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Darboux transformation and exact solutions of nls-mb equations

Abstract. A nonlinear wave is one of the basic objects of physics. They are inherent to plasma physics and solid state physics, gravity and nuclear physics, field theory and optics, hydrodynamics and aerodynamics, kinetics of chemical reactions and population dynamics. It is well known that the constuction of explicit solutions for an integrable system plays a significant part in the definition and explanation of nonlinear phenomena. In this article, we will focus on integrable nonlinear Schrodinger and Maxwell-Bloch equations (NLS-MB) that represents the propagation of optical impulses in an inhomogeneous fibreglass with erbium-doped losses or amplification due to an external potential. Lax representation of NLS-MB will be given. Based on relevant Lax pair, Darboux transformation for NLS-MB will be obtained. Exact solutions will be derived through the Darboux transformation. Graphs of the obtained solutions will be constructed. By using our approach one can find also other differerent exact solutions of NLS-MB equations.
Keywords: NLS-MB equations, Lax pair, Darboux transformation, soliton, exact solutions.

Introduction

Nonlinear wave equations represent common and significant phenomena arising in different physical contexts, including plasma, acoustics, optics, and waves on water, and as a consequence they continue to attract considerable attention from researchers. Frequently nonlinear waves are mathematically defined by nonlinear partial differential equations. In certain physical modes, these PDEs are completely integrable and, as a consequence, have a remarkably deep mathematical structure. For instance, complete integrability is related to the existence of an endless number of conservation laws; the existence of a Lax pair makes it possible to develop and use different analytical tools. The behavior of solutions of integrable nonlinear equations shows a lot of interesting phenomena. These PDEs admit multiple types of exact solutions, including bound states, and solutions of finite kind and solitons. Furthermore, these equations are interesting not only from a mathematical viewpoint, but also significant from a practical point of view, since they determinate equations for many particular physical conditions.

Over the past few years, long-range optical fiber communication has attracted great interest from scientists around the world. The transmission of soliton pulses in ultrafast communication systems plays an especially important role and is considered the tool of the future to achieve low loss, efficiency and high speed communication. Many equations have been studied by mathematicians and physicists as models for fiber optic communication.To take into account the influence of large pulse widths, the dynamics of the system is controlled by the coupled system of the nonlinear Schrodinger equation and Maxwell-Bloch equation (NLS-MB) [1]. The nonlinear Schrödinger equation (NSE) describes the propagation of optical pulses through nonlinear optical fibers in the picosecond range [2]. For soliton-based communication systems, fiber attenuation must be compensated for to be more competitive, reliable, and cost-effective than traditional systems. Maxwell-Block (MB) equations describe a type of pulse propagation called self-induced transparency (SIT) [3]. This type of pulse reaches a steady state, where the width, energy and shape of the pulse remain unchanged after several classical absorption lengths, and the pulse velocity is much lower than the speed of light in this medium. Together they are known as the NLS-MB equations.

In this article we will construct the Darboux transform of the Lax pair of the NLS-MB equation, new exact solutions will be directly constructed starting from the seed solution. Because obtaining of exact solutions to the integrable equations is one of the most essential and meaningful topics.

Materials and Methods

There are some methods for constructing solutions, such as the inverse scattering transformation method [4], the Hirota bilinear transformation method [5], The Backlund [6] and Darboux transformation (DT) methods [7-11], the Fokas approach [12], the long-time asymptotic approach [13], and so on. Among them, the Darboux transformation is the most effective method for finding explicit solutions to integrable equations. The unique advantage of DT in solving integrable equations is that their solutions are constructed using a purely algebraic procedure.

This article consists of three main sections. In the first section Lax representation of the integrable NLS-MB equations will be introduced. Then we will give the detailed proof of the Darboux transformation for NLS-MB equations in section 2. In section 3 we will derive new and different kind of solutions based on obtained Darboux transformation and construct their graphs. Last section devoted to conclusion.

