

D. Serikbaev 

Al-Farabi Kazakh National University, Almaty, Kazakhstan
 Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
 Ghent University, Ghent, Belgium
 e-mail: daurenbek.serikbaev@ugent.be

Gas Dynamics type Burgers Equation with Convolutional Nonlinearity

Abstract. In this paper, we consider an initial value problem for the Burgers' equation with convolution type weak nonlinearity for the Sturm–Liouville operator. We prove that this problem has an explicit solution in the form of series.

To achieve our goals, we use methods that correspond to various fields of mathematics, such as the theory of partial differential equations, mathematical physics, and functional analysis. In particular, we use the Fourier analysis method to establish the existence of solutions to this problem on the Sobolev space. As far as we know, it is the first result obtained for the convolution type Burgers' equation.

Since, we use the Fourier analysis method we gave the properties of Fourier transform when acting on convolution, and also gave a property of fractional order of the Sturm–Liouville operator.

The generalized solutions of the convolution type weak nonlinear Burgers' equation with the initial Cauchy condition are studied.

Key words: weak nonlinear equation, Burgers' equation, convolution, initial value problem.

Introduction

$$u_t + uu_x = \mu u_{xx}, \quad (4)$$

In this paper, we study convolution type Burgers' equations for the operators Sturm–Liouville

$$u_t(t, x) - u_{xx}(t, x) = \int_0^\pi u(t, x - y) \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} u(t, y) dy, \quad (1)$$

for $(t, x) \in \{t > 0, 0 < x < \pi\}$ with Cauchy condition

$$u(0, x) = \varphi(x) \quad (2)$$

and with boundary condition

$$u(t, 0) = u(t, \pi) = 0, \quad (3)$$

where $\varphi(x)$ is the given function from any separable Hilbert space.

Before describing our results, let's dwell on the historical facts about the Burgers equations and convolution type equations.

The differential equation

was first introduced by Burgers in [1] as a simple model for the equations of motion of a viscous fluid, and it is usually referred to as Burgers' equation. In [2, 3] the general properties of Equation (4) are investigated, and it is shown that with the help of a certain substitution it can be reduced to the heat equation. Specifically, if $u_0(x) = u(0, x)$ is continuous and $\int_0^x u_0(y) dy = o(x^2)$ as $|x| \rightarrow \infty$, then they proved the existence and uniqueness of the solution of the problem.

In [4, 5] using the Fourier method solved Burgers' type equation.

The Burgers equation has been applied in many fields, such as fluid mechanics, nonlinear acoustics, gas dynamics, and so on. For example, the Burgers equation is a nonlinear partial differential equation for modeling the propagation and reflection of a shock wave. It is also the simplest nonlinear model equation of a diffusion wave in hydrodynamics. At the same time, it is also a model equation of the Navier-Stokes equations without a stress term.

In this paper, our goal is to show existence of solution of the initial value problem (1)–(2), by using Fourier spectral method. In our knowledge, it

is the first result obtained for the convolution type Burgers' equation (1).

In [7, 8, 9, 10], the Fourier method is widely used to solve several types of problems of evolution equations for positive operators. In [6], convolution operators and self-adjoint positive operators on an abstract Hilbert space are considered in more detail.

Preliminaries

The differential operator of second order in $L^2(0, \pi)$ generated by the differential expression

$$Ly(x) = -y''(x), x \in (0, \pi),$$

with the boundary condition

$$y(0) = y(\pi) = 0,$$

has the following eigenvalues

$$\lambda_k = k^2, k \in \mathbb{N},$$

and the corresponding system of eigenfunction

$$y_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, k \in \mathbb{N}. \tag{5}$$

Definition 1. The space $W^{2,2}[0, \pi]$ is the Hilbert space consisting of all elements of $L^2[0, \pi]$ having generalized derivatives up to order 2 in L^2 , i.e.

$$W^{2,2}[0, \pi] = \{f \in L^2[0, \pi] \mid f', f'' \in L^2[0, \pi]\}$$

and the norm is defined by

$$\|f\|_{W^{2,2}[0,\pi]}^2 = \|f\|_{L^2[0,\pi]}^2 + \left\| \frac{d^2 f}{dx^2} \right\|_{L^2[0,\pi]}^2.$$

• $W_0^{2,2}[0, \pi]$ is the subspace $W^{2,2}[0, \pi]$, defined as the closure with respect to

$\|\cdot\|_{W^{2,2}[0,\pi]}$ of all functions twice continuously differentiable in $[0, \pi]$ that vanish at the points 0 and π .

