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Lax-Wendroff Difference Scheme with Richardson Extrapolation Method for One Dimensional Wave Equation Subjected To Integral Condition

Abstract. In this paper, the Lax-Wendroff difference scheme has been presented for solving the one-dimensional wave equation with integral boundary conditions. First, the given solution domain is discretized and the derivative involving the spatial variable x is replaced by the central finite difference approximation of functional values at each grid point by using Taylor series expansion. Then, for solving the resulting second-order linear ordinary differential equation, the displacement function is discretized in the direction of a temporal variable by using Taylor series expansion and the Lax-Wendroff difference scheme is developed, then it gives a system of algebraic equations. The derivative of the initial condition is also discretized by using the central finite difference method. Then the obtained system of algebraic equations is solved by the matrix inverse method. The stability and convergent analysis of the scheme are investigated. The established convergence of the scheme is further accelerated by applying the Richardson extrapolation which yields fourth-order convergent in spatial variable and sixth-order convergent in a temporal variable. To validate the applicability of the proposed method, three model examples are considered and solved for different values of the mesh sizes in both directions. Numerical results are presented in tables in terms of maximum absolute error, L_2 and L_∞ norm. The numerical results presented in tables and graphs confirm that the approximate solution is in good agreement with the exact solution.

Key words: Hyperbolic equation, one-dimensional wave equation, Lax-Wendroff difference scheme, Taylor series methods, Richardson extrapolation, stability, and convergence.

Introduction

The hyperbolic partial differential equation with an integral condition arises in many physical phenomena [10]. Over the last few years, it has become increasingly apparent that many physical phenomena can be described in terms of hyperbolic partial differential equations with an integral condition replacing the classic boundary condition [4]. These types of equation appear in a variety of physical problems such as in the study of thermoelasticity and plasma physics, and chemical heterogeneity [2, 4], acoustic waveguides [20], heterogeneous physical properties and/or complex geometry, seismology (study of earthquakes, regional and global seismology, accurate calculation of synthetic seismograms [23], water waves, sound waves, radio waves, light wave and seismic waves [14]. Hyperbolic partial differential equations play a very important role in modern applied mathematics due to their deep physical background. This hyperbolic differential equation subject to an integral conservation condition in one space dimensional, feature in the mathematical modeling of many phenomena [7]. It also arises in a broad spectrum of applications where wave motion is involved, for example; optics, acoustics, oil and gas dynamics, and vibrating string to name but a few [14]. Waves have distinct properties specific to their type but also exhibit characteristics in common with more abstract waves such as sound waves and light waves [6, 8, 3]. Seismic waves may be simulated by a viscous type of wave equation which is known to be difficult to obtain the analytical solution [14]. This is where the numerical solution to such types of equations is needed to solve practical problems, [14, 19].

Due to the wide range of the application of the wave equation, several numerical methods have been developed. Even though many numerical methods were applied to solve these types of equations. Accordingly, more efficient and simpler numerical methods are required to solve wave equations. The FDM is becoming increasingly more important in the seismic industry and structural modeling due to its relative accuracy and computational efficiency [14]. Numerical modeling of wave propagation in the presence of surface topography requires the incorporation of non-flat boundaries. Embedding irregular boundaries into a finite-difference method is an old problem [11]. It can be avoided altogether with finite-element or finite-volume or mesh-free methods, but usually at a higher computational cost. Given the existing, highly optimized finite-difference code base, a local modification of finite-difference stencils close to the irregular boundary is more attractive [9]. An obvious approach for dealing with topography is the introduction of an air layer. The extreme density contrast may require some smoothing to preserve stability [3, 12]. The presence of an airwave can be suppressed by setting the sound speed in the air equal to that at the surface, using constant extrapolation [9]. Again the numerical methods have been widely implemented to model acoustic and hydrodynamic waves [1]. This method of solving for the elastic wave eigenmodes in acoustic waveguides of an arbitrary cross-section is presented. Operating under the assumptions of linear, isotropic materials, it utilizes a finite-difference method on a staggered grid to solve for the acoustic Eigenmodes (field and frequency) of the vector-field elastic wave equation with a given propagation constant [20].

Therefore growing attention is being paid to the development; analysis and implementation of numerical methods for the solution of hyperbolic partial differential equations. Most of the researchers have studied the numerical solutions of the 1D wave equation. Shazalina et al [24] presented numerical techniques based Cubic Trigonometric B-spline Approach solving the 1D wave equation. Ang [22] solved the same problem using a scheme based on an integrodifferential equation and local interpolating functions. Dehghan [21] presented numerical techniques based on the three-level explicit finite difference schemes for solving the 1D hyperbolic partial differential equation. He used three-level techniques are based on two second-order (one explicit and one weighted) schemes and a fourth-order technique (a weighted explicit). Also. Saadatmandi [27] is developed the shifted Legendre Tau technique for the solution of the studied model. The approach in their work consists of reducing the problem to a set of algebraic equations by expanding the approximate solution as a shifted Legendre function with unknown coefficients. The integral and derivative operational matrices are given. These matrices together with the tau method are then utilized to evaluate the unknown coefficients of shifted Legendre functions. Mehdi Dehghan and Ali Shokri [10] are presented numerical techniques based on the mesh less schemes for solving the one-dimensional wave equation with an integral condition using radial basis functions. Wang et. al. [17] presented the solution of the one-dimensional wave Equation by using the Lagrangian meshfree finite difference particle method with variable smoothing length. Ang [22] developed a numerical technique based on an integrodifferential equation and local interpolating functions for solving the one-dimensional wave equation subject to a nonlocal conservation condition and suitably prescribed initial boundary conditions. Ramezani et.al. [28] Combined finite difference and spectral methods to solve the one-dimensional wave equation with an integral condition. They used that the time variable is approximated using a finite difference scheme. But the spectral method is employed for discretizing the space variable. The main idea behind their approach is that high-order results can be obtained. Hikmet et.al [7] present a numerical solution of the One-dimensional wave equation with an Integral conservation condition is presented by the method of the non-polynomial cubic spline. Beilin [29] presented the existence and uniqueness of the solution of the one-dimensional wave equation with integral boundary conditions. Recently, much attention has been paid in the literature to the development, analysis, and implementation of accurate methods.