Literature review

For the first time Maimistov and Manykin [4] derived the coupled NLS-MB system to consider the propagation of ultrashort pulses in a light guide with a two-level resonant medium with Kerr nonlinearity. Many scientists have worked on this, achieving significant results [5,6]. These equations has also been reduced using the Painleve analysis [7]. In addition, the Lax pair and the multisoliton solution of the NLS-MB equations were proposed by Kakei and Satsuma [8]. There has been a lot of research done on the NLS-MB equations recently. The multisoliton solution is given in reference [9]. Single soliton and single respiratory solutions of the NLS-MB equations were obtained using the Darboux transformation (DT) [10-12].

The Darboux transform (DT) of an integrable system was first proposed by Matveev and Salle [6] . The main idea is that they construct a DT for a linear system and an adjoint system. They then join the two DTs together and find the double DT (i.e. BDT). Moreover, the BDT of some integrable equations was constructed in [14, 15]. Numerous successful implementations of Darboux transformation in various fields of physics and applied mathematics ensure its importance from an applied point of view [17, 18]. It is proved that this method, based on lax pairs, is one of the most productive algorithmic procedures for obtaining explicit solutions to nonlinear evolution equations. An effective way to create obvious solutions for many integrated systems is Darboux Transformation (DT) [6-10].

1. Lax representation of the integrable NLS-MB

If an optical pulse propagates through a nonlinear waveguide, the evolution of the pulse is determined by the NLS-MB equations. The NLS-MB equations are written as [1, 4].

$$q_t = i[\tfrac{1}{2} q_{xx} + |q|^2 q] + 2p, \tag{1}$$

$$p_x = 2i\omega_0 p + 2q\eta, \tag{2}$$

$$\eta_x = -(qp^* + q^* p). \tag{3}$$

where:

q – the complex field envelope;

p – measure of polarization of the resonant medium;

η – inverse population between two levels of wave functions of two energy levels of resonant atoms;

ω – the real constant parameter, it corresponds to the frequency;

$*$ – is the complex conjugate.

NLS-MB equations' linear eigenvalue problem is expressed as

$$\Psi_x = U\Psi \tag{4}$$

$$\Psi_t = V\Psi \tag{5}$$

where

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \tag{6}$$

$$U = \begin{bmatrix} \lambda & q \\ -q^* & -\lambda \end{bmatrix} \equiv \lambda\sigma_3 + U_0, \tag{7}$$

$$V = i\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\lambda^2 + \begin{bmatrix} 0 & q \\ -q^* & 0 \end{bmatrix}\lambda + \frac{1}{2}\begin{bmatrix} |q|^2 & q_x \\ q_x^* & -|q|^2 \end{bmatrix}\right) + + \frac{1}{\lambda - i\omega_0}\begin{bmatrix} \eta & -p \\ -p^* & -\eta \end{bmatrix} \equiv i\sigma_3\lambda^2 + i\lambda V_1 + \frac{i}{2}V_0 + \frac{1}{\lambda - i\omega_0}V_{-1} \tag{8}$$

here

λ – the complex eigenvalue parameter constant;

U and V – the Lax pair of NLS-MB equations.

2. Darboux transformation for the NLS-MB equations.

In this section Darboux transformation will be introduced for the integrable NLS-MB equations. Firstly, to construct Darboux transformation of NLS-MB equation, we consider the transformation about linear function Ψ.

$$\Psi' = T\Psi = (\lambda I - S)\Psi \tag{9}$$

therefore

$$\Psi'_x = U'\Psi' \tag{10}$$

$$\Psi'_t = V'\Psi' \tag{11}$$

where $S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. U' and V' depend on q', p', η', λ and it dependence is the same as the dependence of p, p, η, λ on U and V. To hold equations (10) and (11) , T must satisfy

$$T_x + TU = U'T \tag{12}$$

$$T_t + TV = V'T \tag{13}$$

The relation between q, p, η and new solutions q', p', η' which is called Darboux transformation can be got by using equations (12) and (13). From equation (12) we have

$$-S_x + \lambda^2 I\sigma_3 + \lambda I U_0 - \lambda S\sigma_3 - SU_0 - -\lambda^2\sigma_3 I + \lambda\sigma_3 S - \lambda U'_0 I + U'_0 S = 0 \tag{14}$$

