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Experimental Data and the Nonlinear Inverse Problem of Heat Transfer

Abstract. In the paper the development a method for finding the nonlinear heat-conducting characteristics of the soil is being presented. Two-layer container complexes have been created, the side faces of which are thermally insulated so that the 1D heat equation can be used. In order not to solve the boundary value problem with a contact discontinuity and lose the accuracy of the method's solution, a temperature sensor was placed at the junction of two media, and a mixed boundary value problem is solved in each area (container). To provide the initial data with an inverse coefficient problem, two temperature sensors are used: one sensor was placed at the open boundary of the container and recorded the soil temperature at this boundary, and the second sensor was placed at a short distance from the boundary, which recorded the air temperature. The measurements were carried out on the time interval $(0, t_{max})$. First, the initial-boundary value problem of thermal conductivity with nonlinear coefficients of thermal conductivity, heat capacity, heat transfer, and material density are studied numerically. The nonlinear initial-boundary value problem is solved by the finite difference method. Based on the measured data of the complex, special functionals are constructed and the thermal conductivity coefficient k, density ρ, specific heat capacity c, heat transfer coefficient h are found, which depend on the temperature of the material. Based on the experimentally measured data, the corresponding functional is minimized on each time interval using the gradient descent method. All thermophysical characteristics for a container with clay were found with a relative error of 5%.

Key words: thermal conductivity, nonlinearity, difference problem, iteration, convergence, inverse problem.

Introduction

Heat transfer processes are one of the main sections of modern science and are of great practical importance in industrial energy. Determining the parameters of heat protection systems and obtaining a solution to the problem of thermal design are directly related to the calculations of thermal fields in the soil and ground. In turn, this requires knowledge of the thermophysical characteristics of the soil [1, 2]. The thermophysical properties of the soil play an essential role in the structure of the thermal field of the earth's crust. At the same time, the thermal field of the Earth is largely determined by the processes associated with prospecting, exploration, development of oil, gas and thermal water deposits, operation of main oil and gas pipelines and underground storage facilities. Optimization and analysis of thermal and moisture characteristics of building components is an important engineering tool [3]. In addition, studies of the thermophysical parameters of soil are of great importance in the gas industry for solving thermodynamic problems related to temperature forecasting when drilling deep and ultra-deep wells, calculating gas reserves, predicting the temperature of fluids at the mouth of production wells, assessing reservoir filtration parameters, and thermal treatment. productive horizons, as well as for transportation and underground storage of gas [4]. Nowadays, theoretical models for finding the thermophysical characteristics of inhomogeneous composite media do not have sufficient accuracy. Therefore, the main source of information about thermophysical properties is the performance of a physical experiment [5,6]. For the theoretical basis of the method for finding the thermophysical characteristics of a medium, the law of conservation of energy is used, the consequence of which is a nonlinear differential equation of heat conduction [1, 7, 8]. Where the thermal conductivity coefficient χ, density ρ, specific heat capacity c, heat transfer coefficient h depends on the temperature of the material and determine the process of heat transfer in the medium. Temperature is one of the main factors affecting the thermal conductivity of the soil. It has been established that the nature of the influence of temperature on the thermophysical parameters of the soil-soil is nonlinear [9-11]. In this regard, there is an urgent need to solve the inverse problem of the nonlinear heat equation.

Therefore, the purpose of the study is to conduct a thermophysical experiment and develop methodological support for determining thermophysical coefficients based on solving a nonlinear inverse problem of heat conduction [12, 13]. On the basis of the above mathematical model, the direct problem of heat transfer by input parameters is solved. Then the temperature field in the medium or in the material is determined. The physical-mathematical model and experimental temperature values at the accessible soil-ground boundary make it possible to find thermophysical characteristics in inverse coefficient problems of heat transfer [14-16]. The difficulty here is that the experimental temperature data are obtained from unknown thermophysical characteristics, which are calculated in the inverse problem with a predetermined accuracy. In addition, it should be taken into account that the initial approximations of the thermophysical coefficients specified in the iterative algorithm can differ significantly (several times) from the true values used to measure the experimental temperature data. On this basis, it is necessary to develop such algorithms that would eventually give almost zero functional discrepancy even with a significant deviation of the initial values of the unknown thermophysical characteristics from the true ones [17]. It is also necessary to verify the stability of the algorithm [18-21]. In turn, in this study, based on the nonlinear heat equation and experimental data, a method for solving the inverse nonlinear coefficient problem is proposed. The basis of the method is the minimization of the quadratic residual functional between numerical and experimental temperature values. Minimization of the functional is carried out by the method of gradient descent. When determining the damping factor (descent step), the fastest descent method is used.

The article is organized as follows: Section 2 presents a demonstration of a mathematical model for describing the physical phenomenon of heat conduction. The discretization of the computational domain and the model is also shown. Section 3 provides a description of the experiment, characteristics and installation of the experimental equipment. The soil, consisting of two layers, soil 1 and soil 2, is in a controlled environment – a thermally insulated container. The end faces of the container from the inner side are in contact with the soil, and the outer sides are directed to the boundary condition, which depends on the environment. In Section 4, the reliability of numerical predictions is assessed by comparing them with experimental observations. The description of the obtained results is given, and the graphs of the obtained data are shown.

Mathematical model

Formulation of the problem. Figure 1 illustrates a two-layer container, the side faces of which are thermally insulated, and the end faces are in contact with the environment (air). Considering these limitations, instead of the three-dimensional heat equation, we can consider the one-dimensional non-stationary equation

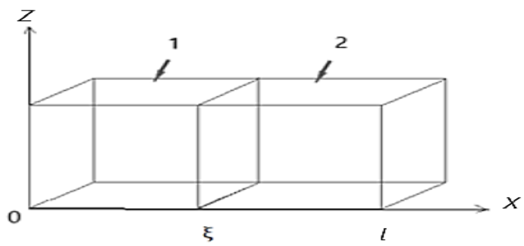


Figure 1 – Two-layer container

$$c(u)\rho(u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right),$$
$$x \in (0, \xi) \times (\xi, l), t \in (0{,}4\mathrm{t}_{\max})$$

The ambient temperature at the left boundary of the region at $x = 0$ will be denoted by $u_{ins}(t)$, and at the right boundary at $x = l$ we will denote by $u_{out}(t)$. In engineering calculations, the parameters c, ρ and k are usually considered to be constants. However, many scientists conclude that the study of nonlinear processes is of great practical interest. Since most processes occurring in nature are non-linear. Taking into account the nonlinearity of equation greatly complicates the mathematical formulation of the problem. Denote by $u(x, t)$ – distribution the temperature inside the complex of containers, where x- is the coordinate of the complex along the Ox axis, t-is the current time. At the initial time of observation, the temperature of both layers of the body is: $t = 0, u(x,0) = u_0(x), x \in (0,l)$.

The boundary conditions that determine the features of the process on the wall surface are given as follows:

The left and right boundaries of the region $\Omega = (0,\xi) \times (\xi, l)$ are in contact with the gaseous medium (air), so it is advisable to formulate a boundary condition of the third kind on these boundaries – the relationship between the heat flux due to thermal conductivity from a solid wall and the heat flux from a gaseous medium. Thus, the boundary conditions on the left and right boundaries are written as follows:

$$x = 0:\ k_1(u)\frac{\partial u}{\partial x} = h_{ins}(u)\big(u - u_{ins}(t)\big),$$

$$x = l:\ k_2(u)\frac{\partial u}{\partial x} = -h_{out}(u)\big(u - u_{out}(t)\big),$$

where $u_{ins}(t), u_{out}(t) -$ are ambient temperatures; $h_{ins}(u), h_{out}(u) -$ heat transfer coefficients; $k_1(u), k_2(u)$ – thermal conductivity coefficients of the medium "1" and "2" (Figure 1).

