7	0.0627E-7	0.5564E-7
34	0.85	0.2372E-7	0.6226E-7	0.5616E-7
35	0.875	0.2973E-7	0.6022E-7	0.5533E-7
36	0.9	0.3707E-7	0.5628E-7	0.5281E-7
37	0.925	0.4597E-7	0.5009E-7	0.4825E-7
38	0.95	0.5668E-7	0.4119E-7	0.4120E-7
39	0.975	0.6951E-7	0.2907E-7	0.3114E-7
40	1	0.8484E-7	0.1306E-7	0.1744E-7

4 Conclusion

In this paper, we developed a numerical algorithm of the Dzhumabaev's parameterization method for solving the linear two-point BVP for the system of loaded DEPCAG. This technique can be applied to various kinds of functional-differential equations.

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