Collecting different degrees of λ, we get the following set of identities

$$\lambda: IU_0 - S\sigma_3 + \sigma_3 S - U'_0 I = 0 \tag{15}$$

which further leads to

$$U'_0 = U_0 + [\sigma_3, S], \tag{16}$$

$$\lambda^0: S_x = \sigma_3 S^2 + \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} S - -S\sigma_3 S - S\begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \tag{17}$$

from above several identities, we can get

$$q' = q + 2s_{21}, \tag{18}$$

$$q^{*\prime} = q^* + 2s_{21}, \tag{19}$$

and S should have a condition $s_{12} = s_{21}^*$. By (13), following identity can be obtained

$$-S_t + i\lambda^3 I\sigma_3 + i\lambda^2 IV_1 + \frac{i}{2}\lambda IV_0 + \frac{\lambda}{\lambda - i\omega_0}IV_{-1} - iS\sigma_3\lambda^2 - i\lambda SV_1 - \frac{i}{2}SV_0 - -\frac{1}{\lambda - i\omega_0}SV_{-1} - i\lambda^3\sigma_3 I + i\lambda_2\sigma_3 S - i\lambda_2 V'_1 I + +i\lambda V'_1 S - \frac{i}{2}\lambda V'_0 I + \frac{i}{2}V'_0 S - \frac{\lambda}{\lambda - i\omega_0}V'_{-1} I + \frac{1}{\lambda - i\omega_0}V'_{-1} S = 0 \tag{20}$$

Comparing the coefficient of λ^i $(i = 0,1,2)$ of the two sides of equation (20) as we did before with equation (14), we have

$$\lambda^2: IV_1 - iS\sigma_3 + i\sigma_3 S - iV'_1 I = 0 \tag{21}$$

which leads to

$$V'_1 = V_1 + [\sigma_3, S], \tag{22}$$

$$\lambda: \frac{i}{2} IV_0 - ISV_1 + IV'_1 S - \frac{i}{2} V'_0 I = 0 \qquad (23)$$

$$\lambda^0: S_t = V_{-1} - V'_{-1} + \frac{i}{2} V'_0 S - \frac{i}{2} SV_0, \qquad (24)$$

$$\frac{1}{\lambda - i\omega_0}: i\omega_0 IV_{-1} - SV_{-1} - SV_{-1} - $$
$$-i\omega_0 V'_{-1} I + V'_{-1} S = 0 \qquad (25)$$

further it leads to

$$V'_{-1} = (S - i\omega_0 I) V_{-1} (S - i\omega_0 I)^{-1} \qquad (26)$$

Thus, from the above identities, after simplifications, several important equations (16-19), (21-26) were obtained that lead to Darboux transformations for the NLS-MB system later.

Now in order to determine the values of p', $p^{*\prime}$ and η' we put into Eq. (26) values of S, V_{-1}, V'_{-1} and get

$$p' = \frac{2\eta(s_{11} - i\omega_0)s_{12} - p^* s_{12}^2 + p(s_{11} - i\omega_0)^2}{\triangle} \qquad (27)$$

$$p^{*\prime} = \frac{-2\eta(s_{22} - i\omega_0)s_{21} + p^*(s_{11} - i\omega_0)^2 - p s_{21}^2}{\triangle} \qquad (28)$$

$$\eta' = \frac{\eta[(s_{11} - i\omega_0)(s_{22} - i\omega_0) + s_{12}s_{21}] - p^* s_{12}(s_{22} - i\omega_0) + p(s_{11} - i\omega_0)s_{21}}{\triangle} \qquad (29)$$

