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Control of Vibrations of a Beam with Nonlocal Boundary Conditions

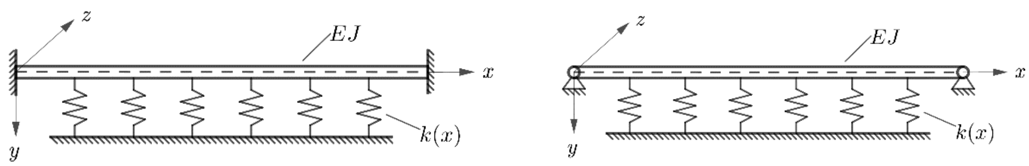
Abstract. In this article is considered the models of uniform Euler-Bernoulli beams with an arbitrary variable coefficient of foundation on a finite segment. The variable of foundation corresponds to the Winkler model. The control problem the first eigenvalues of the beam vibration is investigated. Two types of fastenings at the ends are considered: clamped-clamped and hinged-hinged. The control is based on the Kanguzhin algorithm through integral perturbations of one of the boundary conditions of the original problem. Conditions for the boundary parameters for controlling the first eigenvalues are found. First, a result is formulated regarding the control of the first eigenvalue of the oscillation of the Euler-Bernoulli beam with hinge fastening at both ends. The result is then extended to control with several eigenvalues for this beam, which are important from the point of view of the application. Such questions are especially relevant when studying the resonant natural frequencies of a mechanical system. A similar result was obtained for a Euler-Bernoulli beam with clamped fastening at both ends. Such results of eigenvalue control of a mechanical system contribute to the creation of various non-destructive testing devices that are widely used in technical acoustics.

Key words: Vibration controls, eigenvalues, characteristic determinant, Euler-Bernoulli beam.

Introduction

Oscillations of mechanical systems are described by a differential equation and initial-boundary conditions. The differential equation contains information about the qualitative properties of the object under consideration, such as the physical properties of the material, the basic laws of vibration, according to which the vibration occurs. Initial-boundary conditions describe the behavior at the initial moment of time and the state at the boundaries of the object under study. Depending on the mechanical problems, different questions arise in the control of the vibrations of mechanical systems. Questions of the isospectral problem may arise, which require the coincidence of the spectrum of different two problems [1, 2]. In such problems, the goal is achieved mainly through additional conditions on the physical parameters of the object, for example, in [1], conditions on the density were found. Also, questions arise on the extreme properties of the eigenvalues and the question of the behavior of the eigenvalues during the destruction or defects of systems [3, 4]. In such tasks, lumped elements at internal points and features of the geometric structures of an object play an important role. In [5, 6], the question was investigated whether it is possible to change the spectrum of the boundary value problem to a predetermined one by changing only one of the boundary conditions. In such problems, it is important to take into account the dependence of the boundary conditions on the spectral parameter and the singularity of the original problem as the coincidence of the spectrum with the entire complex plane. Along with the above questions, problems of control of some eigenvalues also arise [7-10]. In such problems, one of the important methods is the integral perturbation of the original problem. Basis properties and the question of the asymptotic behavior of the eigenvalues for differential operators with integral perturbations for an arbitrary order were considered in [11, 12], and for the second order in [13-17]. Similar questions arise in the problems of aeroelasticity [18, P. 291] and MEMS resonators [19, Ch. 1].

In this paper, we investigate the problem of controlling the first eigenvalues of the Euler-Bernoulli beam vibration with hinged and clamped fixings at both ends, see Fig. 1. Control is achieved due to integral perturbations of the boundary conditions of the original problem. The research methods of this work are conceptually close to the methods of work [9, 10]. Conditions on the boundary parameters for controlling the first eigenvalues are found (see Theorems 1, 2, 3).



