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An Initial-Boundary Value Problem for Kelvin-Voigt Equations with (p(x), q(x)6 m(x)) Structure

Abstract. A proof of a existence global in time of solutions of initial-boundary value problems for nonlinear equations mostly is not easy, even in some cases it is impossible. However, by establishing some qualitative properties of its solutions, one can find answers to such questions. For example, by establishing the blowing up in a finite time property of a solution, one can show that a solution does not exist globally in time. Thus, in last years, the investigating the qualitative properties of solutions such as localization and/or blow up in a finite time, has been developing rapidly.

In this work, we study the nonlinear initial-boundary value problem for the generalized Kelvin-Voigt equations describing the motion of incompressible viscoelastic non-Newtonian fluids. The equations generalized by replacing the diffusion and relaxation terms in equation with p(x)-Laplacian and q(x)-Laplacian, respectively, and adding a nonlinear absorption term with variable exponents and coefficients. A definition of a weak solution is given. Under suitable conditions for variable exponents and coefficients, and data of the problem, the blowing up of weak solution is established.
Key words: Kelvin-Voigt equation, blow up, p-Laplacian, damping term.

1. Introduction

In this work, we study the following initial-boundary value problem for the modified Kelvin-Voigt equations (without convective term) perturbed by $p(x), q(x)$- Laplacian diffusion, relaxation and damping term with variable exponents and coefficients

$$\vec{v}_t + \nabla \pi = div\left(\chi(x)\left|D\vec{v}\right|^{q(x)-2} D\vec{v}_t + \mu(x)\left|D\vec{v}\right|^{p(x)-2} D\vec{v}\right) + \gamma(x)\left|\vec{v}\right|^{m(x)-2}\vec{v}, \quad (x,t)\in Q_T, \tag{1}$$

$$div\vec{v} = 0, (x,t)\in Q_T \tag{2}$$

that supplemented by the following initial and boundary conditions

$$\vec{v}(x,t)\big|_{t=0} = \vec{v}_0(x), x\in\Omega, \tag{3}$$

$$\vec{v}(x,t)\big|_{\Gamma_T} = 0. \tag{4}$$

Here $\Omega\subset R^n$, $n\geq 2$, is a bounded domain with a smooth boundary $\partial\Omega$ and $Q_T = \Omega\times(0,T)$ is the bounded cylinder with lateral $\Gamma_T = \partial\Omega\times(0,T)$, $D(\vec{v}) = \frac{1}{2}\left(\nabla\vec{v} + \nabla\vec{v}^T\right)$ is the rate of the strain tensor, the vector function $\vec{v}(x,t) = (v_1, v_2, ..., v_n)$ is a velocity field, the scalar function $\pi(x,t)$ is a pressure, μ is a viscosity kinematic coefficient, and χ is a viscosity relaxation coefficient. The coefficients χ, μ, γ and the exponents q, p, m are given measurable functions on Ω, such that

$$\begin{aligned} &0<\mu^- \leq \mu(x) \leq \mu^+ <\infty,\ 0<p^- \leq p(x)\leq p^+ <\infty,\\ &0<\chi^- \leq \chi(x)\leq \chi^+ <\infty,\ 0<q^- \leq q(x)\leq q^+ <\infty,\\ &0<\gamma^- \leq \gamma(x)\leq \gamma^+ <\infty,\ 0<m^- \leq m(x)\leq m^+ <\infty, \end{aligned} \tag{5}$$

where "$+$" and "$-$" on power denote the $ess\,\boldsymbol{sup}$ and $ess\,\boldsymbol{inf}$ values on Ω of corresponding functions, for example, for the function $\sigma(x)$:

$$\sigma^+ := ess\,\underset{x\in\Omega}{\boldsymbol{sup}}\,\sigma(x),\ \sigma^- := ess\,\underset{x\in\Omega}{\boldsymbol{inf}}\,\sigma(x).$$

The system of equations (1)-(2) with $p = q = 2$ and $\gamma = 0$ and with constant coefficients is called the classical linear Kelvin-Voigt equations and it is used as the model of the motion of incompressible non-Newtonian fluids [1-3]. The name of the Kelvin-Voigt equations has been appeared in works of Oskolkov [4-8], though neither Kelvin nor Voigt have suggested any system of equations and these equations have been used in some cases even before the above Oskolkov's works. For instance, in 1966, Ladyzhenskaya [9] has suggested these classical Kelvin-Voigt equations as a regularization to the 3-dimensional Navier-Stokes equations to ensure the existence of unique global solutions, see also [2, 10-11] and references therein.