where

$$\triangle = (s_{11} - i\omega_0)(s_{22} - i\omega_0) - s_{12}s_{21}.$$

The main step is to find the exact value of S expressed by solving the column of equations (4) and (5). Suppose, that

$$S = H \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} H^{-1} \equiv H\Lambda H^{-1} \qquad (30)$$

$$H = \begin{pmatrix} \Psi_1(\lambda_1, x, t) & \Psi_1(\lambda_2, x, t) \\ \Psi_2(\lambda_1, x, t) & \Psi_2(\lambda_2, x, t) \end{pmatrix}, \qquad (31)$$

where λ_1 and λ_1 are complex constants, $\det H \neq 0$. From equations (4) and (37) we have

$$H_x = \sigma_3 H\Lambda + \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} H \qquad (32)$$

At the same time, from equations (5) and (31) can be derived:

$$H_t = \sigma_3 H\Lambda^2 + iV_1 H\Lambda + \frac{i}{2} V_0 H +$$
$$+V_{-1} H \begin{pmatrix} \frac{1}{\lambda_1 - i\omega_0} & 0 \\ 0 & \frac{1}{\lambda_1 - i\omega_0} \end{pmatrix} \qquad (33)$$

Then we can verify by direct calculation that S defined by equation (30) actually satisfies equations (18) (24). To satisfy the S' and V'_{-1} constraints as above, we obtain

$$\lambda_2 = -\lambda_1^* \qquad (34)$$

$$H = \begin{pmatrix} \Psi_1(\lambda_1, x, t) & -\Psi_2^*(\lambda_1, x, t) \\ \Psi_2(\lambda_1, x, t) & \Psi_1^*(\lambda_1, x, t) \end{pmatrix} \qquad (35)$$

Thus, we replacing equations (35) and (30) again to equations (18) and (26) and obtain the following Darboux transformations for NLS-MB.

$$q' = q + 2\frac{(\lambda_1 + \lambda_1^*)\Psi_1 \Psi_2^*}{\triangle} \qquad (36)$$

$$\begin{aligned} p' &= -\frac{2\eta}{\triangle^2}\left[|\Psi_1|^2\left(1 - \frac{\tilde{}_{1}}{\tilde{}_{2}}\right) - |\Psi_2|^2\left(1 - \frac{\tilde{}_{2}}{\tilde{}_{1}}\right)\right]\Psi_1\Psi_2^* + \\ &+\frac{p^*}{\triangle^2}\left[2 - \frac{\tilde{}_{1}}{\tilde{}_{2}} - \frac{\tilde{}_{2}}{\tilde{}_{1}}\right]\Psi_1^2\Psi_2^{*2} + \frac{p}{\triangle^2}\left[|\Psi_1|^4\frac{\tilde{}_{1}}{\tilde{}_{2}} - 2|\Psi_1|^2|\Psi_2|^2 + |\Psi_2|^4\frac{\tilde{}_{2}}{\tilde{}_{1}}\right] \end{aligned} \qquad (37)$$

$$\begin{aligned} \eta' &= -\frac{p^*}{\triangle^2}\left[|\Psi_1|^2\left(1 - \frac{\tilde{}_{2}}{\tilde{}_{1}}\right) - |\Psi_2|^2\left(1 - \frac{\tilde{}_{1}}{\tilde{}_{2}}\right)\right]\Psi_1\Psi_2^* + \\ &+\frac{p}{\triangle^2}\left[|\psi_1|^2\left(1 - \frac{\tilde{}_{1}}{\tilde{}_{2}}\right) - |\Psi_2|^2\left(1 - \frac{\tilde{}_{2}}{\tilde{}_{1}}\right)\right]\Psi_2\Psi_1^* + \\ &+\frac{\eta}{\triangle^2}\left[|\Psi_1|^4 + |\Psi_2|^4 + 2\left(\frac{\tilde{}_{2}}{\tilde{}_{1}} + \frac{\tilde{}_{1}}{\tilde{}_{2}} - 1\right)|\Psi_1|^2|\Psi_2|^2\right] \end{aligned} \qquad (38)$$

Here $\Psi_i \equiv \Psi_i(\lambda_1, x, t)$, $\tilde{\lambda}_i \equiv \lambda_i - i\omega_0$, $i = 1,2$ and $\triangle = |\Psi_1|^2 + |\Psi_2|^2$.