However, still, the accuracy and stability of the method need attention because the treatment of the method used to solve one-dimensional wave equation is not trivial distribution. Even though the accuracy and stability of the aforementioned methods need attention, they require large memory and long computational time. So the treatments this method presents severe difficulties that have to be addressed to ensure the accuracy and stability of the solution. To this end, this paper aims to develop an accurate and stable Lax-Wendroff difference scheme with Richardson extrapolation numerical method that is capable of solving a one-dimensional wave equation with integral boundary conditions and approximate the exact solution. The convergence present method has been shown in the sense of maximum absolute error, L_∞ and L_2 $norm.$ So that the local behavior of the solution is captured and the agreement of numerical solution with the analytical solution is showed by table and graph. The consistency, stability, and convergent analysis of the present method are also established.

Statement of the problem

Consider that the following linear one-dimensional wave equation with integral boundary condition considered in [2, 5, 10, 16] given by:

$$u_{tt} - u_{xx} = q(x,t), (x,t) \in (a,b) \times (0,T) \quad (1)$$

with initial and boundary condition respectively

$$u(x,0) = \omega_1(x), u_t(x,0) = \omega_2(x), a \le x \le b,$$
$$u(a,t) = \phi_1(t), \int_a^b u(x,t) = \phi_2(t),$$
$$0 \le t \le T \quad (2)$$

Where $q(x,t), \omega_1(x), \omega_2(x), \phi_1(t)$ and $\phi_2(t)$ are smooth function on$[a,b] \times [0,T]$. Now we define a mesh size h and k and the constant grid point by drawing equidistance horizontal and vertical line of distance 'h' and 'k' respectively in 'x' and 't' direction. These lines are called gridlines and the point at which they interacting is known as the mesh point. The mesh point that lies at end of the domain is called the boundary point. The points that lie inside the region are called interiors points. The goal is to approximate the solution '$u_{j\,n}$' at the interior mesh points. Hence we discretized the solution domain as:

$$a = x_0 < x_1 < x_2 < \cdots < x_M = b$$
$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T \quad (3)$$

Where $x_{j+1} = x_j + jh$ and $t_{n+1} = t_n + nk$, $= 0(1)M$, $n = 0(1)N$. M and N are the maximum numbers of grid points respectively in the x and t direction.

Then the present paper is organized as follows. Section two is Methodology (Methods), section three is Stability and convergence analysis, section four is Numerical results. Section five is Discussion; section six is the conclusion of the present framework.

Methods

In this paper, the Lax-Wendroff difference scheme with Richardson extrapolation numerical method is developed to solve a one-dimensional wave equation with integral boundary conditions given in Eq. (1). Hyperbolic equations such as the wave equation in Eq. (1) have two derivatives in time and two in space. An initial condition (both function and its derivative) is required and boundary conditions on both sides of the domain. Contrast this with Laplace's equation for the gravitational potential (∇.H = 0) which is an elliptic partial differential equation. In this case, conditions on the entire boundary are needed and specify the solution everywhere. Either the function or its derivative must be specified on each of the boundaries and changing the conditions at one point will change the solution everywhere.

Lax Wendorff Scheme

A numerical technique was proposed in 1960 by P.D. Lax and B. Wendroff [30, 31, [32] for solving partial differential equations and systems numerically. Despite the impressive developments in numerical methods for partial differential equations from the 1970s onwards, in which the Lax Wendroff method has played a historic role, they are presently (1998) substantial research activities aimed at further improvements of methods [32]. Lax Wendroff's method is also explicit but needs improvement in accuracy in time. This method is an example of explicit time integration where the function that defines governing equation is evaluated at the current time [32]. Purpose of this method to achieve good enough accuracy in time.

Thus we apply Taylor series expansion for both spatial and time derivative of Eq. (1) to develop the present numerical method. Assuming that $u(x,t)$ has continuous higher order partial derivative on the region $[a,b]x[0,T]$. For the sake of simplicity, we use $u(x_j,t_n) = u_{jn}$, $\frac{\partial^p u}{\partial x^p} = \partial_x^p u_{j\,n}$ and $\frac{\partial^p u}{\partial t^p} = \partial_t^p u_{j\,n}$ for $p \ge 1$ is p^{th} order derivatives.

Spatial Discretization

By using Taylor series expansion, we have

$$u_{j+1\,n} = u_{j\,n} + h\partial_x u_{j\,n} + $$
$$+ \frac{h^2}{2!}\partial_x^2 u_{j\,n} + \frac{h^3}{3!}\partial_x^3 u_{j\,n} + \cdots$$
$$u_{j-1\,n} = u_{j\,n} - h\partial_x u_{j\,n} + $$
$$+ \frac{h^2}{2!}\partial_x^2 u_{j\,n} - \frac{h^3}{3!}\partial_x^3 u_{j\,n} + \cdots \quad (4)$$

Subtract the second equation from the first equation, and adding the first equation to the second equation of Eq.(4), respectively we obtain the central finite difference scheme of the first and second-order of $u(x,t)$ concerning spatial variable given by:

$$\partial_x u_x = \frac{u_{j+1} - u_{j-1}}{2h} + \tau_2$$
$$\partial_x^2 u_{j\,n} = \frac{u_{j+1} - 2u_{j\,n} + u_{j-1\,n}}{h^2} + \tau_2 \qquad (5)$$