The various initial-boundary value problems for the classical linear and nonlinear Kelvin-Voigt equations have been studied by several authors, for instance, in [2], [4-11] for homogenous fluids, i.e. when the density is a known constant, and in [12], for nonhomogeneous fluids, i.e. when the density is unknown function.

On the other hand, the equation (1) is the pseudo-parabolic type equation, and the blow up properties of solutions of such equations with p-Laplacian with variable and constant exponents were studied in [13-15] (see the references therein).

In last years, as PDE generalized by p-Laplacian and nonlinear damping terms, an investigation of modified equations of hydrodynamics, in particular, the Navier-Stokes equations modified with p-Laplacian diffusion and with a damping term is rapidly developing, see [16-19].

The system (1)-(4) with a convective term, when all exponents and coefficients are constant, has been studied in [20]-[22], where the existence and uniqueness and the qualitative properties of weak solutions as large time behaviors and blow up in a finite time, are established.

Organization of this paper: in section 2, we introduce functional spaces, the inequalities and preliminary results used in the analysis. Later, in section 3 we state and prove our main result, in which we establish the conditions under which the weak solutions to the investigating problems are blow up in a finite time.

2. Notation and Preliminaries

In this section, we introduce the necessary definitions and preliminary results to state the main results of this paper. For the definitions and notations of the function spaces used throughout the paper and for their properties, we address the reader to e.g. the monographs [19, 25] cited in this work. We just fix the following notations for the functions spaces of mathematical fluid mechanics:

$$\wp := \{v \in C_0^\infty(\Omega) : div\,\vec{v} = 0\},$$
$$H := closure\ of\ \wp\ in\ the\ norm\ of\ L^2(\Omega);$$
$$V_p := closure\ of\ \wp\ in\ the\ norm\ of\ W^{1,p}(\Omega).$$

Let $1 \le p < \infty$ and $\Omega \in R^n,\ n \ge 1$, be a domain. We will use the classical Lebesgue spaces $L^p(\Omega)$ whose norm is denoted by $\|\bullet\|_{p,\Omega}$. For any nonnegative k, $W^{k,p}(\Omega)$ denotes the Sobolev space of all functions $u \in L^p(\Omega)$ such that the weak derivatives $D^\alpha u$ exist, in the generalized sense, and are in $L^p(\Omega)$ for any multi-index α such that $0 \le |\alpha| \le k$.

Let $p : \Omega \to [1,\infty]$ be a measurable function and we define

$$p^- := ess\,\underset{x\in\Omega}{\boldsymbol{inf}}\ p(x),\ \ p^+ := ess\,\underset{x\in\Omega}{\boldsymbol{sup}}\ p(x).$$

Given $p : \Omega \to [1,\infty]$ we denote by $L^{p(\cdot)}(\Omega)$ the space of all measurable functions u in Ω such that its semimodular is finite

$$A_{p(\cdot)} := \int_\Omega |u(x)|^{p(x)}dx < \infty.$$

The space $L^{p(\cdot)}(\Omega)$ is called Lebesgue space with variable exponent equipped with the norm

$$\|u\|_{p(\cdot),\Omega} := \boldsymbol{inf}\left\{\lambda > 0 : A_{p(\cdot)}\left(\frac{u}{\lambda}\right) \le 1\right\},$$

and $L^p(\Omega)$ becomes a Banach space with this norm.

The weak solution to the problem (1)-(4) is understood as the following sense

Definition 1. The vector function $\vec{v}(x,t)$ is called a weak solution to the problem (1)-(4), if:

(i) $\vec{v}(x,t)\in L^{\infty}\left(0,T;H(\Omega)\cap V_{q(x)}(\Omega)\right)\cap L^{p(x)}\left(0,T;V_{p(x)}(\Omega)\right)\cap L^{m(x)}(Q_T)$,

(ii) $\vec{v}(x,0)=\vec{v}_0(x)$ *a.e. in* Ω;

(iii) and for every $\varphi(x)\in H(\Omega)\cap V_{q(x)}(\Omega)\cap V_{p(x)}(\Omega)\cap L^{m(x)}(\Omega)$ and for a.e. $t\ge 0$ holds

$$\frac{d}{dt}\int_{\Omega}\left(\vec{v}\cdot\varphi+\chi(x)|D\vec{v}|^{q(x)-2}D\vec{v}:D\varphi\right)dx+\int_{\Omega}\mu(x)|D\vec{v}|^{p(x)-2}D\vec{v}:D\varphi dx=\int_{\Omega}\gamma(x)|\vec{v}|^{m(x)-2}\vec{v}\varphi dx. \quad (6)$$

3. Main result

In this section, we establish the conditions for the coefficients, exponents and data of the problem,
that a weak solution to the problem (1)-(4) blows up in a finite time, i.e. the weak solution does not exist globally in time.