Usually, on the contact surface of the layers $x = \xi$ a boundary condition is set that determines the equality of temperatures and heat fluxes at the junction of materials:

$$u_1(\xi, t) = u_2(\xi, t),$$
$$k_1(u)\frac{\partial u_1}{\partial x}(\xi, t) = k_2(u)\frac{\partial u_2}{\partial x}(\xi, t). \qquad (1)$$

Here $u_1(x, t)$ и $u_2(x, t)$ – are the temperatures of the material layers in contact. When solving problems with contact conditions of the form (1), the rate of convergence of a homogeneous difference scheme becomes very low [19]. Therefore, to avoid this problem, we placed a separate sensor at the point $x = \xi$ which measures the change in soil temperature at the point of contact of two media. Due to this, the original task is split into two tasks, i.e. using the measured data in each container, its own problem of nonlinear thermal conductivity is solved. In the future, we will state the problem only on the left container shown in Fig.1.

In addition to $u_{ins}(t), u_{out}(t)$, the initial temperature values $T_\xi(t), t \in [0, t_{max}]$. For convenience of notation, we introduce the notation $h_{ins}(u) = h_1(u)$.

Problem. Using the measured values $u_{ins}(t), T_{ins}(t), T_\xi(t), t \in [0, t_{max}]$, it is required to develop a method for finding the temperature $u(x,t)$ and all the thermophysical parameters of the soil.

In the region of $Q_1 = (0,\xi) \times (0, t_{max})$ we studied the following system of equations.

$$c_1(u)\rho_1(u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k_1(u)\frac{\partial u}{\partial x}\right),$$

$$u(x,0) = u_0(x),$$

$$k_1(u)\frac{\partial u}{\partial x} = h_1(u)\big(u - u_{ins}(t)\big), x = 0,$$

$$u(\xi, t) = T_\xi(t).$$

Grid method. Section $(0,\xi)$ is divided into I equal parts with a step $\Delta x = {}^{\xi}/_{I}$. Then $\xi = I\Delta x$, where I -is the node number of the contact point $x = \xi$. And the segment $(0, t_{max})$ is divided into m equal parts with $\Delta t = {}^{t_{max}}/_{m}$.

As a result of this action, we get a grid:

$$\omega = \{x_i = i\Delta x, t_j = j\Delta t;\ i = 0,1, \dots, I;\ j = 0, 1, \dots, m\},$$

In the present work, a method has been developed for finding the soil parameters $k_s(u), c_s(u), \rho_s(u), h_s(u), s = 1,2$. In this case, the measured values of the ambient temperature are used as initial information $u_{ins}(t), u_{out}(t)$ и $u_0(t)$ −initial temperature distribution at time $t = 0$. And also $T_\xi(t) -$ soil temperature at the contact point of two media $x = \xi$. To compile the functional, $T_{ins}(t), T_{out}(t) -$ measured values of soil temperature at the boundary of the considered area are used.

Development of iteration methods

In the grid region ω the difference scheme is studied:

$$\rho_1\left(u_i^{j+1}\right)\cdot c_1\left(u_i^{j+1}\right)\frac{u_i^{j+1}-u_i^j}{\Delta t} = \frac{1}{\Delta x}\left(k_1\left(u_{i+\frac{1}{2}}^{j+1}\right)\frac{u_{i+1}^{j+1}-u_i^{j+1}}{\Delta x} - k_1\left(u_{i-\frac{1}{2}}^{j+1}\right)\frac{u_i^{j+1}-u_{i-1}^{j+1}}{\Delta x}\right),$$

$$i = 1,2 \dots, I - 1;\ j = 0,1, \dots, m - 1;$$

$$u_i^0 = u_0(x_i), i = 0,1, \dots, I; \quad (2)$$

$$u_I^{j+1} = T_\xi(t_{j+1}), j = 0,1, \dots, m - 1;$$

$$k_1\left(u_{\frac{1}{2}}^{j+1}\right)\frac{u_1^{j+1} - u_0^{j+1}}{\Delta x} = h_1\left(u_0^{j+1}\right)\left(u_0^{j+1} - u_{ins}^{j+1}\right);$$

where

$$u_{i+\frac{1}{2}} = \frac{u_{i+1}^{j+1} + u_i^{j+1}}{2}, i = 0,1, \dots, I - 1.$$

Let's rewrite difference equation of the system (2) in the form:

$$F\left(u_{i+1}^{j+1}, u_i^{j+1}, u_{i-1}^{j+1}\right) \equiv Z \cdot \left(k\left(u_{i+\frac{1}{2}}^{j+1}\right)\left(u_{i+1}^{j+1} - u_i^{j+1}\right) - k\left(u_{i-\frac{1}{2}}^{j+1}\right)\left(u_i^{j+1} - u_{i-1}^{j+1}\right)\right) -$$

$$-\rho\left(u_i^{j+1}\right)c\left(u_i^{j+1}\right)\left(u_i^{j+1} - u_i^j\right) = 0,$$

$$i = 1, 2, \dots, I - 1;\ j = 0, 1, \dots, m - 1$$

where $Z = \frac{\Delta t}{(\Delta x)^2}$.

Let's $x^s = \left(u_{i+1}^{s,j+1}, u_i^{s,j+1}, u_{i-1}^{s,j+1}\right)$.

When $s = 0, x^0$ will be the initial approximation of the system (3). Then, applying the Newton method for system (3), the following approximation of the unknown grid function is obtained:

$$\frac{\partial F(x^s)}{\partial u_{i+1}^{j+1}}\left(u_{i+1}^{s+1,j+1} - u_{i+1}^{s,j+1}\right) + \frac{\partial F(x^s)}{\partial u_i^{j+1}}\left(u_i^{s+1,j+1} - u_i^{s,j+1}\right) + \frac{\partial F(x^s)}{\partial u_{i-1}^{j+1}}\left(u_{i-1}^{s+1,j+1} - u_{i-1}^{s,j+1}\right) + F(x^s) = 0 \quad (4)$$

where s – iteration's number for the Newton's method.