Where $\tau_1 = -\frac{h^2}{6}\partial_x^3 u_{j\,n}$ $\tau_2 = -\frac{h^2}{12}\partial_x^4 u_{j\,n}$ are respectively their local truncation error terms. Now substituting the second equation of Eq. (5) into Eq. (1) we obtain the system of second-order ordinary differential equation in the temporal variable given in the form:

$$\frac{d^2 u(x_j,t)}{dt^2} = \frac{u_{j+1} - 2u_{j\,n} + u_{j-1\,n}}{h^2} + q(x_j, t) + \tau_2 \qquad (6)$$

Subject to the initial and boundary conditions

$$u(x_j, 0) = \omega_1(x_j), \frac{du(x_j,0)}{dt} = \omega_2(x)\,, a \le x_j \le b,$$
$$u(a,t) = \phi_1(t)\,,$$
$$\int_a^b u(x_j, t) = \phi_2(t)\,, 0 \le t \le T \qquad (7)$$

Temporal Discretization

Since Lax Wendroff's method is an explicit method that uses to improvement in accuracy in time. This method is an example of explicit time integration. So to discretize the derivative involve with the temporal variable and develop the lax Wendorff scheme, first we expand the functional value $\boldsymbol{u(x, t+k) = u_{j\,n+1}}$ by using Taylor series expansion at $(\boldsymbol{x_j, t_n})$as follow:

$$u_{j\,n+1} = u_{j\,n} + k\partial_t u_{j\,n} + \frac{k^2}{2!}\partial_t^2 u_{j\,n} +$$
$$+ \frac{k^3}{3!}\partial_t^3 u_{j\,n} + \frac{k^4}{4!}\partial_t^4 u_{j\,n} + \cdots \qquad (8)$$

Since the governing hyperbolic partial differential equation in (1) is $u_{tt} = u_{xx} + q(x,t)$. Then assuming that $q(x,t) = 0$, and $u_{tt} = u_{xx}$, this implies that $u_t = u_x$ .Thus by using this idea and truncating third-order derivative and above, from Eq. (8) we obtain:

$$u_{j\,n+1} = u_{j\,n} + k\partial_x u_{j\,n} + \frac{k^2}{2!}\partial_x^2 u_{j\,n} + \tau_3 \qquad (9)$$

where $\tau_3 = \frac{k^3}{3!}\partial_t^3 u_{j\,n}$ is the local truncation of the expiation series. Now substituting the first equation of Eq. (5) and Eq.(6) into Eq.(9) we obtain,

$$u_{j\,n+1} = u_{j\,n} + \alpha(u_{j+1} - u_{j-1}) +$$
$$+2\alpha^2(u_{j+1} - 2u_{j\,n} + u_{j-1\,n}) + G_{j\,n} + \tau_{j\,n} \qquad (10)$$

where $\alpha = \frac{k}{2h}$, $G_{j\,n} = 2\alpha^2 q(x_j, t_n)$ and $\tau_{j\,n} = \alpha\tau_1 + 2\alpha^2\tau_2 + \tau_3$ is local truncation error term of obtained difference equation for the full discretization of the one-dimensional hyperbolic partial differential equation given in (1). This local truncation error is:

$$\tau_{j\,n} = h^2\left(-\frac{\alpha}{6}\partial_x^3 u_{j\,n} - \frac{\alpha^2}{6}\partial_x^4 u_{j\,n}\right) +$$
$$+ \frac{k^3}{6}\partial_t^3 u_{j\,n} = \;O(h^2 + k^3) \qquad (11)$$

Thus the desired lax-Wendorff scheme that we use to solve second order linear one-dimensional wave equation in Eq.(1) is obtained by truncating the truncation error term $\tau_{j\,n}$, from difference equation in Eq.(10), the scheme is:

$$u_{j\,n+1} = u_{j\,n} + \alpha(u_{j+1\,n} - u_{j-1\,n}) +$$
$$+2\alpha^2(u_{j+1\,n} - 2u_{j\,n} + u_{j-1\,n}) + G_{j\,n}$$

Or

$$u_{j\,n+1} = \alpha(1+2\alpha)u_{j+1\,n} +$$
$$+(1-4\alpha^2)\partial_t^3 u_{j\,n} + \alpha(2\alpha - 1)u_{j-1\,n} + G_{j\,n} \;(12)$$

with the order of accuracy is $O(h^2 + k^3)$.

Implementation of Initial Condition

Since the initial condition in Eq. (7) is existing as the derivative in the temporal variable. so we discretize this initial condition by using central finite difference approximation for $n = 0$. It is given by.

$$\frac{u_{j,1} - u_{j,-1}}{2k} = \omega_2(x_j)$$
$$u_{j+1,1} = u_{j,-1} + 2k\omega_2(x_j) \qquad (13)$$

In Eq.(13) both $u_{j+1,1}$ and $u_{j,-1}$ are unknown value. So it is difficult to use this discrete scheme of the initial condition. Therefore to use this discrete scheme we must avoid unknown old value $u_{j,-1}$ from Eq. (13). To avoid this value let us substitute $n = 0$ into central finite difference discretization equation of one-dimensional wave equation in Eq. (1) we obtain:

$$u_{j,1} = 2u_{j,0} - u_{j,-1} + \\ +\alpha^2\left(u_{j+1,0} - 2u_{j,0} + u_{j-1,0}\right) + k^2 q(x_j, t_n) \quad (14)$$

Now adding Eq.(13) into Eq.(14) and using $u(x_j, 0) = u_{j,0} = \omega_1(x_j)$ we obtain :

$$u_{j,1} = \omega_1(x_j) + k\omega_2(x_j) + \\ + 0.5\alpha^2\left(\omega_1(x_{j+1}) - 2\omega_1(x_j) + \omega_1(x_{j-1})\right) + \\ +0.5k^2 q(x_j, t_n) \quad (15)$$

Hence the grid point at $j-1$ and $j+1$ are indicated the beyond of boundary grid point of the solution domain. Hence the scheme in Eq.(15) is only used the initial condition at the interior grind point, because due to the existence of boundary conditions in Eq.(7). Therefore by using the finite discrete scheme given in Eqs. (12) and (15) give the M-1 system of the equation that gives an accurate numerical solution of the one-dimensional wave equation given in Eq.(1) implicitly using the matrix inverse method.