a) clamped-clamped b) hinged-hinged

Figure 1 – Euler-Bernoulli beams with two types of anchorage

Problem statement and main results

Let $k(x), x \in (0; l)$ be a real-valued summable symmetric function with respect to the point $x = \frac{l}{2}$, i.e. $k(x) = k(l - x), x \in \left[0; \frac{l}{2}\right]$. In the sequel, $k(x)$ will mean the variable coefficient of the beam foundation. Before presenting the main result, we recall that the equation of transverse vibrations of a homogeneous Euler-Bernoulli beam of length l at $0 < x < l,\ t > 0$ has the following form

$$\rho A \frac{\partial^2 \mathrm{w}(x,t)}{\partial t^2} + EJ \frac{\partial^4 \mathrm{w}(x,t)}{\partial x^4} + k(x) w(x,t) = 0,$$

where $\mathrm{w}(x, t)$ is the transverse displacement; ρ is the density of material; A is the cross-sectional area; E is the elastic modulus of material; J is the moment of inertia of the cross-sectional area of relative to around the z axis.

We denote $\lambda = \dfrac{\omega^2 \rho A}{EJ}$. In the new notation, the problem of transverse vibrations of a beam with hinged fastening at both ends by replacing $\mathrm{w}(x,t) = y(x)\sin(\omega t)$ is reduced to the following spectral problem:

$$y^{IV}(x) + \frac{k(x)}{EJ} y(x) = \lambda y(x),\ 0 < x < l, \quad (1)$$

$$y(x)\big|_{x=0} = 0,\ y(x)\big|_{x=l} = 0,$$
$$y''(x)\big|_{x=0} = 0,\ y''(x)\big|_{x=l} = 0. \quad (2)$$

The operator corresponding to the spectral problem (1)-(2) is denoted by $Ky(x) = \lambda y(x)$. For $k(x) \equiv 0$, the eigenvalues of the operator K are calculated explicitly $\lambda_n = \left(\dfrac{\pi n}{l}\right)^4, n = 1, 2, \ldots$ [20, P. 269]. Spectral properties with respect to symmetric equivalence of the operator K were investigated in [21]. The system of eigenfunctions $\{y_n(x)\}_{n=1}^{\infty}$ of the operator K forms an orthonormal basis of $L_2(0, l)$.

Problem 1: Let $(B_1 - \mu I)$ be the operator in $L_2(0, l)$ correspond to the problem:

$$u^{IV}(x) + \frac{k(x)}{EJ} u(x) = \mu u(x), \quad 0 < x < l, \quad (3)$$

$$u(x)\big|_{x=0} = \alpha_1 \int_0^l u(x) y_1(x) dx,$$
$$u(x)\big|_{x=l} = 0,\ u''(x)\big|_{x=0} = 0,$$
$$u''(x)\big|_{x=l} = 0, \quad (4)$$

where α_1 is a nonzero real number. $y_1(x)$ is an eigenfunction of problem (1), (2) corresponding to the first eigenvalue λ_1. Select the boundary parameter α_1 so that the eigenvalues of the operator B_1 are outside the interval $(-\lambda_2, \lambda_2)$.

The operator B_1 can be considered a perturbation of the operator K, since only the domain $D(K)$ of the operator K has changed.

The main result is

Theorem 1. If the boundary parameter α is chosen so that the inequality holds

$$(\lambda_2 - \lambda_1) \le \alpha_1 y_1'''(0), \quad (5)$$

then the eigenvalues $\{\mu_n\}_{n=1}^{\infty}$ of the operator B_1 are determined by the formula $\mu_n = \lambda_n$ for $n \ge 2$ and μ_1 is the only real root outside the interval $(-\lambda_2, \lambda_2)$ of the equation

$$1 = \frac{\alpha_1 y_1'''(0)}{\mu - \lambda_1}.$$

Auxiliary statements. To prove the theorem, we need some auxiliary statements. It is obviously, the operator K is self-adjoint. All eigenvalues of self-adjoint operator K are real. The eigenfunctions of a self-adjoint operator corresponding to different eigenvalues are orthogonal [22, Theorem 3, Corollary, P. 31].