Expanding the brackets, (4) is reduced to tridiagonal system:

$$A_i\left(u_{i+1}^{j+1}\right)^{s+1} + B_i\left(u_i^{j+1}\right)^{s+1} + C_i\left(u_{i-1}^{j+1}\right)^{s+1} = D_i^{\ s},$$
$$i = 1, \dots, N - 1;\ j = 0, \dots, m - 1,$$

where the coefficients are equal:

$$A_i = \frac{\partial F(x^s)}{\partial u_{i+1}^{j+1}}, B_i = \frac{\partial F(x^s)}{\partial u_i^{j+1}}, C_i = \frac{\partial F(x^s)}{\partial u_{i-1}^{j+1}},$$

$$D_i^s = -F(x^s) + A_i\left(u_{i+1}^{j+1}\right)^s +$$

$$+B_i\left(u_i^{j+1}\right)^s + C_i\left(u_{i-1}^{j+1}\right)^s$$

Then, expanding equation (3), the corresponding derivatives were found:

$$F\left(u_{i+1}^{j+1}, u_i^{j+1}, u_{i-1}^{j+1}\right) =$$

$$= \left[k_0 + k_1\left(u_{i+\frac{1}{2}}^{j+1}\right) + k_2\left(u_{i+\frac{1}{2}}^{j+1}\right)^2 + k_3\left(u_{i+\frac{1}{2}}^{j+1}\right)^3\right] \cdot \cdot \left(u_{i+1}^{j+1} - u_i^{j+1}\right)Z -$$

$$-\left[k_0 + k_1\left(u_{i-\frac{1}{2}}^{j+1}\right) + k_2\left(u_{i-\frac{1}{2}}^{j+1}\right)^2 + k_3\left(u_{i-\frac{1}{2}}^{j+1}\right)^3\right]\cdot$$

$$\cdot\left(u_i^{j+1} - u_{i-1}^{j+1}\right)Z - c\left(u_i^{j+1}\right)\rho\left(u_i^{j+1}\right)\left(u_i^{j+1} - u_i^j\right) = 0,\ i = 1,..,I-1,$$

$$\frac{\partial F}{\partial u_{i+1}^{j+1}} = Z\cdot\left[\frac{3k_3}{2}\left(u_{i+\frac{1}{2}}^{j+1}\right)^2 + k_2\left(u_{i+\frac{1}{2}}^{j+1}\right) + \frac{k_1}{2}\right]\cdot\left(u_{i+1}^{j+1} - u_i^{j+1}\right) + Z\cdot k\left(u_{i+\frac{1}{2}}^{j+1}\right),$$

$$\frac{\partial F}{\partial u_i^{j+1}} = Z\cdot\left[\frac{3k_3}{2}\left(u_{i+\frac{1}{2}}^{j+1}\right)^2 + k_2\left(u_{i+\frac{1}{2}}^{j+1}\right) + \frac{k_1}{2}\right]\left(u_{i+1}^{j+1} - u_i^{j+1}\right) -$$

$$-Z\cdot k\left(u_{i+\frac{1}{2}}^{j+1}\right) - Z\cdot\left[\frac{3k_3}{2}\left(u_{i-\frac{1}{2}}^{j+1}\right)^2 + k_2\left(u_{i-\frac{1}{2}}^{j+1}\right) + \frac{k_1}{2}\right]\cdot$$

$$\cdot\left(u_i^{j+1} - u_{i-1}^{j+1}\right) - Z\cdot k\left(u_{i-\frac{1}{2}}^{j+1}\right) - \frac{\partial c}{\partial u_i^{j+1}}\rho\left(u_i^{j+1}\right)\left(u_i^{j+1} - u_i^j\right) -$$

$$-\frac{\partial\rho}{\partial u_i^{j+1}}c\left(u_i^{j+1}\right)\left(u_i^{j+1} - u_i^j\right) - \rho\left(u_i^{j+1}\right)c\left(u_i^{j+1}\right),$$

$$\frac{\partial F}{\partial u_{i-1}^{j+1}} = -S\cdot\left[\frac{3k_3}{2}\left(u_{i-\frac{1}{2}}^{j+1}\right)^2 + k_2\left(u_{i-\frac{1}{2}}^{j+1}\right) + \frac{k_1}{2}\right]\left(u_i^{j+1} - u_{i-1}^{j+1}\right) + S\cdot k\left(u_{i-\frac{1}{2}}^{j+1}\right),$$

Similarly, the boundary conditions are revealed, considering the dependence of the thermal conductivity and heat transfer coefficient on temperature:

$$k\left(u_{\frac{1}{2}}^{j+1}\right)\frac{u_1^{j+1} - u_0^{j+1}}{\Delta x} = h\left(u_0^{j+1}\right)\left(u_0^{j+1} - u_{ins}^{j+1}\right).$$

Let's rewrite it in the form:

$$H\left(u_0^{j+1}, u_1^{j+1}\right) \equiv$$

$$\equiv k\left(u_{\frac{1}{2}}^{j+1}\right)\frac{u_1^{j+1} - u_0^{j+1}}{\Delta x}h\left(u_0^{j+1}\right)\left(u_0^{j+1} - u_{ins}^{j+1}\right) = 0.$$

To apply Newton's method, we find the corresponding derivatives:

$$\frac{\partial H\left(u_0^{j+1,s}, u_1^{j+1,s}\right)}{\partial u_0^{j+1}}\left(u_0^{j+1,s+1} - u_0^{j+1,s}\right) + \frac{\partial H\left(u_0^{j+1,s}, u_1^{j+1,s}\right)}{\partial u_1^{j+1}}\left(u_1^{j+1,s+1} - u_1^{j+1,s}\right) + H\left(u_0^{j+1,s}, u_1^{j+1,s}\right) = 0.$$

Let's expand the derivatives in the following form:

$$\frac{\partial H\left(u_0^{j+1,s}, u_1^{j+1,s}\right)}{\partial u_0^{j+1}} = D_0 = \frac{\partial k}{\partial u_0^{j+1}}\cdot\frac{u_1^{j+1} - u_0^{j+1}}{\Delta x} - \frac{k\left(u_{\frac{1}{2}}^{j+1}\right)}{\Delta x} - \frac{\partial h}{\partial u_0^{j+1}}\left(u_0^{j+1} - u_{ins}^{j+1}\right) - h\left(u_0^{j+1}\right),$$

where

$$\frac{\partial k}{\partial u_0^{j+1}} = \frac{3k_3}{2}\left(u_{\frac{1}{2}}^{j+1}\right)^2 + k_2 \cdot u_{\frac{1}{2}}^{j+1} + \frac{k_1}{2}$$

$$\frac{\partial h}{\partial u_0^{j+1}} = h_1.$$

$$\frac{\partial H\left(u_0^{j+1,s}, u_1^{j+1,s}\right)}{\partial u_1^{j+1}} = E_0 = \frac{\partial k}{\partial u_1^{j+1}} \cdot \frac{u_1^{j+1} - u_0^{j+1}}{\Delta x} + \frac{k\left(u_{\frac{1}{2}}^{j+1}\right)}{\Delta x},$$

where

$$\frac{\partial k}{\partial u_1^{j+1}} = \frac{3k_3}{2}\left(u_{\frac{1}{2}}^{j+1}\right)^2 + k_2 \cdot u_{\frac{1}{2}}^{j+1} + \frac{k_1}{2}.$$

We find the initial values for the recursive formula of the Thomas method:

$$D_0 u_0^{j+1,s+1} - D_0 u_0^{j+1,s} + E_0 u_1^{j+1,s+1} - E_0 u_1^{j+1,s} +$$

$$+H\left(u_0^{j+1,s}, u_1^{j+1,s}\right) = 0,$$

$$u_0^{j+1,s+1} = \frac{-E_0 u_1^{j+1,s+1}}{D_0} + u_0^{j+1,s} + \frac{E_0 u_1^{j+1,s}}{D_0} - \frac{H\left(u_0^{j+1,s}, u_1^{j+1,s}\right)}{D_0}.$$

From here we get

$$\alpha_1 = \frac{-E_0}{D_0},$$

$$\beta_1 = u_0^{j+1,s} + \frac{E_0 u_1^{j+1,s}}{D_0} - \frac{H\left(u_0^{j+1,k}, u_1^{j+1,s}\right)}{D_0}.$$