Richardson Extrapolation method

Extrapolation is an extremely powerful tool available to numerical analysts for improving the performance of a wide variety of mathematical methods. It is an incredibly powerful technique for increasing speed and accuracy in various numerical tasks in scientific computing [33]. It is also used to speed up the rate of convergence for numerical methods.

Theorem: Let $D_{p-1}(h,k)$ and $D_{p-1}(2h,2k)$ are two finite differences approximate value of the partial derivative of $u(x,t)$ with an order of accuracy is $O(h^{2p} + k^{2p})$ such that for operator $\mathcal{L}u(x_0, t_0)$ define as

$$\mathcal{L}u(x_0, t_0) = D_{p-1}(h,k) + c_1(h^{2p} + k^{2p}) + c_2(h^{2p+2} + k^{2p+2}) + c_3(h^{2p+2} + k^{2p+2}) + \cdots \\ \mathcal{L}u(x_0, t_0) = D_{p-1}(2h,2k) + 2^{2p}c_1(h^{2p} + k^{2p}) + 2^{2p+2}c_2(h^{2p+2} + k^{2p+2}) + \cdots \quad (16)$$

Where p is the order of the differential equation. Then from the difference scheme in (16), we can drive the improved central finite difference scheme for the solution of the first-order partial derivative of $u(x,t)$ at$(x_0,\ t_0)$ which is given by:

$$\mathcal{L}u(x_0, t_0) = D_p(h,k) + O(h^{2p+2} + k^{2p+2}) = \\ = \frac{4^p D_{p-1}(h,k) - D_{p-1}(2h,2k)}{4^p - 1} + O(h^{2p} + k^{2p})$$

Thus from this general, the central finite difference approximation of second-order partial derivative of $u(x,t)$ at $(x_j,\ t_j)$ for $j \neq 0$ and $n \neq 0$ given by:

$$\mathcal{L}u(x_j, t_n) = D_p(h,k) + O(h^{2p+2} + k^{2p+2}) = \\ = \frac{u(x + 2^{-p}h, t) - 2u(x,t) + u(x - 2^{-p}h, t)}{2 \times 2^{-p}h^2} + \\ + \frac{u(x, t+2^{-p}k) - 2u(x,t) + u(x, t-2^{-p}k)}{2 \times 2^{-p}k^2} \quad (17)$$

Until $|D_{p+1} - D_p| > |D_p - D_{p-1}|$ or $|D_{p+1} - D_p| < tolerance$ where tolerance is provided for $p = 1(1)N$ with the order of accuracy is $O(h^{2p} + k^{2p})$.

Now the truncation error terms of our formulated Lax-Wendorff scheme in Eq. (12) is $O(h^2 + k^3)$. So the absolute error between two solutions at the grid point (x_j, t_n) is satisfied:

$$\left|u(x_j, t_n) - u_{j,n}\right| < \mathbf{C}(h^2 + k^3) \quad (18)$$

where $u(x_j, t_n)$ and $u_{j,n}$ are respectively exact and numerical solution of one-dimensional wave equation given in Eq. (1) and $\mathbf{C}$ is constant that independent from step length 'h' and time step 'k.'

Let us consider that $\Omega_{j\,n}$ is the set of grid points that we obtain by using the mesh size h and k in Eq. (3). Consider that Eq.(18) work for any mesh size $h, k \neq 0$, which implies that for $(x_j, t_n) \in \Omega_{j\,n}$:

$$u(x_j, t_n) - u_{j,n} < \mathbf{C}(h^2 + k^3) + \mathcal{R}_{j,n} \quad (19)$$

Where $\mathcal{R}_{j,n}$ is reminder term in this interval for which $u_{j,n}$ is approximated at each grid point $\Omega_{j\,n}$. Let $\Omega_{2j\,2n}$ is the set of grid point that obtained by bisecting each mesh point in $\Omega_{j\,n}$ and let $u_{2j,2n}$ is approximated numerical value at each bisected grid point. Then, Eq.(18) also work for ${}^{h}/_{2}, {}^{k}/_{2} \neq 0$ which implies that:

$$u(x_j,t_n) - u_{2j,2n} < \mathbf{C}\left(\left(\frac{h}{2}\right)^2 + \left(\frac{k}{2}\right)^3\right) + \mathcal{R}_{2j,2n}$$
$$8u(x_j,t_n) - 8u_{2j,2n} <$$
$$< \mathbf{C}(2h^2 + k^3) + 8\mathcal{R}_{2j,2n} \tag{20}$$

Where $\mathcal{R}_{2j,2n}$ is reminder term in this interval for which $u_{2j,2n}$ is approximated at each grid points $\Omega_{2j\,2n}$. Now subtracting Eq.(19) from Eq.(20) we obtain:

$$7u(x_j,t_n) - 8u_{2j,2n} + u_{j,n} < \mathbf{C}\,\boldsymbol{h}^2 + \Re$$
$$\left(u(x_j,t_n)\right)^{exact} - {}^1\!/_7\left(8u_{2j,2n} - u_{j,n}\right) <$$
$$< {}^1\!/_7\left(\mathbf{C}\,\boldsymbol{h}^2 + \Re\right)$$