• The L^2 -scalar product of two real functions $f, g: [0, \pi] \rightarrow \mathbb{R}$ is defined by

$$(f, g)_{L^2} = \int_0^\pi f(x)g(x)dx.$$

• The fractional power of the Sturm–Liouville operator is defined by

$$\begin{aligned} & \left(\left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} f \right) (x) = \\ & = \sum_{k=1}^\infty k \hat{f}(k) y_k(x), \forall f \in L^2[0, \pi]. \end{aligned} \tag{6}$$

Definition 2. The Fourier transform $\mathcal{F}: L^2[0, \pi] \rightarrow l^2$ is defined as follows

$$\begin{aligned} (\mathcal{F}f)(k) &= \hat{f}(k) = \\ &= \int_0^\pi f(x)y_k(x)dx, \forall f \in L^2[0, \pi]. \end{aligned}$$

Definition 3. The inverse Fourier transform \mathcal{F}^{-1} is defined as follows

$$\begin{aligned} (\mathcal{F}^{-1}\hat{f})(x) &= f(x) = \\ &= \sum_{k=1}^\infty \hat{f}(k)y_k(x), \forall f \in L^2[0, \pi]. \end{aligned}$$

• The Fourier transform of $f \star g$ is the product of the Fourier transforms of f and g :

$$(\widehat{f \star g})(k) = \hat{f}(k)\hat{g}(k) \tag{7}$$

Main results

In this section, we study the existence of a solution to problem (1)-(2).

Theorem 1. Let $\varphi \in \{\varphi \in W_0^{2,2}[0, \pi] : \|\varphi\|_{W_0^{2,2}[0,\pi]} < 1\}$. Then solution $u \in C^1([0, \infty); L^2[0, \pi]) \cap C([0, \infty); W_0^{2,2}[0, \pi])$ of the problem (1)-(2) is exist and can be written in the following form

$$u(t, x) = \sum_{k=1}^\infty \frac{ke^{-k^2t}\varphi_k}{\varphi_k(e^{-k^2t} - 1) + k} \sin kx,$$

where $\varphi_k = (\varphi, y_k)_{L^2[0,\pi]}, k \in \mathbb{N}$.

Proof. Since the system of eigenvalue (5) is an orthonormal basis in $L^2(0, \pi)$, we seek the function $u(t, x)$ in the form

$$u(t, x) = \sum_{k=1}^\infty \hat{u}(t, k) \sin kx, \tag{8}$$

here $\hat{u}(t, k)$ is an unknown function, and it defined by

$$\hat{u}(t, k) = \sqrt{\frac{2}{\pi}} \int_0^\pi u(t, x) \sin kx dx.$$

Substituting (8) into the equation (1)–(2) and in view of the formulas (6), (7), we get the following ordinary differential equation corresponding to the function $u(t, k)$:

$$\hat{u}(t, k) + k^2 \hat{u}(t, k) = k \hat{u}^2(t, k), t > 0, \quad (9)$$

$$u(0, k) = \hat{\varphi}(k), \quad (10)$$

for $\forall k \in \mathbb{N}$, where $\hat{\varphi}(k) = \sqrt{\frac{2}{\pi}} \int_0^\pi \varphi(x) \sin kx dx$. (9) is the homogeneous Riccati equation. Solution of this Riccati equation corresponding to the Cauchy condition (10) is

$$\hat{u}(t, k) = \frac{ke^{-k^2t}\hat{\varphi}(k)}{\hat{\varphi}(k)(e^{-k^2t}-1)+k}, k \in \mathbb{N}. \quad (11)$$

Putting (11) into (8) we get a solution of the equation (1)–(2) in the form

$$u(t, x) = \sum_{k=1}^{\infty} \frac{ke^{-k^2t}\hat{\varphi}(k)}{\hat{\varphi}(k)(e^{-k^2t}-1)+k} \sin kx. \quad (12)$$

By calculating the Plancherel norm, we arrive at

$$\|u(t, \cdot)\|_{L^2}^2 = \sum_{k=1}^{\infty} \frac{k^2 e^{-2k^2t} |\hat{\varphi}(k)|^2}{|(e^{-k^2t}-1)\hat{\varphi}(k)+k|^2}.$$