Lemma 1. We have the identity

$$(\mu-\lambda_n)\int_0^l u(x)y_n(x)dx = u(0)y_n'''(0).$$

Proof of Lemma 1. The right-hand side of the identity can be written as follows

$$(\mu-\lambda_n)\int_0^l u(x)y_n(x)dx = \int_0^l \mu u(x)y_n(x)dx - \int_0^l u(x)\lambda_n y_n(x)dx =$$

[Taking into account (1) and (3), we have]

$$= \int_0^l \left(u^{IV}(x) + \frac{k(x)}{EJ}u(x) \right) y_n(x)dx - \int_0^l u(x)\left(y_n^{IV}(x) + \frac{k(x)}{EJ}y_n(x) \right)dx,$$

Direct calculation shows that the first term is equal to

$$\int_0^l \left(u^{IV}(x) + \frac{k(x)}{EJ}u(x) \right) y_n(x)dx = u(0)y_n'''(0) + \int_0^l u(x)\left(y_n^{IV}(x) + \frac{k(x)}{EJ}y_n(x) \right)dx.$$

Taking into account the last relation, we obtain the proof of Lemma 1. Lemma 1 is proved.

Proof of Theorem 1. Taking into account Lemma 1, for further calculations we rewrite the perturbed boundary conditions (4) of the operator B_1 using the forms $V_{k-1}, k = 1, 2, 3, 4$

$$\begin{aligned} V_0[u] &:= f(\alpha,\mu,\lambda_1)u(x)\big|_{x=0} = 0, \\ V_1[u] &:= u(x)\big|_{x=l} = 0, \\ V_2[u] &:= u''(x)\big|_{x=0} = 0, \\ V_3[u] &:= u''(x)\big|_{x=l} = 0, \end{aligned} \qquad (6)$$

where $f(\alpha_1,\mu,\lambda_1) = \dfrac{\mu-\lambda_1-\alpha_1 y_1'''(0)}{\mu-\lambda_1}, \mu \neq \lambda_1$.

It is known that the question of finding the eigenvalues of problem (3), (6) is reduced to finding the zeros of the characteristic determinant [22, pp. 1-27]

$$\Delta(\mu) = \begin{vmatrix} V_0[u_1] & V_1[u_1] & V_2[u_1] \\ V_0[u_2] & V_1[u_2] & V_2[u_2] \\ V_0[u_3] & V_1[u_3] & V_2[u_3] \end{vmatrix} \qquad (7)$$

where $V_{k-1}, k = 1, 2, 3, 4$ are boundary forms that correspond to boundary conditions (6), $\{u_k(\mu,x)\}_{k=1}^3$ are the fundamental system of solutions to equation (3) generated by the conditions

$$u_k^{(m-1)}(\mu,x)\big|_{x=a} = \begin{cases} 0, & \text{if } k \neq m, \\ 1, & \text{if } k = m, \end{cases} \quad k, m = 1, 2, 3,$$

where a is an arbitrary point of the segment $[0, l]$. The characteristic determinant (7) for the operator B_1 has the form

$$\Delta(\mu) = f(\alpha_1,\mu,\lambda_1)\,\Delta_K(\lambda), \qquad (8)$$

where $\Delta_K(\lambda)$ is the characteristic determinant of the operator K. The first part of Theorem 1 follows from relation (8). Let us prove the second part of Theorem 1. Let us calculate the first eigenvalue μ_1 and prove that it is outside the interval $(-\lambda_2, \lambda_2)$. To do this, we find the single root of the function $f(\alpha_1,\mu,\lambda_1)$ with respect to the spectral parameter μ: $\mu_1 = \lambda_1 + \alpha_1 y_1'''(0)$. It follows from condition (5) for the boundary parameter α that the second part of Theorem 1. Theorem 1 is proved.