Differentiation with respect to a parameter

1) In the area $(0,\xi) \times (0, t_{max})$ the discrete problem is solved

$$\rho_1\left(u_i^{j+1}\right) \cdot c_1\left(u_i^{j+1}\right) u_{i,\bar{t}}^{j+1} = \left(k_1\left(u_{i+\frac{1}{2}}^{j+1}\right) u_{ix}^{j+1}\right)_{\bar{x}}$$

$$i = 1,2 \ldots, I-1,\ j = 0,1,\ldots,m-1,\quad u_i^0 = u_0(x_i),\ \mathrm{i} = 0,1,\ldots,I, \tag{5}$$

$$u_I^{j+1} = T_\xi\left(t_{j+1}\right),\ j = 0,1,\ldots,m-1$$

$$k_1\left(u_{\frac{1}{2}}^{j+1}\right) u_{1\bar{x}}^{j+1} = h_1\left(u_0^{j+1}\right)\left(u_0^{j+1} - u_{ins}^{j+1}\right), \qquad j = 0,1,\ldots,m-1.$$

We consider that the coefficient $k_1(u)$ is represented as

$$k_1(u) = k_{10} + k_{11}u + k_{12}u^2 + k_{13}u^3.$$

Assuming that the solution to problem (4) continuously depends on $k_1(u)$) and has a derivative with respect to $k_1(u)$, we differentiate system (4) with respect to the parameter

$$k_{1s}(u),\ s = 0,1,2,3$$

Let's denote

$$\frac{\partial u_i^{j+1}}{\partial k_{1s}} = y_i^{j+1}(s), i = 0,1,\ldots,I,$$

$$j = 0,1,\ldots,m-1, s = 0,1,2,3$$

Then

$$\frac{\partial \rho_1(u_i^{j+1})}{\partial k_{1s}} = \rho_1'(u_i^{j+1}) y_i^{j+1}(s),$$

$$\frac{\partial c(u_i^{j+1})}{\partial k_{1s}} = c_1'(u_i^{j+1}) y_i^{j+1}(s),$$

$$\frac{\partial k_1(u_i^{j+1})}{\partial k_{1s}} = \left(u_i^{j+1}\right)^s + k_1'(u_i^{j+1}) y_i^{j+1}(s),$$

$$\frac{\partial h_1(u_0^{j+1})}{\partial k_{1s}} = h_1'(u_0^{j+1})\, u_0^{j+1}(s).$$

After differentiating system (4) with respect to $k_{1s}, s = 0,1,2,3,$ various problems follow depending on s. These tasks can be written in a single form as follows

$$[c_1'(u_i^{j+1})\rho_1(u_i^{j+1}) + \rho_1'(u_i^{j+1})c_1(u_i^{j+1})]\, y_i^{j+1}(s)\, u_{i,\bar{t}}^{j+1} + c_1(u_i^{j+1})\rho_1\,(u_i^{j+1}) y_{i,\bar{t}}^{j+1}(s) =$$

$$= \left[k_1\left(u_{i+\frac{1}{2}}^{j+1}\right) y_{ix}^{j+1}(s)\right]_{\bar{x}} + \left[\left(\left(u_{\frac{1}{2}}^{j+1}\right)^s + k_1'\left(u_{i+\frac{1}{2}}^{j+1}\right)\frac{y_{i+1}^{j+1}(s) + y_i^{j+1}(s)}{2}\right) u_{i,\bar{x}}^{j+1}\right]_{\bar{x}}$$

$$i = 1,2 \dots, I-1, \ = 0,1,\dots,m-1,$$

$$y_i^0 = 0,\ i = 0,1,\dots,I,\ u_I^{j+1} = 0,\ j = 0,1,\dots,m-1,$$

$$k_1\left(u_{\frac{1}{2}}^{j+1}\right) y_{1,x}^{j+1}(s) + \left(\left(u_{\frac{1}{2}}^{j+1}\right)^s + k_1'\left(u_{\frac{1}{2}}^{j+1}\right)\frac{y_1^{j+1}(s) + y_0^{j+1}(s)}{2}\right) u_{1\bar{x}}^{j+1} =$$

$$= h_1'(u_0^{j+1})(u_0^{j+1} - u_{ins}^{j+1}) + h_1(u_0^{j+1}) y_0^{j+1}(s),\ j = 0,1,\dots,m-1.$$

The values of the coefficients $k_{1s}, s = 0,1,2,3$of the coefficient of thermal conductivity of soil $k_1(u)$ will be found from the condition of the minimum of the functional

$$J(k_1(u)) = \sum_{j=0}^{m-1}\left(u_0^{j+1}(k_1) - T_0^{j+1}\right)^2 \Delta t$$

Direct differentiation of the last equality with respect to $k_{1s}, s = 0,1,2,3$ gives us the gradient of the composed functional written as

$$\nabla J(k_{1s}) =$$
$$= 2\sum_{j=0}^{m-1}\left(u_0^{j+1}(k_1) - T_0^{j+1}\right) y_0^{j+1}(s)\Delta t,$$
$$s = 0,1,2,3. \quad (6)$$

Then

$$J(k_1(u)) = \sum_{s=0}^{3} J(k_{1s})$$

Knowing the explicit expression for the gradient of the functional, the parameters of the functions $k_1(u)$ are defined as follows

$$k_{1s}(n+1) = k_{1s}(n) + \mu_1(s)\nabla J(k_{1s}(n)),\ =$$
$$= 0,1,2,3.$$

To determine the damping factor $\mu_1(s)$ of the functional

$$J(k_{1s}(n+1)) =$$
$$= \sum_{j=0}^{m-1}\left(u_0^{j+1}(k_{1s}(n+1)) - T_0^{j+1}\right)^2 \Delta t.$$

Minimize by parameter $\mu_1(s)$. For this we use the expansion

$$u_0^{j+1}(k_{1s}(n+1)) =$$
$$= u_0^{j+1}(k_{1s}(n)) + \mu_1(s)\nabla J(k_{1s}(n)) =$$
$$= u_0^{j+1}(k_{1s}(n)) +$$

$$+\frac{\partial u_0^{j+1}\big(k_{1s}(n)\big)}{\partial k_{1s}}\upmu_1(s)\nabla J\big(k_{1s}(n)\big)+o\big(\upmu_1(s)\big)^2.$$

Using this expansion from (6) after some transformations, we obtain the parameter of the fastest descent in the form:

$$\upmu_1(s)=$$

$$=-\frac{\sum_{j=0}^{m-1}\big(u_0^{j+1}\big(k_{1s}(n)\big)-T_0^{j+1}\big)y_0^{j+1}(s)\Delta t}{\sum_{j=0}^{m-1}\big(y_0^{j+1}\big)^2\Delta t\,\nabla J\big(k_{1s}(n)\big)},$$

$$s=0,1,2,3. \qquad (7)$$

Using (7), we write out the final calculation formula for each coefficient of the function $k_1(u)$ in the following form

$$k_{1s}(n+1)=k_{1s}(n)-$$

$$-\frac{\sum_{j=0}^{m-1}\big(u_0^{j+1}\big(k_{1s}(n)\big)-T_0^{j+1}\big)y_0^{j+1}(s)\Delta t}{\sum_{j=0}^{m-1}\big(y_0^{j+1}\big)^2\Delta t},$$

$$s=0,1,2,3.$$

2) To determine the specific heat coefficient $c_1(u)$ we represent it as

$$c_1(u)=c_{10}+c_{11}u.$$

This is the most commonly used dependence in practice [1].