3. Exact solutions of the NLS-MB equations

In this section our aim is to derive new and different solutions of NLS-MB using obtained DT. Firstly, in order to costruct one-soliton solution we assume seed solutions as $q = 0, p = 0, \eta = 1$, then we take eigenfunctions in the following form:

$$\Psi_1 = \mathrm{e}^{\lambda_1 x + \left(\mathrm{i}\lambda_1{}^2 + \frac{1}{\lambda_1 + \omega_0}\right)\mathrm{t} + \delta_1}, \tag{44}$$

$$\Psi_2 = \mathrm{e}^{-\lambda_1 x - \left(\mathrm{i}\lambda_1{}^2 + \frac{1}{\lambda_1 - \omega_0}\right)\mathrm{t} + \delta_1} \tag{45}$$

where

δ_1 – arbitrary fixed real constant and $\lambda_1 = \mathrm{a} + \mathrm{ib}$. Substituting these two eigenfunctions into the the following Darboux transformations given in (36-38) and choosing $a = 0.5$, $b = 0.5$ $\omega_0 = 1.5$, $\delta_1 = 1$ then one-solitone solution of NLS-MB can be obtained, the evolution of which is shown in the figure. 1 clearly shows that q, p, and η are bright solitons, because their waves under the flat non-vanishing plane.

$$q' = \frac{(2.0 + 0.0\, i)\mathrm{e}^{-\vartheta+\varphi+1}\mathrm{e}^{\tau-\omega+1}}{\mathrm{e}^{-\tau+\omega+1.0}\mathrm{e}^{-\vartheta+\varphi+1} + \mathrm{e}^{\vartheta-\varphi+1}\mathrm{e}^{\tau-\omega+1}}; \tag{46}$$

$$\begin{aligned} p' &= [(1.600000000 + 0.0\, i)\mathrm{e}^{\tau-\omega+1}\mathrm{e}^{-\vartheta+\varphi+1}\mathrm{e}^{\tau-\omega+1} - (0.5 - 1.0\, i) \times \\ &\times \mathrm{e}^{-\tau+\omega+1}\mathrm{e}^{-\vartheta+\varphi+1})]/[\mathrm{e}^{-\tau+\omega+1.0}\mathrm{e}^{-\vartheta+\varphi+1} + \mathrm{e}^{\vartheta-\varphi+1}\mathrm{e}^{\tau-\omega+1}]^2 \end{aligned} \tag{47}$$

$$\begin{aligned} \eta' = \; & [(-0.2000000000 - 0.6000000000\, i)\mathrm{e}^{-\vartheta+\varphi+1.0}\mathrm{e}^{-\tau+\omega+1.0} - \\ & -(0.6000000000 + 0.2000000000\, i)\mathrm{e}^{\tau-\omega+1.0}\mathrm{e}^{\vartheta-\varphi+1.0}\left(\mathrm{e}^{-\vartheta+\varphi+1.0}\right)^{\mathrm{e}^{-\tau+\omega+1.0}} + \\ & +(1.200000000 + 0.6000000000\, i)\left(\mathrm{e}^{-\vartheta+\varphi+1.0}\right)^2(\mathrm{e}^{-\tau+\omega+1.0})^2 \times \\ & \times (1.0 + 0.2000000000\, i)\mathrm{e}^{\tau-\omega+1.0}\mathrm{e}^{\vartheta-\varphi+1.0}\mathrm{e}^{-\vartheta+\varphi+1.0}\mathrm{e}^{-\vartheta+\varphi+1.0} + \\ & +1.0\left(\mathrm{e}^{\vartheta-\varphi+1.0}\right)^2(\mathrm{e}^{\tau-\omega+1.0})^2)]/(\mathrm{e}^{-\vartheta+\varphi+1.0}\mathrm{e}^{-\tau+\omega+1.0} + \mathrm{e}^{\tau-\omega+1.0}\mathrm{e}^{\vartheta-\varphi+1.0})^2 \end{aligned} \tag{48}$$