The previous Theorem 1 can be extended to control with several eigenvalues, which are important from the point of view of the application [7-10]. Let

$(B_2 - \mu I)$ be the operator in $L_2(0,l)$ correspond to the problem:

$$u^{IV}(x) + \frac{k(x)}{EJ}u(x) = \mu u(x),\ 0<x<l,$$
$$u(x)\big|_{x=0} = \sum_{n=1}^{2}\alpha_2\int_0^l u(x)y_n(x)dx,$$
$$u(x)\big|_{x=l} = 0,\ u''(x)\big|_{x=0} = 0,$$
$$u''(x)\big|_{x=l} = 0,$$

where $\alpha_n, n=1,2$ are nonzero real numbers. $y_n(x)$ are eigenfunctions of problem (1), (2) corresponding to the first two eigenvalues λ_1 and λ_2 . Select the boundary parameters $\alpha_n, n=1,2$ α_1 so that the eigenvalues of the operator B_2 are outside the interval $(-\lambda_3, \lambda_3)$.

Generalization of Theorem 1 can be formulated as follows:

Theorem 2. The eigenvalues $\{\mu_n\}_{n=1}^{\infty}$ of the operator B_2 are determined by the formula

$$\mu_n = \lambda_n \text{ for } n \geq 3,$$

and μ_1 and μ_2 are the roots of the equation

$$1 = \frac{\alpha_1 y_1'''(0)}{\mu - \lambda_1} + \frac{\alpha_2 y_2'''(0)}{\mu - \lambda_2}. \quad (9)$$

In this case, $\alpha_n, n=1,2$ is chosen so that the roots of equation (9) are outside the interval $(-\lambda_3, \lambda_3)$.

The proof of Theorem 2 is similar to Theorem 1.

Next, we consider the clamped fixed spectral problem for the Euler-Bernoulli equation. In this case, the spectral problem is described as the boundary value problem with differential equation (1) and the following boundary conditions

$$y(x)\big|_{x=0} = 0,\ y(x)\big|_{x=l} = 0,$$
$$y'(x)\big|_{x=0} = 0,\ y'(x)\big|_{x=l} = 0. \quad (10)$$

Problem 2: Let $(B_3 - \mu I)$ be the operator in $L_2(0,l)$ correspond to the spectral problem with differential equation (1) and the following internal boundary conditions

$$u(x)\big|_{x=0} = 0,$$
$$u(x)\big|_{x=l} = \alpha_3\int_0^l u(x)y_1(x)dx,$$
$$u'(x)\big|_{x=0} = 0,\ u'(x)\big|_{x=l} = 0, \quad (11)$$

where α_3 is a nonzero real number. $y_1(x)$ is an eigenfunction of problem (1), (10) corresponding to the first eigenvalue λ_1. Select the boundary parameter α_3 from (11) so that the eigenvalues of the operator B_3 are outside the interval $(-\lambda_2, \lambda_2)$. Similarly to Theorem 1, we can formulate the following statement

Theorem 3. If the boundary parameter α_3 is chosen so that the inequality holds

$$(\lambda_2 - \lambda_1) \leq -\alpha_3 y_1'''(l), \quad (12)$$

then the eigenvalues $\{\mu_n\}_{n=1}^{\infty}$ of the operator B_3 are determined by the formula $\mu_n = \lambda_n$ for $n \geq 2$ and μ_1 is the only real root outside the interval $(-\lambda_2, \lambda_2)$ of the equation

$$1 = -\frac{\alpha_3 y_1'''(l)}{\mu - \lambda_1}.$$

The proof of Theorem 3 is similar to the proof of Theorem 1 and is based on the following

Lemma 2. We have the identity

$$(\mu - \lambda_n)\int_0^l u(x)y_n(x)dx = -u(l)y_n'''(l).$$

Lemma 2 is proved similarly to Lemma 1. Note that spectral properties with respect to symmetric equivalence of spectral problem (1), (10) with axial load without $k(x)$ were investigated in [23].

Conclusions

The well-known algorithm for controlling the first eigenvalues for second order differential operators developed by Professor Kanguzhin is adapted for the vibration of a homogeneous beam with hinged and clamped fixings at both ends. On the basis of the adapted algorithm, the conditions for the boundary parameters for the control of the first eigenvalues are written out.

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