Where $\Re = 8\mathcal{R}_{2j,2n} - \mathcal{R}_{j,n}$ total reminder terms of two schemes. Negating the right-hand side terms of the above difference equation, we obtain:

$$\breve{u}_{j\,n} = {}^1\!/_7\left(8u_{2j,2n} - u_{j,n}\right) \tag{20}$$

Where $\breve{u}_{j\,n} = \left(u(x_j,t_n)\right)^{exact}$ is also an approximation of the exact solution (x_j,t_n). Therefore using the above theorem, the order of truncation error for the scheme in Eq. (20) is $\mathbf{O}(h^4 + k^6)$. Thus, the approximate solution in Eq. (20) is satisfies

$$\left|u(x_j,t_n) - \breve{u}_{j\,n}\right| \le \mathbf{C}(h^4 + k^6) \tag{21}$$

Thus the extrapolation method accelerates the convergence (accuracy) of the present method to approximate the exact solution with fourth-order in special variable six orders in the time variable.

Stability and convergent Analysis

Because hyperbolic equations often describe the motion and development of waves, Fourier analysis is of great value in studying the accuracy of methods as well as their stability [34]. The modulus of $\lambda(k)$ describes the damping and the argument describes the dispersion in the scheme, i.e., the extent to which the wave speed varies with the frequency [34]. The Fourier analysis (Von-Neumann) stability analysis technique is applied to investigate the stability of the proposed method. Such an approach has been used by many researchers like [15, 18, 25 26, 35]. Now assume that the solution of the given problem at the points (x_j,t_n) is

$$u_{j\,n} = \lambda^n e^{ij\theta} \tag{22}$$

Where $i = \sqrt{-1}$,$\theta = k\pi/N,\ k \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ which is $\mathbb{R}$ set of a real number and $\mathbb{C}$ set of a complex number. Substituting Eq. (22) into Eq. (12), we obtain:

$$\lambda^{n+1}e^{ij\theta} = \alpha(1+2\alpha)\lambda^n e^{i(j+1)\theta} + (1-4\alpha^2)\lambda^n e^{ij\theta} + \alpha(2\alpha-1)\lambda^n e^{i(j-1)\theta} + G_{j\,n}$$
$$\lambda = \alpha(1+2\alpha)e^{i\theta} + (1-4\alpha^2) + \alpha(2\alpha-1)e^{-i\theta} + G_{j\,n}$$
$$\lambda = 1-4\alpha^2 + \alpha(1+2\alpha)[cos\theta + isin\theta] + \alpha(2\alpha-1)[cos\theta - isin\theta] + G_{j\,n}$$
$$\lambda\,(k) = 1 - 4\alpha^2\left(1 - cos\theta - \beta G_{j\,n}\right) + i2\alpha sin\theta) \tag{23}$$

Where $\beta = 1/2\alpha$ and λ(k) is an amplification factor of the proposed scheme. From this, we see that the modules of this amplification factor are less than one($|\,\lambda\,(k)| < 1$).

Theorem 1. Let **Z** has a power series expansion in the power of variable δ as follow

$$Z \approx c_1\delta + c_2\delta^2 + c_3\delta^3 + c_4\delta^4 + \cdots$$

for $\delta \to 0$, then $\tan^{-1}(Z)$ is equal to the power series expansion of Z, then Z lies in the unit circle.

Proof: Let $u_{j,n}$ be the finite difference approximation of first-order partial differential equation obtained by using the finite discrete scheme of the form:

$$u_{j\,n+1} = u_{j\,n} - \alpha\Delta_- u_{j\,n}$$

where Δ_- indicate backward difference operator and $\alpha = \Delta x/\Delta t$. Then the Von Neumann stability analysis gives

$$Z(k) = 1 - \alpha\left(1 - e^{i\theta}\right) =$$
$$= 1 - \alpha(1 - cos\theta + isin\theta)$$

$$\left(Z(k)\right)^2 = 1 - 4\alpha(1-\alpha)\sin^2(\theta/2)$$

Thus from this for all ‘k’ and $0 \le \alpha \le 1$ we have $|Z(k)| < 1$. Hence the argument of amplification factor $Z(k)$ is defined by

$$\mathrm{Arg}\big(Z(k)\big) \approx \tan^{-1}\left(\frac{\alpha \sin(\theta)}{1-\alpha(1-cos\theta)}\right) \approx$$
$$\approx -\alpha\theta\big(1-1/6(1-\alpha^2)\big)\theta^2 + \cdots$$

Hence the theorem is proved. Now using the idea of this theorem, the arguments of the amplification factor in Eq.(23) is

$$Arg\ \mathrm{Arg}(\lambda\,(k)) \approx \tan^{-1}\left(\frac{2\alpha \sin(\theta)}{1-4\alpha^2\big(1-cos\theta-\beta G_{j\,n}\big)}\right)\Bigg) \approx$$
$$\approx \tan^{-1}\left(\frac{2\alpha\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{24}-\frac{\theta^6}{720}+\cdots\right)}{1-4\alpha^2+4\alpha^2\left(\theta-\frac{\theta^3}{6}+\frac{\theta^5}{120}-\frac{\theta^7}{5040}+\cdots\right)+4\alpha^2\beta G_{j\,n}}\right) \cong$$
$$\cong \frac{2\alpha\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{24}-\frac{\theta^6}{720}+\cdots\right)}{1-4\alpha^2+4\alpha^2\left(\theta-\frac{\theta^3}{6}+\frac{\theta^5}{120}-\frac{\theta^7}{5040}+\cdots\right)+4\alpha^2\beta G_{j\,n}} =$$
$$= \frac{7\alpha(720-360\theta^2+30\theta^4-\theta^6+\cdots)}{2520-2\alpha(5040-5040\theta+840\theta^3-42\theta^5+\theta^7-\cdots)+4\alpha^2\beta G_{j\,n}}$$

Since $0 \le \alpha = k/h \le 1$ and $\theta = k\pi/N$. Hence for $k \to 0$, $\theta \to 0$. still, we have seen that $|\lambda\,(k)| < 1$. Thus $\lambda\,(k)$ are lies inside the unit circle. Therefore, the lax-Wendorff finite difference scheme given in Eq. (12) is stable for the wave equation.