here

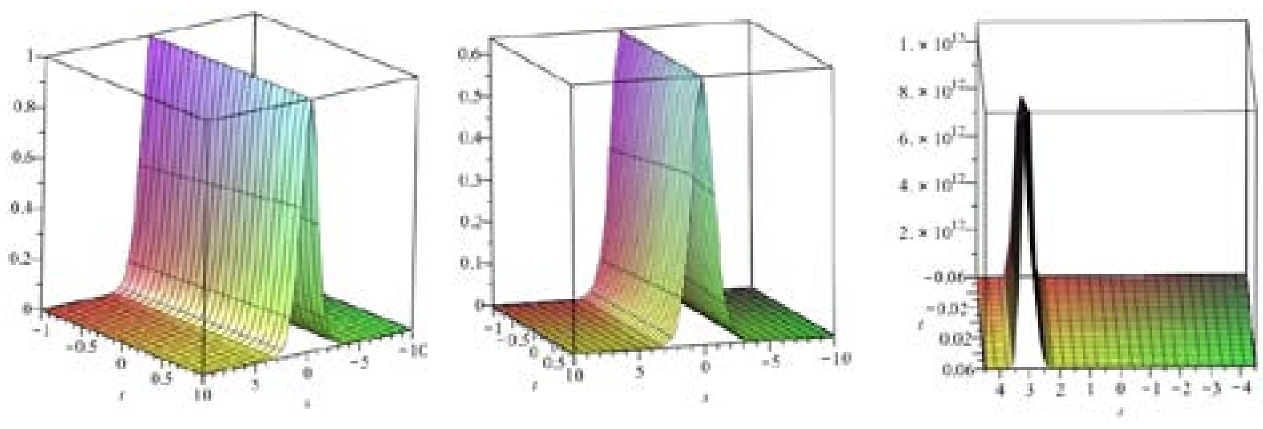
$$\tau = (0.0294117647 - 0.1176470588\, i)t$$
$$\vartheta = (0.0294117647 + 0.1176470588\, i)t$$

and

$$\omega = (0.5 - 0.5\, i)x$$
$$\varphi = (0.5 + 0.5\, i)x$$

Graphs of one-soliton solutions of the NLS-MB equations are shown in Figure 1:

($|q|^2$) ($|p|^2$) ($|\eta|^2$)