Theorem 1. Let the conditions (5) be fulfilled and for the exponents $p(x), q(x), m(x)$ hold the conditions:

$$p^+\le m^- \text{ and } m^->\boldsymbol{max}\{2,q^+\}. \quad (7)$$

Let us assume, that also $\vec{v}_0\in V^{p(x)}(\Omega)\cap L^{m(x)}(\Omega)$ and

$$\int_{\Omega}\left(\frac{\gamma(x)}{m(x)}|\vec{v}_0|^{m(x)}-\frac{\mu(x)}{p(x)}|D\vec{v}_0|^{p(x)}\right)dx\ge 0. \quad (8)$$

Then there exists a finite time $T_{\max}<\infty$ (defined by (18)) such that a weak solution to problem (1)-(4) blows up.

Proof. The proof of Theorem 1 is based on the methods, presented in [23-24].
Let us first introduce the following functional

$$\Phi(t)=\int_0^t\left(\frac{1}{2}\|\vec{v}\|_2^2+\int_{\Omega}\frac{\chi(x)}{q(x)}|D\vec{v}|^{q(x)}dx\right)d\tau.$$

Under the conditions of Theorem 1, for every nontrivial solution of (1)-(4) and for all $t>0$

$$\Phi'(t)=\frac{1}{2}\|\vec{v}\|_2^2+\int_{\Omega}\frac{\chi(x)}{q(x)}|D\vec{v}|^{q(x)}dx\ge 0. \quad (9)$$

Testing now (6) by $\vec{v}$ and using

$$\frac{d}{dt}\left(\int_{\Omega}\frac{\chi(x)}{q(x)}|D\vec{v}|^{q(x)}dx\right)=\int_{\Omega}\chi(x)|D\vec{v}|^{q(x)-2}D\vec{v}:D\vec{v}_t dx.$$

we have

$$\frac{d}{dt}\left(\frac{1}{2}\|\vec{v}\|_{2,\Omega}^2+\int_{\Omega}\frac{\chi(x)}{q(x)}|D\vec{v}|^{q(x)}dx\right)= =\int_{\Omega}\left(\gamma(x)|\vec{v}|^{m(x)}-\mu(x)|D\vec{v}|^{p(x)}\right)dx. \quad (10)$$

Combining (9) and (10), we obtain

$$\Phi''(t)=\int_{\Omega}\left(\gamma(x)|\vec{v}|^{m(x)}-\mu(x)|D(\vec{v})|^{p(x)}\right)dx. \quad (11)$$

Next, taking $\varphi=\vec{v}_t$ in (6) for all $t\ge 0$, we get

$$\|\vec{v}_t\|_{2,\Omega}^2+\int_{\Omega}\chi(x)|D\vec{v}|^{q(x)-2}|D\vec{v}_t|^2dx= =\frac{d}{dt}\left(\int_{\Omega}\left(\frac{\gamma(x)}{m(x)}|\vec{v}|^{m(x)}-\frac{\mu(x)}{p(x)}|D\vec{v}|^{p(x)}\right)dx\right). \quad (12)$$

Integrating (12) by τ from 0 to t and applying the assumption (8), we get

$$\int_0^t \left(\|\vec{v}_t\|_2^2 + \int_\Omega \chi(x) |D\vec{v}|^{q(x)-2} |D\vec{v}_t|^2 dx \right) d\tau =$$

$$\int_\Omega \left(\frac{\gamma(x)}{m(x)} |\vec{v}(t)|^{m(x)} - \frac{\mu(x)}{p(x)} |D\vec{v}(t)|^{p(x)} \right) dx - \int_\Omega \left(\frac{\gamma(x)}{m(x)} |\vec{v}_0|^{m(x)} - \frac{\mu(x)}{p(x)} |D\vec{v}_0|^{p(x)} \right) dx < \tag{13}$$

$$\int_\Omega \left(\frac{\gamma(x)}{m(x)} |\vec{v}(t)|^{m(x)} - \frac{\mu(x)}{p(x)} |D\vec{v}(t)|^{p(x)} \right) dx, \ \forall t > 0.$$

