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T.A. Nauryz

Kazakh-British Technical University, Almaty, Kazakhstan
Email: targyn.nauryz@gmail.com

Similarity Solution of Two-Phase Cylindrical Stefan Solidification Problem

Abstract. The mathematical model of determining temperature fields in cylindrical domain with solidification process is represented. The solidification process of cylinder due to cooling is constructed by two-phase cylindrical Stefan problem for liquid and solid zones with freezing interface. Respect to strength of the heat sink at the center of cylindrical material boundary condition $\lim_{r\to 0}[2\pi r\lambda_1 \frac{\partial\theta}{\partial r}]=\theta_0$ is an important to determine temperature in solid domain. The analytical solution of the problem is introduced with method of similarity principle which enables us to reduce free boundary problem to ordinary differential equations. Temperature solutions of solid and liquid zones are represented by special function which called exponential integral equation. The free boundary at freezing interface and temperatures at two phases are determined. Lemmas about exponential integral functions are introduced and used to prove that obtained operator function is contraction operator. Upper boundness of the exponential integral function is checked graphically. It is shown that existence of uniqueness of solution exists.

Key words: Stefan problem, similarity principle, special function, exponential integral.

Introduction

Solution of the Stefan problems with similarity principle is widely developed in recent years. The one- and two-phase Stefan problems with temperature dependence coefficients are considered in papers [1]-[3]. The inverse Stefan problems for determining time-dependent heat conductivity with shifted Chebyshev polynomials [4] and using Kumar function finding latent heat depending on the position [5] are considered by method of similarity principle.

Mathematical model of the solidification problem in cylindrical domain is considered with two-phase Stefan type cylindrical problem in which temperature in liquid and solid zones and free boundary at freezing interface have to be determined. The similarity principle in this paper is used to modeling temperature field in liquid and solid cylindrical metal zone. The method is very useful to solve this free boundary problem by reducing it to nonlinear ordinary differential equation, then to obtain new solution form with special function which called exponential integration function. This is one of the main result of this paper. The second result of this paper is that obtained solution form is contraction operator and proved that there exists unique solution.

Mathematical model of the problem.

Mathematical model of the process of solidification of a cylindrical material and cooling along cylindrical axis $r = 0$ is constructed with two-phase cylindrical Stefan problem. Two domains are modeled, for solid zone $D_1 : (0 \le r \le \alpha(t))$ temperature is θ_1 and for liquid zone $D_2 : (\alpha(t) \le r \le \infty)$ temperature is θ_2 and it is required to solve the following Stefan's problem

$$\frac{\partial\theta_1}{\partial t} = a_1^2\left(\frac{\partial^2\theta_1}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_1}{\partial r}\right), \quad 0 < r < \alpha(t), \tag{1}$$

$$\frac{\partial\theta_2}{\partial t} = a_2^2\left(\frac{\partial^2\theta_2}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_2}{\partial r}\right), \quad \alpha(t) < r < \infty, \tag{2}$$

The initial conditions are

$$\theta_2(r,0) = \theta_0, \quad \alpha(0) = 0. \tag{3}$$

The boundary condition at the center $r = 0$ is based on the strength of the heat sink. It is expressed as

$$\lim_{r\to 0}[2\pi r\lambda_1 \frac{\partial\theta_1}{\partial r}] = \theta_0. \tag{4}$$

The remaining boundary conditions are

$$\theta_1(\alpha(t),t) = \theta_r, \tag{5}$$

$$\theta_2(\alpha(t),t) = \theta_r, \tag{6}$$

where θ_r is freezing temperature and at infinity we have

$$\theta_2(\infty,t) = \theta_0. \tag{7}$$

The Stefan's condition is

$$-\lambda_1 \frac{\partial\theta_1(\alpha(t),t)}{\partial r} = -\lambda_2 \frac{\partial\theta_2(\alpha(t),t)}{\partial r} + L\gamma \frac{d\alpha}{dt}. \tag{8}$$

The mathematical model of heat problems in electrical contact processes are also can be considered without condition (4) with θ_1 as liquid zone's temperature and θ_2 for solid zone if phenomena is performed between two cylindrical contact materials. In most situation S.N. Kharin considered spherical Stefan problem as in electrical contact spot heat distributed spherically in contact materials, see [6]-[10].