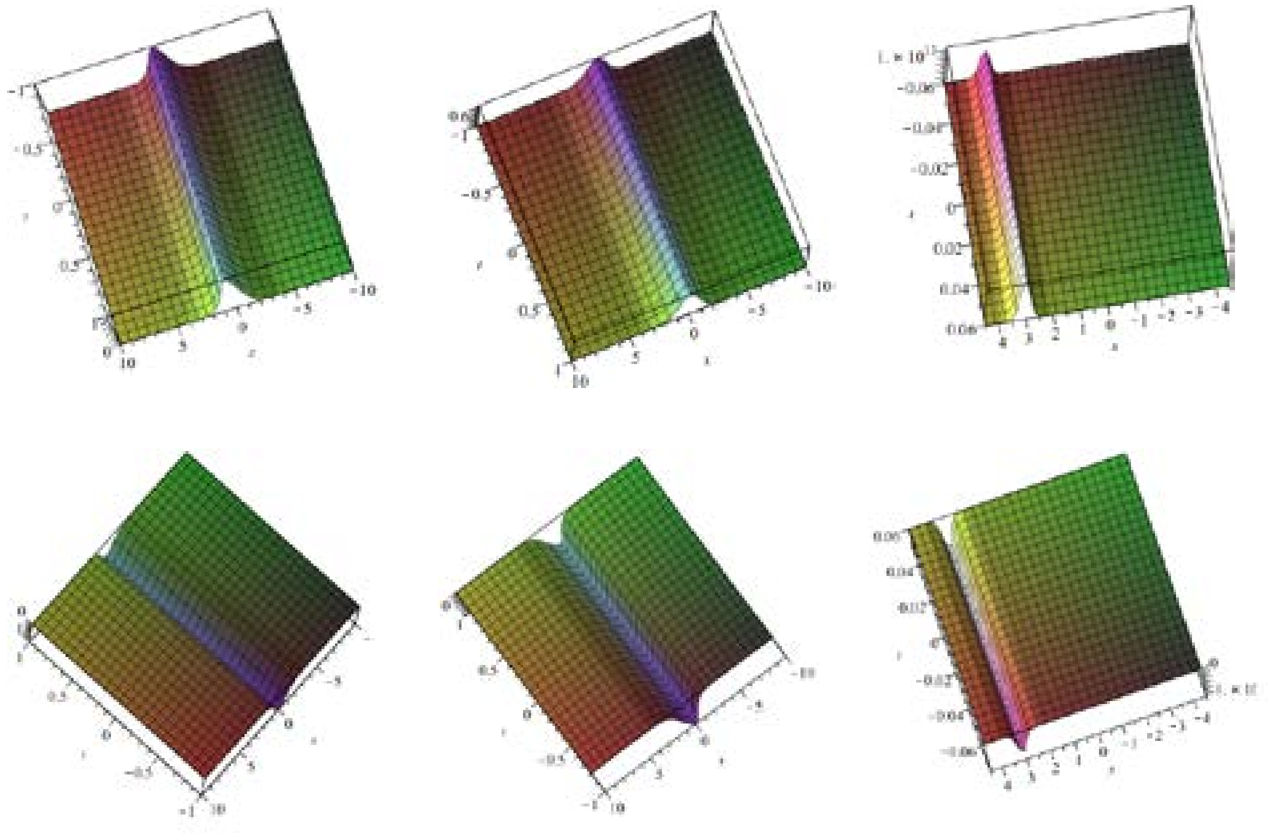


Figure 1 – One-soliton solutions of the NLS-MB equations

We obtained another type of solution for the NLS-MB equations by taking the seed solutions as $= 0, p = 0, \eta = 1$.

$$\Psi_1 = x + \mathrm{it} + 1, \tag{49}$$

$$\Psi_2 = -x - \mathrm{it} \tag{50}$$

Substituting the eigenfunctions Ψ_1 and Ψ_2 from (48)-(49) and the eigenvalues $\lambda_1 = \mathrm{a} + \mathrm{ib}$ and $a = 0.5$, $b = 0.5$, $\omega_0 = 1.5$ into the Darboux transformation (36)-(38), we obtain exact solutions for the system of equations (1)-(3) in the following form:

$$q' = -2.0\ \frac{(x+it+1)(-x+it)}{-2\,t^2-2\,x^2-2\,x-1}; \tag{51}$$

$$p' = -1.600000000\ \frac{(-x+it)(x+it+1)\left(0.5-1.0\,i-2.0\,it^2-2.0\,ix^2+1.0\,x-2.0\,ix\right)}{(-2\,t^2-2\,x^2-2\,x-1)^2}; \tag{52}$$

$$\begin{aligned} \eta' = \ & (-0.8000000000 + 0.0\,i)\times \\ & \times\left[\left(\frac{(0.5+0.5\,i)(x+it+1)(x-it+1)-(0.5-0.5\,i)(-x-it)(-x+it)}{(x+it+1)(x-it+1)+(-x-it)(-x+it)} - 1.5\,i\right)\times\right. \\ & \times\left(\frac{(0.5+0.5\,i)(-x-it)(-x+it)-(0.5-0.5\,i)(x+it+1)^{x-it+1}}{(x+it+1)(x-it+1)+(-x-it)(-x+it)} - 1.5\,i\right)+ \\ & \left.+\frac{(1.0+0.0\,i)(x+it+1)(-x+it)(x-it+1)(-x-it)}{\left((x+it+1)(x-it+1)+(-x-it)(-x+it)\right)^2}\right] \end{aligned} \tag{53}$$