Now the discrete problem is composed in the region $(0,\xi)\times(t_{max},2t_{max})$ and has the form

$$\rho_1\big(u_i^{j+1}\big)\cdot c_1\left(u_i^{j+1}\right)u_{i,\bar t}^{j+1}=$$

$$=\left(k_1\left(u_{i+\frac{1}{2}}^{j+1}\right)u_{ix}^{j+1}\right)_{\bar x},$$

$$i=1,2\ldots,I-1,j=m,m+1,\ldots,2m-1,$$

$u_i^m=u_0(x_i)$ – solution of problem (5) for $j=m-1,i=0,1,\ldots,I$,

$$u_I^{j+1}=T_\xi\big(t_{j+1}\big),\quad j=m,m+1,\ldots,2m-1, \quad (8)$$

$$k_1\left(u_{\frac{1}{2}}^{j+1}\right)u_{1\bar x}^{j+1}=h_1\big(u_0^{j+1}\big)\big(u_0^{j+1}-u_{ins}^{j+1}\big),$$

$$j=m,m+1,\ldots,2m-1.$$

In this case, all the coefficients of the functions $\rho_1(u)$, $k_1(u)$ and $h_1(u)$ are taken from the current iteration level, and the coefficients of the function $c_1(u)$ are changed, calculating the minimum of the functional

$$J\big(c+1(u)\big)=\sum_{j=m}^{2m-1}\big(u_0^{j+1}(c_1)-T_0^{j+1}\big)^2\Delta t=$$

$$=J\big(c_{10}(u)\big)+J\big(c_{11}(u)\big)=$$

$$=\sum_{j=m}^{2m-1}\big(u_0^{j+1}(c_{10})-T_0^{j+1}\big)^2\Delta t+$$

$$+\sum_{j=m}^{2m-1}\big(u_0^{j+1}(c_{11})-T_0^{j+1}\big)^2\Delta t.$$

Assuming the continuous dependence of the solution of the problem u_i^{j+1} on the parameters c_{10} and c_{11}, and, assuming the existence of a derivative of the function u_i^{j+1} with respect to the named parameters, we differentiate (8) with respect to c_{1s}, $s=0,1$.

As early as we introduce the notation

$$\frac{\partial u_i^{j+1}}{\partial c_{1s}}=y_i^{j+1}(s),i=0,1,\ldots,I,$$
$$j=m,m+1,\ldots,2m-1,\quad s=0,1.$$

And given that

$$\frac{\partial c_1\big(u_i^{j+1}\big)}{\partial c_{1s}}=\big(u_i^{j+1}\big)^s+c_1'\big(u_i^{j+1}\big)y_i^{j+1}(s),s=0,1.$$

We compose a system with respect to the unknowns $y_i^{j+1}(s)$ in the following form

$$\left(\left(u_i^{j+1}\right)^s + c_1'\left(u_i^{j+1}\right)y_i^{j+1}(s)\right)\rho_1\left(u_i^{j+1}\right)u_{i,\bar{t}}^{j+1} + \rho_1'\left(u_i^{j+1}\right)y_i^{j+1}(s)c_1\left(u_i^{j+1}\right)u_{i,\bar{t}}^{j+1} +$$

$$+c_1\left(u_i^{j+1}\right)\rho_1\left(u_i^{j+1}\right)y_{i,\bar{t}}^{j+1}(s) =$$

$$= \left[k_1\left(u_{i+\frac{1}{2}}^{j+1}\right)y_{ix}^{j+1}(s)\right]_{\bar{x}} + \left[\left(k_1'\left(u_{i+\frac{1}{2}}^{j+1}\right)\frac{y_{i+1}^{j+1}(s) + y_i^{j+1}(s)}{2}\right)u_{ix}^{j+1}\right]_{\bar{x}}$$

$$i = 1,2 \ldots, I-1, j = m, m+1, \ldots, 2m-1,$$

$$y_i^m = 0, \quad i = 0,1, \ldots, I, y_I^{j+1} = 0, j = m, m+1, \ldots, 2m-1,$$

$$k_1\left(u_{\frac{1}{2}}^{j+1}\right)y_{1,\bar{x}}^{j+1}(s) + \left(k_1'\left(u_{\frac{1}{2}}^{j+1}\right)\frac{y_1^{j+1}(s) + y_0^{j+1}(s)}{2}\right)u_{1\bar{x}}^{j+1} =$$

$$= h_1'\left(u_0^{j+1}\right)\left(u_0^{j+1} - u_{ins}^{j+1}\right) + h_1\left(u_0^{j+1}\right)y_0^{j+1}(s), j = m, m+1, \ldots, 2m-1.$$

Here $s = 0,1$.

Repeating all the calculations done when deriving the calculation formula k_{1s}, we derive the calculation formula for c_{1s} in the following form

$$c_{1s}(n+1) = c_{1s}(n) -$$

$$-\frac{\sum_{j=m}^{2m-1}\left(u_0^{j+1}\left(c_{1s}(n)\right) - T_0^{j+1}\right)y_0^{j+1}(s,n)\Delta t}{\sum_{j=m}^{2m-1}\left(y_0^{j+1}(s,n)\right)^2\Delta t},$$

$$s = 0,1$$

The corresponding functional has the form

$$J\left(c_{1s}(n)\right) = \sum_{j=m}^{2m-1}\left(u_0^{j+1}\left(c_{1s}(n)\right) - T_0^{j+1}\right)^2\Delta t,$$

$$s = 0,1.$$

3) Assuming the dependence of the specific density $\rho_1(u)$ in the form

$$\rho_1(u) = \rho_{10} + \rho_{11}u$$

and given that $\frac{\partial u_i^{j+1}}{\partial \rho_{1s}} = y_i^{j+1}(s)$, we compose the corresponding discrete problem. This time in the area $(0,\xi) \times (2t_{max}, 3t_{max})$ in the form

$$\left(\left(u_i^{j+1}\right)^s + \rho_1'\left(u_i^{j+1}\right)y_i^{j+1}(s)\right)c_1\left(u_i^{j+1}\right)u_{i,\bar{t}}^{j+1} + c_1'\left(u_i^{j+1}\right)y_i^{j+1}(s)\rho_1\left(u_i^{j+1}\right)u_{i,\bar{t}}^{j+1} +$$

$$+c_1\left(u_i^{j+1}\right)\rho_1\left(u_i^{j+1}\right)y_{i,\bar{t}}^{j+1}(s) =$$

$$= \left[k_1\left(u_{i+\frac{1}{2}}^{j+1}\right)y_{ix}^{j+1}(s)\right]_{\bar{x}} + \left[k_1'\left(u_{i+\frac{1}{2}}^{j+1}\right)\frac{y_{i+1}^{j+1}(s) + y_i^{j+1}(s)}{2}u_{ix}^{j+1}\right]_{\bar{x}}$$

$$i = 1,2 \ldots, I-1, j = 2m, m+1, \ldots, 3m-1, s = 0,1,$$

$$y_i^{2m} = 0, i = 0,1, \ldots, I; y_I^{j+1} = 0, \quad j = 2m, m+1, \ldots, 3m-1$$

$$k_1\left(u_{\frac{1}{2}}^{j+1}\right)y_{1,\bar{x}}^{j+1}(s)+\left(k_1'\left(u_{\frac{1}{2}}^{j+1}\right)\frac{y_1^{j+1}(s)+y_0^{j+1}(s)}{2}\right)u_{1\bar{x}}^{j+1}=$$

$$=h_1'(u_0^{j+1})(u_0^{j+1}-u_{ins}^{j+1})+h_1(u_0^{j+1})y_0^{j+1}(s),\quad j=2m,m+1,\dots,3m-1.$$

Here $s=0,1$.