Theorem 2: The difference equation given in the form of Eq.(12) is stable if for which the eigenvalues of the coefficient matrix of the system of the differential equation are satisfied $Real(\lambda_j) < 0$. **Proof:** See reference [15]

Since from the principal part of the local truncation error, the derived local truncation error for the proposed scheme is

$$\tau_{j\,n} = h^2\left(-\frac{\alpha}{6}\partial_x^3 u_{j\,n} - \frac{\alpha^2}{6}\partial_x^4 u_{j\,n}\right) +$$
$$+\frac{k^3}{6}\partial_t^3 u_{j\,n} + O\big(h^2+k^3\big)$$

Thus, $\tau_{j\,n} \to 0$ as $h, k \to 0$. So that, the scheme is consistent with the order of. Hence the scheme is convergent.

Criteria for Investigating the Accuracy of the Method

This section presented the criteria that the accuracy of the present method is investigated. Since there are two types, Round-off errors and Truncation errors occur when differential equations are solved numerically. Rounding errors originate from the fact that computers can only represent numbers using a fixed and limited number of significant figures. Thus, such numbers or cannot be represented exactly in computer memory. The discrepancy introduced by this limitation is to call a Round-off error. Truncation errors in numerical analysis arise when approximations are used to estimate some quantity. The accuracy of the solution will depend on how small we make the step size, h, and time step k. To test the performance of the proposed method, maximum absolute error, L_2 and L_∞ norms are used to measure the accuracy of the method. These norms are calculated by:

$$L_\infty = \max_{1\le n\le N}\left|u(x_j\,,t_n) - u_{j\,n}\right|,$$
$$L_2 = \sqrt{\frac{1}{N}\sum_{j=0}^{N}\left|u(x_j\,,t_n) - u_{j\,n}\right|^2}$$

where M is the maximum number of step, $u(x_j\,,t_n)$ is the exact solution and $u_{j\,n}$ approximation solution of the wave equation in Eq.(1) at the grid point $(x_j\,,t_n)$.

Results

To test the validity of the proposed method, we have considered the following three model problem considered in [2, 5, 10, 16]. Numerical results and errors are computed and the outcomes are represented tabular and graphically.

Example 1: consider the classical wave equation considered in [2, 5]

$$u_{tt} = u_{xx}\,, (x,t) \in (0,1) \times (0,T)$$

Subjected to initial and boundary condition

$$u(x,0) = \cos(\pi x)\,, u_t(x,0) = 0\,, 0 \le x \le 1\,,$$
$$u(0,t) = \cos(t)\,, \int_0^1 u(x,t)dt = 0\,,\ 0 \le t \le T$$

An analytical solution is $u(x,t) = \frac{1}{2}\left(\cos\left(\pi(x+t)\right) + \cos\left(\pi(x-t)\right)\right)$

Example 2: consider the wave equation considered in [5]

$$u_{tt} - u_{xx} = \left(\pi^2 + \frac{1}{4}e^{-t/2}\right)\sin(\pi x), (x,t) \in (0,1) \times (0,T)$$

Subjected to initial and boundary condition

$$u(x,0) = \sin(\pi x)\,, u_t(x,0) = {}^{1}/_{2}\ \sin(\pi x),\ 0 \le x \le 1,$$
$$u(0,t) = \cos(\pi t)\,, \int_0^1 u(x,t)dt = \frac{2}{\pi}e^{-t/2}\,,\ 0 \le t \le T$$

An analytical solution is $u(x,t) = e^{-t/2}\sin(\pi x)$

Example 3: consider the wave equation considered in [2]

$$u_{tt} - u_{xx} = 2t - 6x - 2\,, (x,t) \in (0,1) \times (0,T)$$

Subjected to initial and boundary condition

$$u(x,0) = x^2, u_t(x,0) = x - x^2, 0 \le x \le 1,$$
$$u(0,t) = -t^2, \int_0^1 u(x,t)dt = -t^2 + \frac{t}{6} + {}^{1}/_{4},\ 0 \le t \le T$$

An analytical solution is $u(x,t) = (x^2 + t)\,(x - t)$

Table 1 – Comparison of pointwise absolute error for problem give in example one with computations carried out until final time $T = 5$ with mesh size $h=0.01$ *and* $k=0.1$.

x	By the previous method Goh Joan et.al.[16]	S. M. Zin. et.al. [5]	By the present method maximum absolute error
0.2	1.21E-04	1.12E-04	1.7865E-04
0.3	1.15E-04	1.07E-04	1.6803e-05
0.4	6.88E-05	6.40E-05	1.436E-05
0.5	2.03E-13	5.05E-15	4.5452EE-15
0.6	6.88E-05	6.40E-05	2.0236E-05
0.7	1.15E-04	1.07E-04	1.7205E-05
0.8	1.21E-04	1.12E-04	3.1439E-05
0.9	7.97E-05	7.39E-05	3.1214E-05