The solution of the problem (1)-(8) we try to find by using similarity principle

$$\theta_i(r,t) = u_i(\eta), \quad \eta = \frac{r^2}{4a_i^2 t}, \qquad \alpha(t) = \alpha_0\sqrt{t}, \quad i = 1,2. \tag{9}$$

The equations (1) and (2) transforms to following

$$\eta\frac{d^2u_i}{d\eta^2} + (1+\eta)\frac{du_i}{d\eta} = 0, \quad i = 1,2, \tag{10}$$

then domain for θ_1 is $0 < \eta < \dfrac{{\alpha_0}^2}{4{a_1}^2}$ and domain for θ_2 is $\dfrac{{\alpha_0}^2}{4{a_1}^2} < \eta < \infty$.

We can rewrite equation (10) as

$$\frac{\dfrac{d^2u_i}{d\eta^2}}{\dfrac{du_i}{d\eta}} + \frac{1}{\eta} + 1 = 0, \quad i = 1,2. \tag{11}$$

Integrating we get

$$\ln\frac{du_i}{d\eta} + \ln\eta + \eta = \ln C_i, \quad i = 1,2$$

or

$$\ln\left[\frac{du_i}{d\eta}\cdot\eta\cdot e^{\eta}\right] = \ln C_i, \quad i = 1,2,$$

$$\frac{du_i}{d\eta} = \frac{C_i}{\eta}e^{-\eta}, \quad i = 1,2.$$

Second integration gives

$$u_i = A_i\int_\eta^\infty \frac{e^{-\eta}}{\eta}d\eta + B_i, \quad A_i = C_i, \quad i = 1,2.$$

Using exponential integral function

$$Ei(x) = \int_x^\infty \frac{e^{-z}}{z}dz = \int_{-\infty}^x \frac{e^z}{z}dz$$

we get

$$u_i(\eta) = A_i Ei(\eta) + B_i, \quad i = 1,2. \tag{12}$$

If $\alpha(t) = \alpha_0\sqrt{t}$, then the conditions (3)-(8) can be written in the form in terms of variable η

$$\lim_{\eta\to 0}\left(4\pi\lambda_1\eta\frac{du_1}{d\eta}\right) = u_0, \tag{13}$$

$$u_1({\alpha_0}^2/(4{a_1}^2)) = u_f, \tag{14}$$

$$u_2({\alpha_0}^2/(4a_1)^2) = u_f, \tag{15}$$

$$u_2(\infty) = u_0, \tag{16}$$

$$-\lambda_1\left.\frac{du_1}{d\eta}\right|_{\eta=\frac{\alpha_0^2}{4a_1^2}} = -\lambda_2\left.\frac{du_2}{d\eta}\right|_{\eta=\frac{\alpha_0^2}{4a_1^2}} + 2L\gamma {a_1}^2. \tag{17}$$

Satisfying solutions (12), from conditions (13)-(17) we get

$$A_1 = -\frac{u_0}{4\pi\lambda_1}, \; B_1 = u_f + \frac{u_0}{4\pi\lambda_1}Ei\left(\frac{{\alpha_0}^2}{4{a_1}^2}\right), \qquad A_2 = \frac{u_f - u_0}{Ei\left(\dfrac{{\alpha_0}^2}{4{a_1}^2}\right)}, \quad B_2 = u_0. \tag{18}$$

By making substitution (12) and (18) to (9) we obtain solution for problem (1)-(8) as the following form

$$\theta_1(r,t)=\theta_f+ \\ +\frac{\theta_0}{4\pi\lambda_1}\left[Ei\left(\frac{\alpha_0^2}{4a_1^2}\right)-Ei\left(\frac{r^2}{4a_1^2t}\right)\right], \quad (19)$$

$$\theta_2(r,t)=\frac{\theta_f-\theta_0}{Ei\left(\frac{\alpha_0^2}{4a_2^2}\right)}Ei\left(\frac{r^2}{4a_1^2t}\right)+\theta_0. \quad (20)$$