In this case, to calculate the coefficients $\rho_1(u)$, the formula is derived

$$\rho_{1s}(n+1)=\rho_{1s}(n)-$$

$$-\frac{\sum_{j=2m}^{3m-1}(u_0^{j+1}(\rho_{1s}(n))-T_0^{j+1})y_0^{j+1}(s,n)\Delta t}{\sum_{j=2m}^{3m-1}\left(y_0^{j+1}(s,n)\right)^2\Delta t},$$

$$s=0,1.$$

Functional is minimized

$$J(\rho_{1s}(n))=\sum_{j=2m}^{3m-1}\left(u_0^{j+1}(\rho_{1s}(n))-T_0^{j+1}\right)^2\Delta t,$$

$$s=0,1.$$

4) Calculation of the heat transfer coefficient $h_1(u)$.

In practical calculations, power-law dependences of the heat transfer coefficient on the soil temperature on the contact surface of two media are usually used. We'll look at the dependency:

$$h_1(u)=h_{10}+h_{11}u$$

Then

$$\frac{\partial h_1(u_i^{j+1})}{\partial h_{1s}}=\left(u_i^{j+1}\right)^s+h_1'(u_i^{j+1})y_i^{j+1}(s),$$

$$s=0,1,$$

here $\frac{\partial u_i^{j+1}}{\partial h_{1s}}=y_i^{j+1}(s), i=1,2,\dots,I-1, j=3m,m+1,\dots,4m-1, s=0,1.$

In this case, the problem is considered in the area $(0,\xi)\times(3t_{max},4t_{max})$ and the next function coefficients are taken from the current iteration level:

$$\rho_1(u),\ c_1(u)\text{ и }k_1(u).$$

After skipping the difference scheme in the next grid domain

$$\omega_{14}=\{x_\iota=i\Delta x,t_j=j\Delta t;\ i=0,1,\dots,I;\ j=3m,\dots,4m-1\},$$

Let us immediately write out the difference problem for the function y_i^{j+1}. The difference scheme has the form

$$\rho_1'(u_i^{j+1})c_1(u_i^{j+1})y_i^{j+1}(s)u_{i,\bar{t}}^{j+1}+c_1'(u_i^{j+1})\rho_1(u_i^{j+1})y_i^{j+1}(s)u_{i,\bar{t}}^{j+1}+c_1(u_i^{j+1})\rho_1(u_i^{j+1})y_{i,\bar{t}}^{j+1}(s)=$$

$$\left[k_1\left(u_{i+\frac{1}{2}}^{j+1}\right)y_{ix}^{j+1}(s)\right]_{\bar{x}}+\left(k_1'\left(u_{i+\frac{1}{2}}^{j+1}\right)\frac{y_{i+1}^{j+1}(s)+y_i^{j+1}(s)}{2}\right)u_{ix}^{j+1}\Bigg]_{\bar{x}}$$

$$i=1,2\dots,I-1;\quad j=3m,m+1,\dots,4m-1;\ s\ =0,1,$$

$$y_i^{3m}=0,i=0,1,\dots,I;y_I^{j+1}=0,j=3m,m+1,\dots,4m-1,$$

$$k_1\left(u_{\frac{1}{2}}^{j+1}\right)y_{1,\bar{x}}^{j+1}(s)+\left(k_1'\left(u_{\frac{1}{2}}^{j+1}\right)\frac{y_1^{j+1}(s)+y_0^{j+1}(s)}{2}\right)u_{1\bar{x}}^{j+1}=$$

$$=\left(\left(u_0^{j+1}\right)^s+h_1'(u_0^{j+1})y_0^{j+1}(s)\right)(u_0^{j+1}-u_{ins}^{j+1})+h_1(u_0^{j+1})y_0^{j+1}(s),$$

$$j=3m,m+1,\dots,4m-1.$$

Here $s = 0{,}1$.

In this case, by controlling the parameters $h_{1s}, s = 0{,}1$, the functional is minimized

$$J(h_{1s}(n)) = \sum_{j=3m}^{4m-1} \left(u_0^{j+1}(h_{1s}(n)) - T_0^{j+1}\right)^2 \Delta t,$$

$$s = 0{,}1.$$

And the control parameters of optimization processes are determined by the formula

$$h_{1s}(n+1) = h_{1s}(n) - $$

$$- \frac{\sum_{j=3m}^{4m-1}\left(u_0^{j+1}(h_{1s}(n)) - T_0^{j+1}\right)y_0^{j+1}(s,n)\Delta t}{\sum_{j=3m}^{4m-1}\left(y_0^{j+1}(s,n)\right)^2 \Delta t},$$

$$s = 0{,}1.$$

Comment. Everywhere we have assumed that the parameters $\rho_1(u), c_1(u), k_1(u)$ *and* $h_1(u)$ depend on u in the form of a polynomial. However, the considered method is applicable in another form of dependence on u.

Experimental setup

Figure 2 – Containers with soil

Containers with sensors were built for the experiment. Photos of containers are shown in Fig. 2. The side faces of the containers are made of 2 cm thermally insulated material, and the end faces are in contact with the environment (air). In each compartment of the container, 15 cm long, there are various soils. One end side is heated with lamps. The second outer side is affected by the ambient temperature.

3 sensors (C2, C3, C4) are evenly distributed inside the material as shown in Figure 1. They measure temperature with an error of 0.3 degrees Celsius according to the technical data sheet of the sensor. In addition to these sensors, there are 2 more sensors (C1, C5) close to the ends to measure the ambient temperature. The errors of these sensors are the same as those of the previous sensors. The temperature data measurement is taken at intervals of 10 minutes.

For calculations, a two-chamber container was considered and, accordingly, with two materials: sand and black soil. The data were measured over a period of three months, and the physical length of the entire container is determined through the interval $x \in (0, l)$, where $l = 30$ cm. The boundary of the two media is at a distance of $x = 15$ cm and the temperature measurement sensor is also located there. Since there is an exchange with the environment at the end boundaries, Robin boundary conditions were considered for the numerical solution. Measurements at points $x = 0$ см. and $x = 30$ cm determine the temperature at the end boundaries. The temperature values of the measured data can be seen in Fig. 3.

It should be noted that enough time has passed to conduct numerical experiments (about 3 months) from the installation of measuring instruments and the data used in the proposed article. Also, for the initial condition, the interpolation of the measured data was taken.