Applying (7), we get the following inequality

$$\int_\Omega \left(\frac{\gamma(x)}{m(x)} |\vec{v}(t)|^{m(x)} - \frac{\mu(x)}{p(x)} |D\vec{v}(t)|^{p(x)} \right) dx \le \int_\Omega \left(\frac{\gamma(x)}{m^-} |\vec{v}(t)|^{m(x)} - \frac{\mu(x)}{p^+} |D\vec{v}(t)|^{p(x)} \right) dx \le$$

$$\le \frac{1}{m^-} \int_\Omega \left(\gamma(x) |\vec{v}(t)|^{m(x)} - \mu(x) |D\vec{v}(t)|^{p(x)} \right) dx \le \frac{1}{m^-} \Phi''(t), \ \forall t > 0.$$

Then, it follows from (13) that

$$0 < \int_0^t \left(\|\vec{v}_t\|_2^2 + \int_\Omega \chi(x) |D\vec{v}|^{q(x)-2} |D\vec{v}_t|^2 dx \right) d\tau \le \frac{1}{m^-} \Phi''(t). \tag{14}$$

Next, applying the Hölder and Young inequalities together with (5), we derive the following chain of inequalities for $0 \le t' < t$:

$$\left[\Phi'(t) - \Phi'(t') \right]^2 = \left[\int_{t'}^t \Phi''(\tau) d\tau \right]^2 = \left[\int_{t'}^t \left(\int_\Omega \vec{v}\vec{v}_t dx + \int_\Omega \chi(x) |D\vec{v}|^{q(x)-2} D\vec{v} : D\vec{v}_t dx \right) d\tau \right]^2 \le$$

$$\left[\int_{t'}^t \left(\|\vec{v}\|_{2,\Omega} \|\vec{v}_t\|_{2,\Omega} + \left(\int_\Omega \chi |D\vec{v}|^{q(x)} dx \right)^{\frac{1}{2}} \left(\int_\Omega \chi |D\vec{v}|^{q(x)-2} |D\vec{v}_t|^2 dx \right)^{\frac{1}{2}} \right) d\tau \right]^2 \le$$

$$\left[\left(\int_{t'}^t \|\vec{v}\|_{2,\Omega}^2 d\tau \right)^{\frac{1}{2}} \left(\int_{t'}^t \|\vec{v}_t\|_{2,\Omega}^2 d\tau \right)^{\frac{1}{2}} + \left(\int_{t'}^t \int_\Omega \chi |D\vec{v}|^{q(x)} dx d\tau \right)^{\frac{1}{2}} \left(\int_{t'}^t \int_\Omega \chi |D\vec{v}|^{q(x)-2} |D\vec{v}_t|^2 dx d\tau \right)^{\frac{1}{2}} \right]^2 \le \tag{15}$$

$$\le \int_{t'}^t \left(\|\vec{v}\|_2^2 + \int_\Omega \chi |D\vec{v}|^{q(x)} dx \right) d\tau \cdot \int_{t'}^t \left(\|\vec{v}_t\|_2^2 + \int_\Omega \chi |D\vec{v}|^{q(x)-2} |D\vec{v}_t|^2 dx \right) d\tau.$$

It follows from (15) and (1), (2), that

$$\begin{gathered}\left[\Phi'(t)-\Phi'(t')\right]^2 \leq \\ \leq \int_{t'}^{t}\left(2\frac{1}{2}\|\vec{v}\|_2^2+q^+\int_\Omega \frac{\chi(x)}{q(x)}|D\vec{v}|^{q(x)}\,dx\right)d\tau\cdot\int_{t'}^{t}\left(\|\vec{v}_t\|_2^2+\int_\Omega \chi(x)|D\vec{v}|^{q(x)-2}|D\vec{v}_t|^2\,dx\right)d\tau \leq \\ \max\{2,q^+\}\Phi(t)\int_{t'}^{t}\left(\|\vec{v}_t\|_2^2+\int_\Omega \chi(x)|D\vec{v}|^{q(x)-2}|D\vec{v}_t|^2\,dx\right)d\tau \leq \\ \frac{\max\{2,q^+\}}{m^-}\Phi(t)\cdot\Phi''(t),\ \forall t>t'>0.\end{gathered} \tag{16}$$