Using Stefan's condition (17) we get the equation to find a_0

$$-\lambda_1\frac{\theta_0}{\pi}\cdot\frac{a_1^2e^{-\frac{\alpha_0^2}{4a_1}}}{\alpha_0^2}= \\ =-\lambda_2\frac{(\theta_f-\theta_0)}{Ei(\alpha_0^2/(4a_2^2))}\cdot\frac{4a_2^2e^{-\frac{\alpha_0^2}{4a_2^2}}}{\alpha_0^2}+L\gamma a_1^2. \quad (21)$$

Thus the solution of the problem is given by the expressions (19) and (20) where $a(t)$ should be found from the equation (22). The next section we prove the existence of uniqueness of the solution form (12).

Existence of uniqueness

To prove existence of uniqueness of (12) we have to obtain the next following results.

Lemma 1. For any $\eta > 0$ there exists $I(\eta):=\int_{-\eta}^{\eta}\frac{e^{-z}}{z}dz$ and it is finite, it satisfies the following identity

$$I(\eta)=(e^{\eta}-e^{-\eta})\log\eta-(e^{\eta}+e^{-\eta})(\eta\log\eta-\eta)+ \\ +\int_0^{\eta}(e^{z}-e^{-z})(z\log z-z)dz. \quad (22)$$

Additionally, it stays finite when $\eta\to 0$ and $\lim_{\eta\to0}I(\eta)=\lim_{\eta\to0}\int_{-\eta}^{\eta}\frac{e^{-z}}{z}dz=0.$

Proof. Let $\eta > 0$ and $I(\eta):=\int_{-\eta}^{\eta}\frac{e^{-z}}{z}dz$.

One just needs to observe that for z close to 0, $\frac{e^{-z}}{z}\sim\frac{1}{z}$ as $e^{z}\sim 1$, and $\frac{1}{z}$ can be integrated on $[-\eta,\eta]$, even if it is not defined at 0, because it is odd: $\int_{-\eta}^{\eta}\frac{1}{z}dz=\lim_{t\to0}\left(\int_{-\eta}^{t}\frac{1}{z}dz+\int_{t}^{\eta}\frac{1}{z}dz\right)$ (as Cauchy's principal value), and $\int_{-\eta}^{t}\frac{1}{z}dz=-\int_{t}^{\eta}\frac{1}{u}du$ with the change of variable $u=-z$. So $\int_{-\eta}^{\eta}\frac{1}{z}dz=0$ for any $\eta\geq 0$.

But we have to show that $I(\eta)$ exists for any $\eta > 0$ and that $I(\eta)\to0$ for $\eta\to 0$. A first Integration By Part with $u=e^{z}$ and $dv=\frac{1}{z}$ we get

$$I(\eta)=\int_{-\eta}^{\eta}\frac{e^{z}}{z}dz=\left[e^{z}\log|z|\right]_{-\eta}^{\eta}-\int_{-\eta}^{\eta}e^{z}\log|z|\,dz=(e^{\eta}-e^{-\eta})\log\eta- \\ -\int_0^{\eta}(e^{z}+e^{-z})\log|z|\,dz=(e^{\eta}-e^{-\eta})\log\eta-(e^{\eta}+e^{-\eta})(\eta\log\eta-\eta)+\int_0^{\eta}(e^{z}-e^{-z})(z\log z-z)dz.$$

The integral is well defined and finite, as the integrated function is continuous and finite for all z, even at 0. So this proves that $I(\eta)$ is finite for any $\eta > 0$.

If we take $\eta\to 0$, it gives that for each terms in $I(\eta)$,

$$\begin{cases}(e^{\eta}-e^{-\eta})\log\eta\sim((1+\eta)-(1-\eta))\log\eta=2\eta\log\eta\to0,\\ (e^{\eta}+e^{-\eta})(\eta\log\eta-\eta)\sim2(\eta\log\eta-\eta)\to0,\\ \int_0^{\eta}(e^{-z}-e^{-z})(z\log z-z)dz\to0,\end{cases}$$

So $I(\eta)\to 0$.