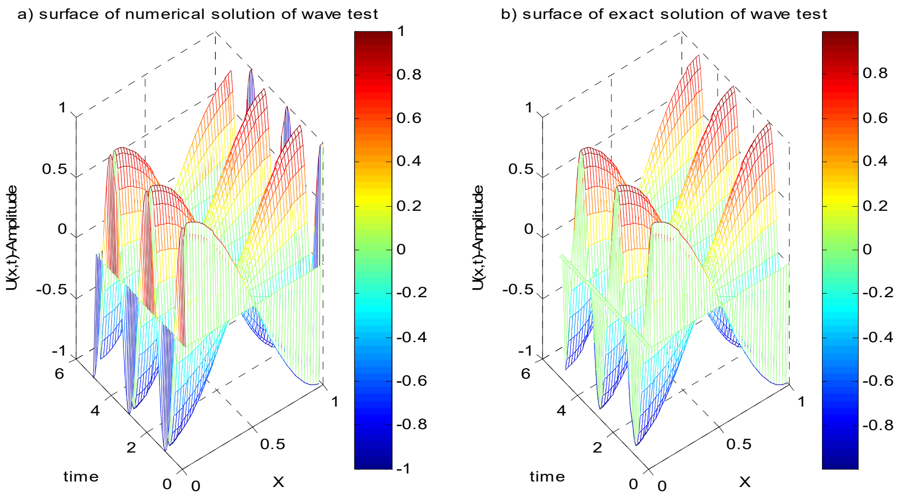

Figure 1: Surface graphs of example 1 showing the physical behavior of the one-dimensional wave equation when $h = 0.02$ and $k = 0.1$

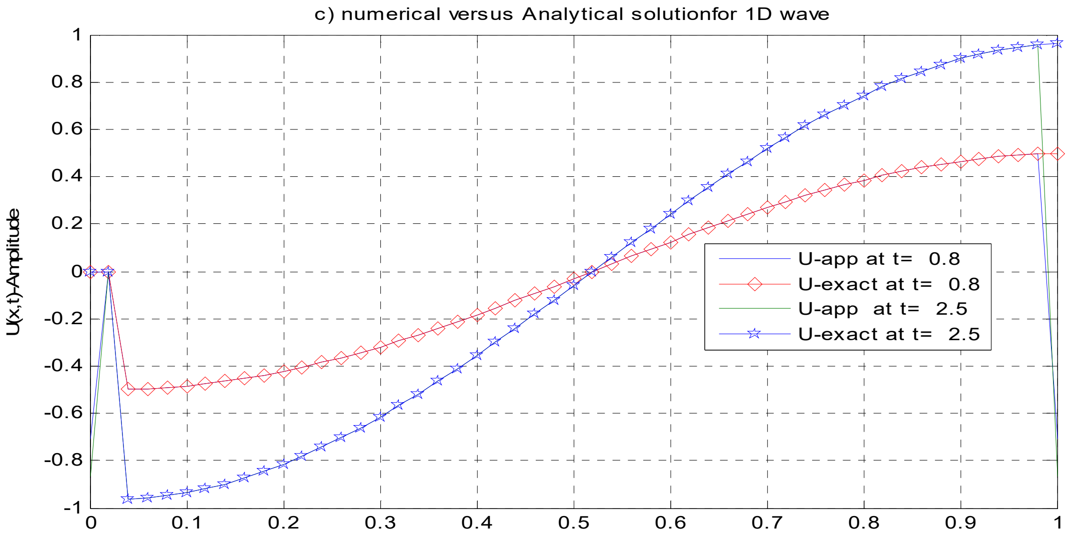

Figure 2 – The physical behavior of the solution for example one for Comparison of the Approximate and Exact solution

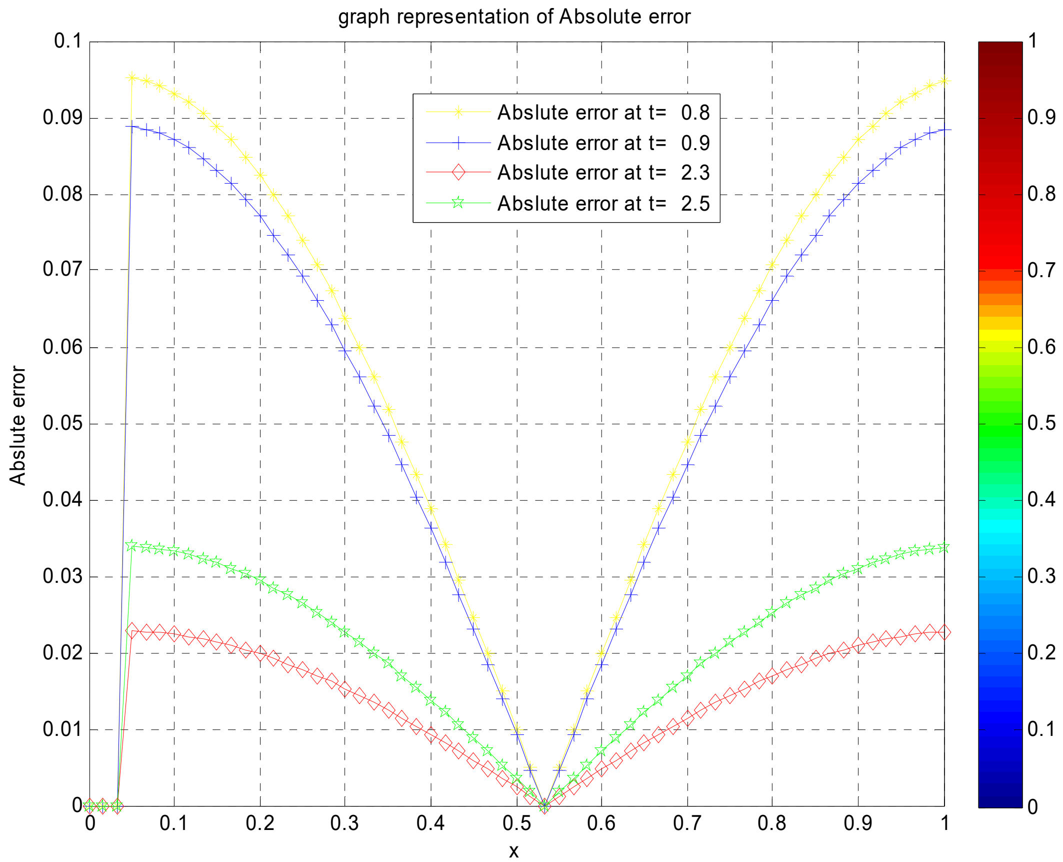

Figure 3 – The physical behavior of Absolute error for solution of example one when $h = 0.01$ and $k = 0.03$