We want to prove that the functional $\Phi(t)$ becomes unbounded (blows up) at a finite moment. Let us assume that for contradiction, the blow-up does not occur in a finite time, i.e. the nontrivial solution $\vec{v}$ exists for all time $t>0$. Since, $\Phi(t),\Phi'(t)$ and $\Phi''(t)$ are nonnegative, there exists a time $t'\geq 0$, such that they are strong positive for all $t\geq t'$, and it is necessary that $\Phi'(t)\to\infty$ as $t\to\infty$. Notice that for every $\sigma\in\left(1,\frac{m^-}{2}\right)$

$$1-\sqrt{\frac{2\sigma}{m^-}}\geq\frac{\Phi'(t')}{\Phi'(t)}\to 0 \ \text{ as } \ t\to\infty.$$

It follows that for every fixed $\sigma\in\left(1,\frac{m^-}{2}\right)$ there exists a moment $t_0>t'$ such that

$$(\Phi'(t)-\Phi'(t'))^2\geq\frac{2\sigma}{m^-}(\Phi'(t))^2$$

for all $t>t_0$, $\Phi(t_0)>0.$

Using (15) and the last inequality, we get

$$\frac{2\sigma}{m^-}(\Phi'(t))^2\leq(\Phi'(t)-\Phi'(t'))^2\leq\frac{2}{m^-}\Phi''(t)\Phi(t)$$

for all $t>t_0$,

Hence

$$\sigma\frac{\Phi'(t)}{\Phi(t)}\leq\frac{\Phi''(t)}{\Phi'(t)}\ \Leftrightarrow\ \left(\ln\Phi^\sigma(t)\right)'\leq\left(\ln\Phi'(t)\right)'\Rightarrow$$

$$\left(\frac{\Phi'(t_0)}{\Phi^\sigma(t_0)}\right)\Phi^\sigma(t)\leq\Phi'(t) \ \text{ for all } \ t>t_0. \tag{17}$$

The direct integration of (17) leads to the inequality

$$\Phi^{\sigma-1}(t)\geq\frac{\Phi^{\sigma-1}(t_0)}{1-(t-t_0)(\sigma-1)\frac{\Phi'(t_0)}{\Phi(t_0)}}\to\infty$$

$$\text{as } \ t\to T_{\max}=t_0+\frac{\Phi(t_0)}{(\sigma-1)\Phi(t_0)}. \tag{18}$$

On the other hand, by using the above assumption on existence of a weak solution $\vec{v}$ to the problem (1)-(4) for all time $t>0$, we obtain that the functional $\Phi(t)$ is bounded at a finite moment $T_{\max}$:

$$\begin{gathered}\infty>T_{\max}\sup_{t\in(0,T)}\left(\frac{1}{2}\|\vec{v}\|_2^2+\frac{\chi}{2}\|\nabla\vec{v}\|_2^2\right)\geq \\ \geq\int_0^t\left(\frac{1}{2}\|\vec{v}\|_2^2+\frac{\chi}{2}\|\nabla\vec{v}\|_2^2\right)d\tau\equiv\Phi(t)\end{gathered}.$$

But this is impossible, because by (18) the functional $\Phi(t)$ is unbounded at a finite moment $T_{\max}$, i.e. $\Phi(t)\to\infty,\ as\ t\to T_{\max}$ and it contradicts the existence of a solution $\vec{v}$ of the problem (1)-(4) for all time $t>0$. Therefore, it follows from this contradiction that the weak solution to the problem (1)-(4) blows up in a finite time, and it completed the proof of the Theorem 1.

4. Conclusion

In this work, the nonlinear initial-boundary value problem for the generalized Kelvin-Voigt equations describing the motion of incompressible viscoelastic non-Newtonian fluids is considered. The equations has been generalized replacing the diffusion and relaxation terms in equation with $p(x)$-Laplacian and $q(x)$-Laplacian, respectively, and adding a nonlinear absorption term with variable exponents and coefficients.

The functional spaces with their norms and some necessary inequalities regarding to the variable exponents have been introduced. Under suitable conditions on exponents and coefficients, and on the data of the problem, the blowing up in a finite time property of weak solutions is established. As it is known from theory of PDE, this property means that the weak solutions of the problem do not exist global in time.

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