Lemma 2. For any $0<\eta\le\frac{\alpha_0^2}{4a_1^2}$, $I(\eta)$ satisfies $I(\eta)\le e^{\eta}-e^{-\eta}$. In particular, $I(\alpha_0^2/(4a_1^2))\le$ $\le e^{\alpha_0^2/(4a_1^2)}-e^{-\alpha_0^2/(4a_1^2)}$.

Proof. For $0<\eta\le\frac{\alpha_0^2}{4a_1^2}$, $\eta\log\eta-\eta\ge-\frac{\alpha_0^2}{4a_1^2}$, and $(e^{-\eta}+e^{-\eta})\log\eta\le 0$, and so the identity (22) gives

$$I(\eta)\le(e^{\eta}-e^{-\eta})+\int_0^{\eta}(e^{z}-e^{-z})(z\log z-z)dz,$$

but $(e^{-z}-e^{-z})(z\log z-z)\le 0$ for all $\eta\in[0,\ \alpha_0^2/(4a_1^2)]$, so $I(\eta)\le e^{\eta}-e^{-\eta}$ as wanted. In particular, we have $I(\alpha_0^2/(4a_1^2))\le e^{\alpha_0^2/(4a_1^2)}-e^{-\alpha_0^2/(4a_1^2)}$.

Lemma 3. For any $0<\eta<\frac{\alpha_0^2}{4a_1^2}$, $\mathrm{Ei}(\eta)\le e^{\eta}$.

Moreover, for $\eta>\frac{\alpha_0^2}{4a_1^2}$, we also have

$$\mathrm{Ei}(\eta)\ge\mathrm{Ei}(\alpha_0^2/(4a_1^2))+\frac{e^{\eta}-e^{\alpha_0^2/(4a_1^2)}}{\eta}\ge-\alpha_0^2/(4a_1^2)+\frac{e^{\alpha_0^2/(4a_1^2)}}{\alpha_0^2/(4a_1^2)}$$

Proof. Let $x\in\mathbb{R}$. First, if $\eta<0$, then $\mathrm{Ei}(\eta)\le 0<e^{\eta}$.

If $0<\eta<\frac{\alpha_0^2}{4a_1^2}$, we can split the integral defining $Ei(\eta)$ as

$$\mathrm{Ei}(\eta)=\int_{-\infty}^{-\eta}\frac{e^{z}}{z}dz+\int_{-\eta}^{\eta}\frac{e^{z}}{z}dz\le I(\eta)\le e^{\eta}-e^{-\eta}\le e^{\eta}.$$

If $\eta>\frac{\alpha_0^2}{4a_1^2}$, we do the same with three terms, and by using

$$\mathrm{Ei}(-\alpha_0^2/(4a_1^2))=\int_{-\infty}^{-\alpha_0^2/(4a_1^2)}\frac{e^{z}}{z}dz\le 0,$$

and

$$I(\alpha_0^2/(4a_1^2))=\int_{-\alpha_0^2/(4a_1^2)}^{\alpha_0^2/(4a_1^2)}\frac{e^{z}}{z}dz\le e^{\alpha_0^2/(4a_1^2)}-e^{-\alpha_0^2/(4a_1^2)}$$

we have

$$\mathrm{Ei}(\eta)=\underbrace{\int_{-\infty}^{-\alpha_0^2/(4a_1^2)}\frac{e^{z}}{z}dz}_{=\mathrm{Ei}(-\alpha_0^2/(4a_1^2))\le 0}+\underbrace{\int_{-\alpha_0^2/(4a_1^2)}^{\alpha_0^2/(4a_1^2)}\frac{e^{z}}{z}dz}_{=I(\alpha_0^2/(4a_1^2))}+\underbrace{\int_{\alpha_0^2/(4a_1^2)}^{\eta}\frac{e^{z}}{z}dz}_{\le e^{\eta}-e^{\alpha_0^2/(4a_1^2)}}\le I(\alpha_0^2/(4a_1^2))+e^{\eta}-e^{\alpha_0^2/(4a_1^2)}$$
$$\le e^{\alpha_0^2/(4a_1^2)}-e^{-\alpha_0^2/(4a_1^2)}+e^{\eta}-e^{\alpha_0^2/(4a_1^2)}=e^{\eta}-e^{-\alpha_0^2/(4a_1^2)}\le e^{\eta}.$$

We can check this inequality $\mathrm{Ei}(x)\le e^{x}$ graphically from Figure 1 where red curve is exponential function and blue curve is exponential integral function.