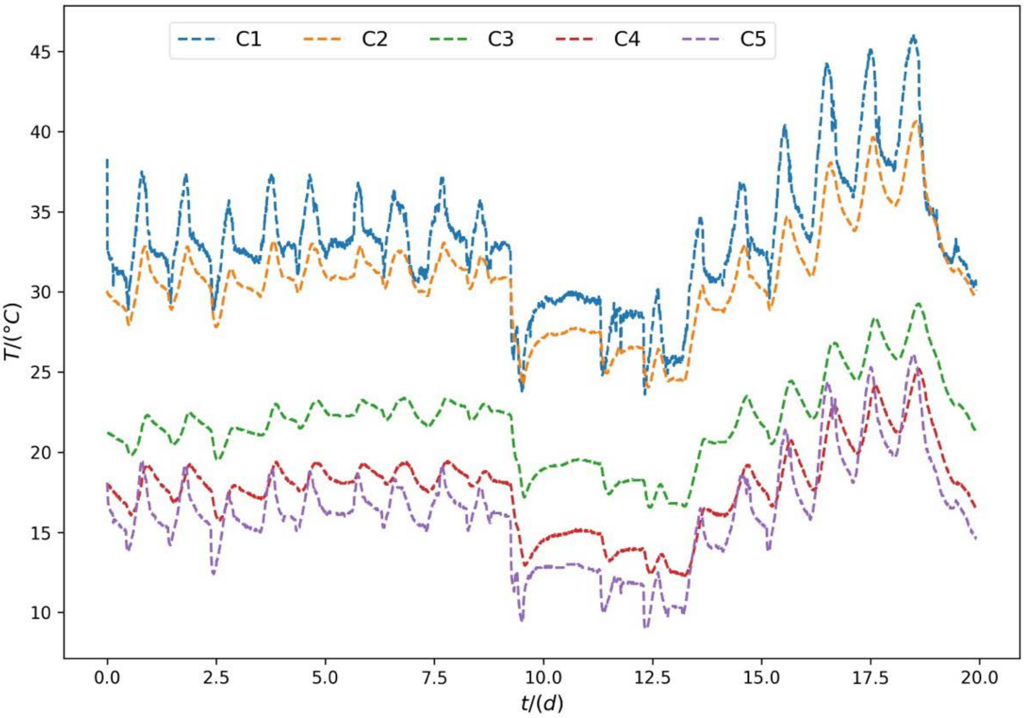
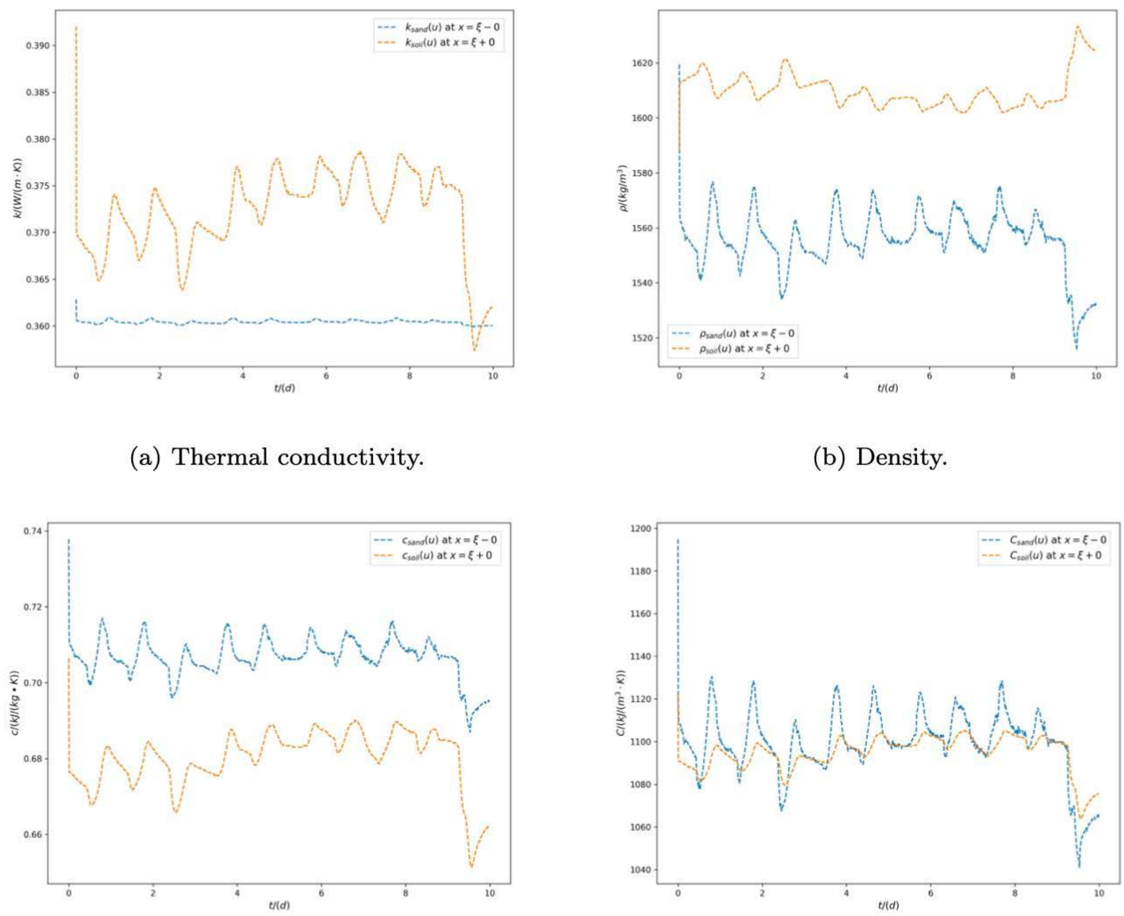


Figure 3 – Containers with soil

Results

The measured temperature data were used to solve a numerical problem to find all thermophysical coefficients (thermal conductivity coefficient, specific heat capacity, specific density and heat transfer coefficient). Thanks to the steepest descent method, the functionals converge fairly quickly and reach a minimum in 6 and 7 iterations. The minimization of the functional continued until the relative error between the nonlinear solution and the experimental data reached ~4.3% for chernozem and ~3.12% for sand, which in turn shows a fairly good accuracy of the solution. If we look at the absolute errors in two environments – ~6.3% and ~5.3%, we see that they also meet our expectations.

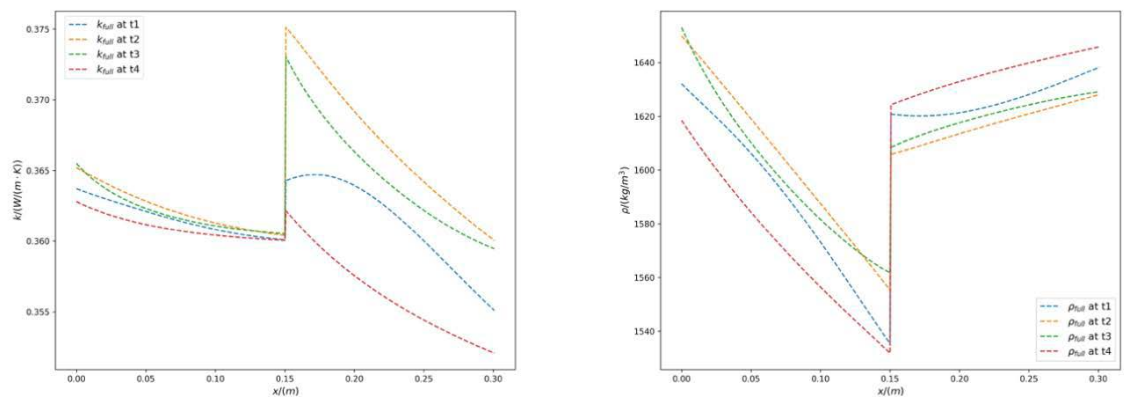
In addition, the values of the coefficients at the contact boundary of two media were considered. Figure 4 illustrates the values of thermal conductivity, density, specific heat capacity and volumetric heat capacity from the left approximation (sand) and the right approximation (soil) to the boundary. As can be seen from the graph, the values of the coefficients at the contact discontinuity differ significantly from each other, but in the case of the volumetric heat capacity coefficient, the values at many points coincide at the boundary of two media, which can prove a continuous volumetric heat capacity at the boundary of two different media. This statement needs further research on other materials. The large difference at the initial points in time is associated with a rough initial approximation of the iterative process for the parameters of thermophysical coefficients.