Finally, by taking into account that φ is a sufficiently small function ($\|\varphi\|_{L^2} < 1$), one obtains

$$\begin{aligned} \|u(t, \cdot)\|_{L^2}^2 &\leq Ce^{-2t} \sum_{k=1}^{\infty} \frac{k^2 |\hat{\varphi}(k)|^2}{|(e^{-k^2t}-1)\hat{\varphi}(k)+k|^2} \\ &\leq Ce^{-2t} \|\varphi\|^2, \end{aligned}$$

or,

$$\|u(t, \cdot)\|_{L^2} \leq Ce^{-t} \|\varphi\|_{L^2},$$

for some positive constant C . This implies $u \in C([0, \infty); L^2[0, \pi])$.

Similarly

$$\begin{aligned} &\left\| \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} u \right\|_{L^2}^2 = \\ &= \sum_{k=1}^{\infty} \frac{k^4 e^{-2k^2t} |\varphi_k|^2}{|(e^{-k^2t}-1)\varphi_k+k|^2} \leq \\ &\leq Ce^{-2t} \left\| \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} \varphi \right\|_{L^2}^2. \end{aligned}$$

That is

$$\left\| \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} u \right\|_{L^2} \leq Ce^{-t} \left\| \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} \varphi \right\|_{L^2}.$$

and, we have

$$\|u_{xx}\|_{L^2} \leq Ce^{-t} \|\varphi_{xx}\|_{L^2}.$$

These imply $u_{xx}, \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} u \in C([0, \infty); L^2[0, \pi])$. Now let us calculate $u_t(t, x)$.

$$u_t(t, x) = \sum_{k=1}^{\infty} \frac{k^3 e^{-k^2t} \hat{\varphi}(k) (\hat{\varphi}(k) - k)}{(\hat{\varphi}(k)(e^{-k^2t}-1)+k)^2} \sin kx.$$

Then, we have

$$\begin{aligned} \|u_t(t, \cdot)\|_{L^2}^2 &= \sum_{k=1}^{\infty} \frac{k^6 e^{-2k^2t} |\hat{\varphi}(k)|^2 |\hat{\varphi}(k) - k|^2}{|\hat{\varphi}(k)(e^{-k^2t}-1)+k|^4} \leq \\ &\leq C \sum_{k=1}^{\infty} \frac{k^6 e^{-2k^2t} |\hat{\varphi}(k)|^2 (|\hat{\varphi}(k)|^2 + k^2)}{|\hat{\varphi}(k)(e^{-k^2t}-1)+k|^4} \leq \\ &\leq C \sum_{k=1}^{\infty} k^2 e^{-2k^2t} |\hat{\varphi}(k)|^2 (|\hat{\varphi}(k)|^2 + k^2) \leq \\ &\leq Ce^{-2t} \sum_{k=1}^{\infty} k^2 |\hat{\varphi}(k)|^4 + Ce^{-2t} \sum_{k=1}^{\infty} k^4 |\hat{\varphi}(k)|^2 \leq \end{aligned}$$

$$\leq Ce^{-2t} \left\| \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} \varphi \right\|_{L^2}^4 + Ce^{-2t} \|\varphi_{xx}\|_{L^2}^2.$$

This implies $u_t \in C([0, \infty); L^2[0, \pi])$ immediately. Here we use the following estimate to the series $\sum_{k=1}^{\infty} k^2 |\hat{\varphi}(k)|^4$:

$$\sum_{k=1}^{\infty} k^2 |\hat{\varphi}(k)|^4 \leq \sum_{k=1}^{\infty} k^4 |\hat{\varphi}(k)|^4 \leq \left(\sum_{k=1}^{\infty} k^2 |\hat{\varphi}(k)|^2 \right)^2 \leq \left\| \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} \varphi \right\|_{L^2}^4.$$

Now we show the convergence of the following convolution product:

$$\left(\left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} u \star u \right) (t, x) = u_t(t, x) - u_{xx}(t, x),$$

it is clear that this formula obtained by the equation (1). From this, we obtain

$$\left\| \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} u \star u \right\|_{C([0, \infty); L^2[0, \pi])} \leq$$

$$\leq \|u_t\|_{C([0, \infty); L^2[0, \pi])} + \|u_{xx}\|_{C([0, \infty); L^2[0, \pi])} < \infty.$$

Theorem 1 is proved.

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