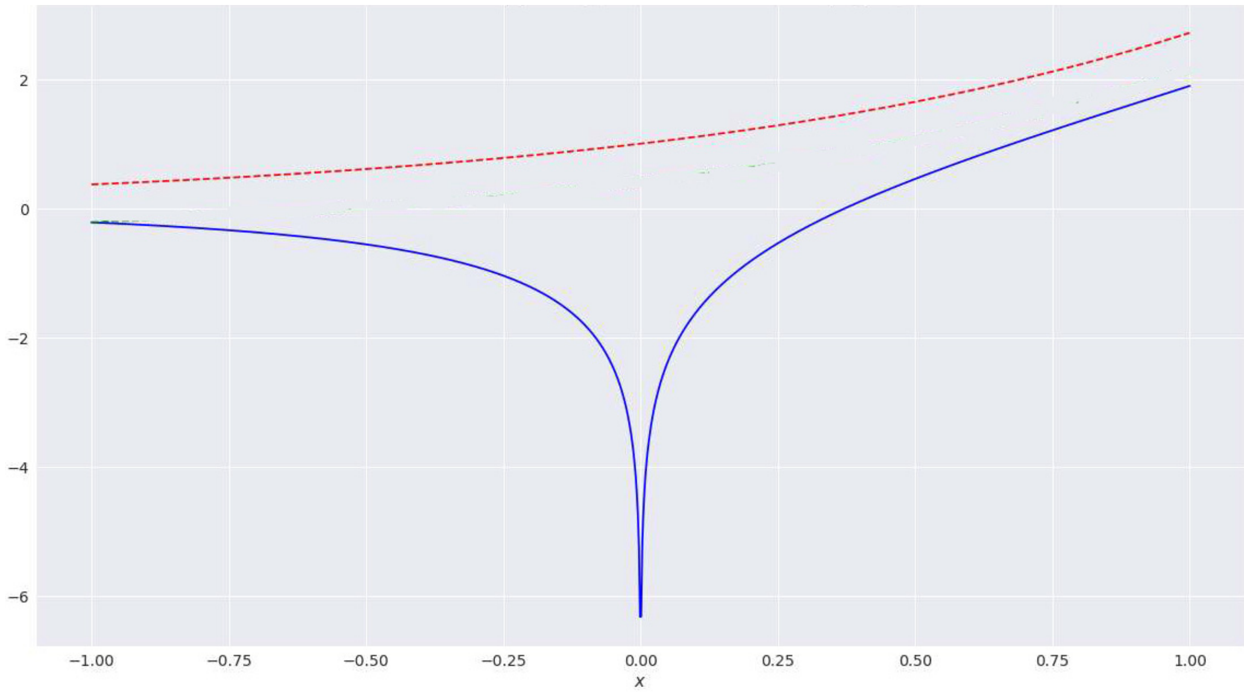

Figure 1 – The Ei function and upper bound function e^x

Theorem 1. Let $0<\eta<\frac{\alpha_0^2}{4a_1^2}$ then equation (12) for $u_1(\eta)$ satisfies a Lipschitz condition, then there exists $k>0$ such that

$$\| u_1(\eta)-u_1(\eta^*)\|<k\|\eta-\eta^*\|$$

for all $\eta,\eta^*\in[0,\alpha_0^2/(4a_1^2)]$, then there exists a unique solution of equation (12).