(a) Thermal conductivity. (b) Density.

(c) Specific heat. (d) Volumetric heat capacity.

Figure 4 – The distribution of a) thermal conductivity b) density c) specific and d) volumetric heat capacity at the boundary of two materials.



(a) Thermal conductivity. (b) Density.

Figure 5 – Distribution of thermal conductivity and density along the container during $t1 = 2.5/d, t2 = 5/d, t3 = 7.5/d, t4 = 10/d$.

Fig.4 and Fig.5 illustrate the values of thermophysical coefficients along the container. The graphs clearly show jumps-discontinuities in the values of thermophysical coefficients at the contact boundary of contact between two media, except for the coefficient of volumetric heat capacity. From here it can also be said that the volumetric heat capacity shows a continuous nature of the values.

Conclusion

In the context of predicting and finding all thermophysical coefficients (thermal conductivity, heat capacity, density and heat transfer), this article proposes an efficient numerical method. In contrast to the methods previously proposed in the literature, this approach allows one-time determination of all thermophysical coefficients in two media with a contact boundary. This approach takes into account the impossibility of finding several coefficients in one time interval. To solve this problem, the entire measured data time is divided into segments equal to the number of coefficients, and the corresponding coefficient is calculated in each segment. In addition, one should not forget that a solution of the nonlinear heat equation is proposed with the heat conductivity coefficient, which is a cubic function, and with the heat capacity, density, and heat transfer coefficients, which are linear functions. The system of nonlinear equations is solved by Newton's method, which ensures high convergence of the solution. The initial approximation for Newton's method is taken from the solution of a linearized difference problem. Also the next approximation for Newton's method, i.e. for a nonlinear difference problem, is found using the Thomas method (sweep), which in turn is unconditionally stable. Finding the thermophysical coefficients is calculated by minimizing the corresponding functional using the steepest descent method. Using the differentiation of a nonlinear difference problem with respect to the desired parameter, the gradient of the functional and the damping coefficient are found in explicit form. With the help of this, the elimination of the solution of the adjoint problem for the solution of inverse problems is achieved. A proof of the quadratic convergence of the iterative scheme for Newton's method is also proposed.

In conclusion, it can be said that research in the field of coefficient inverse problems for non-linear equations should be advanced with detailed experimental measurements, including, for example, moisture, freezing, porosity, etc.

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References

1. Luikov, A.V. *Heat and Mass Transfer in Capillary-Porous Bodies*. 1964.

2. Rysbaiuly B., Adamov A.A. *Matematicheskoe modelirovanie tepla I massoobmennogo processa v mnogosloynom grunte.* [Mathematical modeling of heat and mass transfer process in multilayer soil]. Publishing House «Kazakh Universitety», 2020, 211 p.

3. Desta, T. Z., and Langmans, J., and Roels, S. *Experimental data set for validation of heat, air and moisture transport models of building envelopes.* Building and Environment, 2011.

4. Lipus, M.P., Reinsch, T., Weisenberger, T.B., Kragset, S., Stefansson, A., Bogason, S.G. “Monitoring of a reverse cement job in a high-temperature geothermal environment.” *Geothermal Energy.*

5. Lesnic, D. “Inverse Problems with Applications in Science and Engineering”, *CRC Press*, Abingdon, UK, (2021): 349 p.

6. Jumabekova, A., Berger, J., Foucquier, A., Dulikravich, G.S. “Searching an optimal experiment observation sequence to estimate the thermal properties of a multilayer wall under real climate conditions.” *International Journal of Heat and Mass Transfer*. (2020).

7. Tien-Mo Shih, Chao-Ho Sung and Bao Yang. “A Numerical Method for Solving Nonlinear Heat Transfer Equations.” *Numerical Heat Transfer, Part B: Fundamentals: An International Journal of Computation and Methodology.* (2008).

8. Berger J., Dutykh D., and Mendes N., and Rysbaiuly B. “A new model for simulating heat, air and moisture transport in porous building material.” *International Journal of Heat and Mass Transfer*. 134, (2019): 1041–1060.

9. Travis, M., Matthew, J. “Solving nonlinear heat transfer problems using variation of parameters.” *International Journal of Thermal Sciences.* (2015).

10. Battaglia, Jean-Luc, Maachou, Asma, Malti, Rachid, and Melchior, Pierre. “Nonlinear heat

diffusion simulation using Volterra series expansion." *International Journal of Thermal Sciences.* (2013).

11. Nguyen, Huy Tuana and Pham, Hoang Quanc. "Some extended results on a nonlinear ill-posed heat equation and remarks on a general case of nonlinear terms." *Nonlinear Analysis: Real World Applications.* (2011).

12. Huntul, Mousa, and Lesnic, Daniel. "Determination of the time-dependent convection coefficient in two-dimensional free boundary problems." *Engineering Computations.* (2020).

13. Jumabekova, A., Berger, J., Dutykh, D., and Le Meur, H. "An efficient numerical model for liquid water uptake in porous material and its parameter estimation." *Numerical Heat Transfer, Part A: Applications.* (2019).

14. Hasanov, A. "Simultaneously identifying the thermal conductivity and radiative coefficient in heat equation from Dirichlet and Neumann boundary measured outputs." *Journal of Inverse and Ill-Posed Problems.* (2020).

15. Lesnic, D., Hussein, M.S., Kamynin, V., Kostina, B. "Direct and inverse source problems for degenerate parabolic equations." *Journal of Inverse and Ill-Posed Problems.* (2020).

16. Kabanikhin, S. I., and Shishlenin, M. A. "Theory and numerical methods for solving inverse and ill-posed problems." *Journal of Inverse and Ill-Posed Problems.* (2019).

17. Rysbaiuly, B., Alpar, S., and Mukhametkaliyeva, N., "Nelineynaya obratnaya zadacha teploprovodnosti. Kvasilineynaya dvukhsloynaya obratnaya zadacha teploperedachi." [Nonlinear inverse problem of heat conduction. Quasi-linear two-layer inverse heat transfer problem]. Avtorskoe proizvedenie. №EC-02-21121KZ Certificate of COPYRIGHT 10, 2021.

18. Rysbaiuly, B., "Obratnye zadachi nelineynoy teploprovodnosti". [Inverse Problems of Nonlinear Heat Conduction]. *Издательский дом «Қазақ Университеті»*, (2022): 369.

19. Rysbaev, B., "Ustoychivost raznostnoy schemy dlya teploprovodnogo gaza s kontaktnym razryvom." [Stability of a difference scheme for a heat-conducting gas with a contact discontinuity. Note. methods of fun. analysis in non-classical equations mat. physics]. Novosibirsk: Siberian Branch of the USSR Academy of Sciences, (1988): 17 – 24.

20. Rysbaiuly, B. "Metod. konechnyh raznostey dlya odnomernogo teploprovodnogo vyazkogo szhimaemogo gaza s kontaktnym razryvom." [Finite difference method for one-dimensional heat-conducting viscous compressible gas with a contact discontinuity.] *Siberian journal of computational mathematics*, RAS SB, № 3(4), (2001): 32-43.