Graphs of solutions of the NLS-MB equations are shown in Figure 2:

$(|q|^2)$ $(|p|^2)$ $(|\eta|^2)$

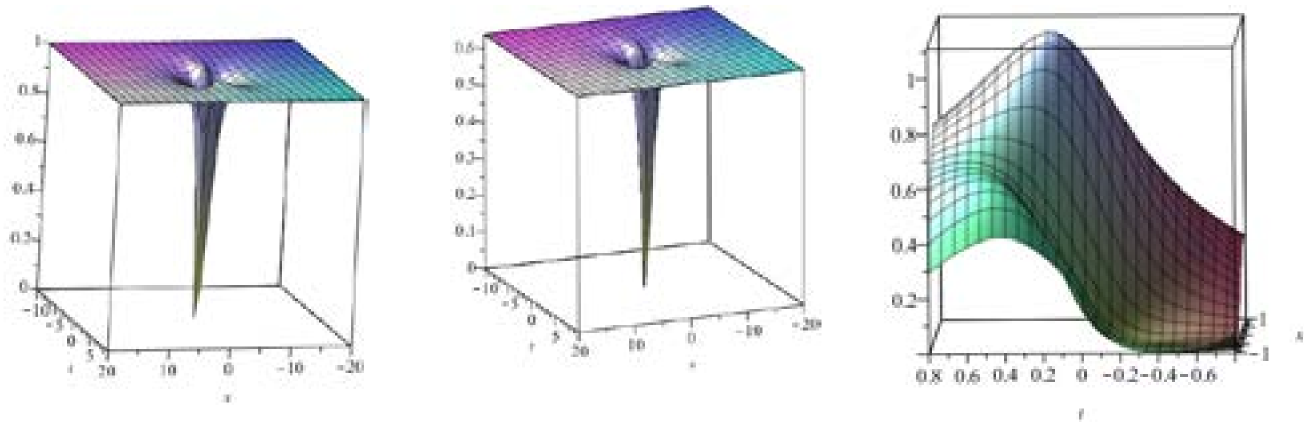


Figure 2 – Exact solutions of the NLS-MB equations

Conclusion

In this article, we studied the nonlinear Schrodinger and Maxwell-Bloch equations (NLS-MB) which describe the propagation of optical solitons in optical fibers with resonant impurities and nonlinear systems doped with erbium.

In the first section we presented a Lax pair formulation for NLS-MB equations. The Lax pair plays an significant role in the research of integrable properties of the NLS equation. The Darboux transformation was constructed and seed solutions were obtained in the second section. The Darboux transformation is the most effective technique of searching for exact solutions of integrable equations. To find Darboux transformation of NLS-MB equation, we considered the transformation about linear function. In the third section on the basis of Darboux transformation, different exact solutions of NLS-MB equations were obtained. Firstly, to construct a one-soliton solution, we took seed solutions as $\boldsymbol{q = 0, p = 0, \eta = 1}$. We chose the appropriate parameter values and built graphs from which you can see the behavior of the solution. Namely, it can be seen from the graph 1 that obtained results corresponded to bright solitons, because their waves under the flat non-vanishing plane. Then in the same section we have obtained other exact solutions. The graphs of these solutions were also presented.

Using our approach a new type of waves can be derived for another integrable coupled system in optics. The nonlinear phenomena studied in this work may be useful in physics, mathematics and other disciplines.

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