Proof. Let $W:C[0,\alpha_0^2/(4a_1^2)]\to C[0,\alpha_0^2/(4a_1^2)]$ be the operator is defined by

$$W(u_1):=u_1(\eta),\quad 0<\eta<\frac{\alpha_0^2}{4a_1^2}.$$

Let $u_1(\eta),u_1(\eta^*)\in C[0,\alpha_0^2/(4a_1^2)]$, by using Lemmas 1–3 and inequality $|\exp(x)-\exp(y)|<|x-y|$ we get

$$\begin{aligned}\|W(u_1(\eta))-W(u_1(\eta^*))\|&=\|u_1(\eta)-u_1(\eta^*)\|=\|A_1\mathrm{Ei}(\eta)+B_1-A_1\mathrm{Ei}(\eta^*)-B_1\|\\&=k\|\mathrm{Ei}(\eta)-\mathrm{Ei}(\eta^*)\|\le k\|e^{\eta}-e^{\eta^*}\|<k\|\eta-\eta^*\|,\end{aligned}$$

for all $\eta,\eta^*\in[0,\alpha_0^2/(4a_1^2)]$. W is a contraction operator and by the fixed point theorem, then there exists a unique solution of integral equation (12).

Existence of uniqueness for $u_2(\eta)$ is proved analogously as in theorem 1.

Conclusion

The temperature in liquid and solid zones (19) and (20) are determined based on similarity principle which enables us to obtain (12). By using fixed point theorem, inequalities about exponential integral function and Lipschitz condition existence of uniqueness of the (12) is proved.

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References

1 A.C. Briozzo, M.F. Natale, D.A. Tarzia, Existence of an exact solution for a one-phase Ste-

fan problem with nonlinear thermal coefficients from Tirskii's method, Nonlinear Analysis 67 (2007), (1989-1998), https://doi.org/10.1016/j.na.2006.07.047.

2 S.N. Kharin, T.A. Nauryz, One-phase spherical Stefan problem with temperature dependent coefficients. Eurasian Mathematical Journal 12(1), (2021): 49-56, https://doi.org/10.32523/2077-9879-2021-12-1-49-56.

3 S.N. Kharin, T.A. Nauryz, Mathematical model of a short arc at the blow-off repulsion of electrical contacts during the transition from metallic phase to gaseous phase. AIP Conference Proceedings 2325, (2021): 02007, https://doi.org/10.1063/5.0040412.

4 J. Bollati, D.A. Tarzia, Explicit solution for Stefan problem with latent heat depending on the position and a convective boundary condition at the fixed face using Kummer functions, arXiv:1610.09338v1[math/AP] (2016).

5 M.J. Huntul, D. Lesnic, An inverse problem of finding the time-dependent thermal conductivity from boundary data, Int. Commun. Heat Mass Transfer 85 (2017): 147-154.

6 S.N. Kharin, T.A. Nauryz, Two phase spherical Stefan inverse problem solution with linear combination of radial heat polynomials and integral error functions in electrical contact process. *International Journal of Mathematics and Physics, 11*(2), 4-13. doi: https://doi.org/10.26577/ijmph.2020.v11.i2.01.

7 S.N. Kharin, "Mathematical models of melting and evaporation in electrical contacts", International Scientific Conference "Functional Analysis and Its Applications", Astana, October 2-5, (2012): Abstracts, 298-299.

8 S.N. Kharin, Yu.R. Shpady, M. Sarsengeldin, "Method for the Solution of the Heat Equation in Domains with Moving Boundaries Using Integral Error Functions ahd Confluent Hypergeometric Functions", International Scientific Conference "Functional Analysis and its Applications", Astana, October 2-5, (2012): Abstracts, 299-300.

9 S.N. Kharin, V.V. Sarsengeldin, "Mathematical Model of Thermoelastic Mechanism of Electrode Erosion During Arcing", Proceedings of the VI International Scientific Conference "The Problems of Differential Equations, Analysis and Algebra", Actobe, 14-17 October, (2012): 245-248.

10 M. Sarsengeldin, S.N. Kharin, Method of the Integral Error Functions for the Solution of the One- and Two-Phase Stefan Problems and Its Application, Filomat 31:4 (2017): 1017–1029 DOI 10.2298/FIL1704017S, Available at: http://www.pmf.ni.ac.rs/filomat.