Table 2 – Comparison of maximum pointwise absolute error for problem give in example two with computations carried out until final time $T = 1$ with mesh size h=0.01 and k=0.0001.

t	My previous method		By present method
	Meghan [10]	S. M. Zin. et.al. [5]	Maximum abs. Error
0.5	1.3371E-03	3.9515E-05	1.7005E-05
1	2.3794E-03	2.008E-04	4.1897E-06

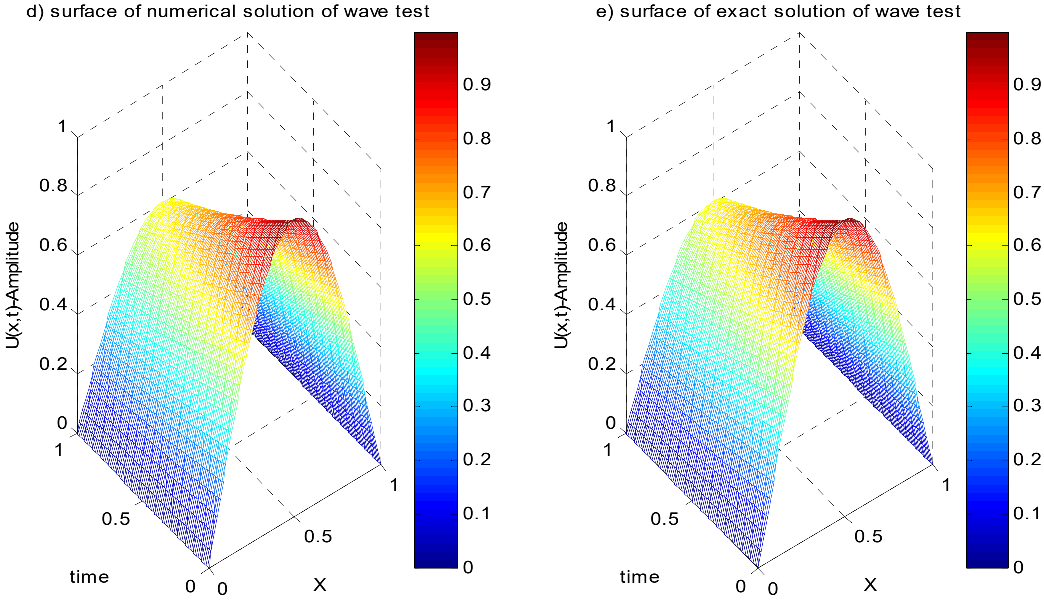

Figure 4 – Surface graphs of example 2 showing the physical behavior of the one-dimensional wave equation when $h = 0.02$ and $k = 0.1$

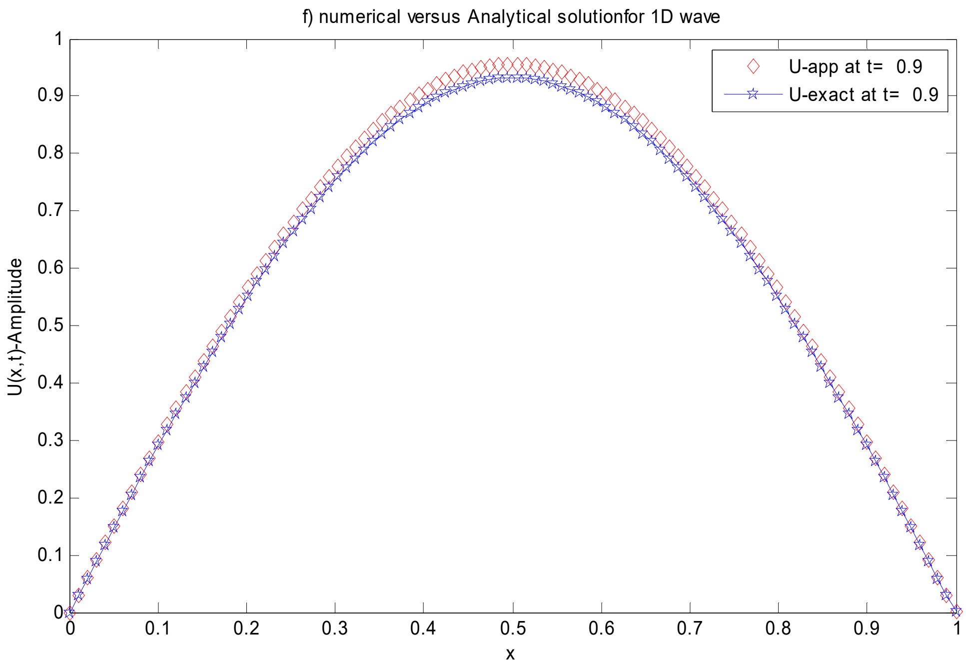

Figure 5 – The physical behavior of the solution for example 2 for Comparison of the Approximate and Exact solution for step length $h = 0.01$ and time step $k = 0.001$

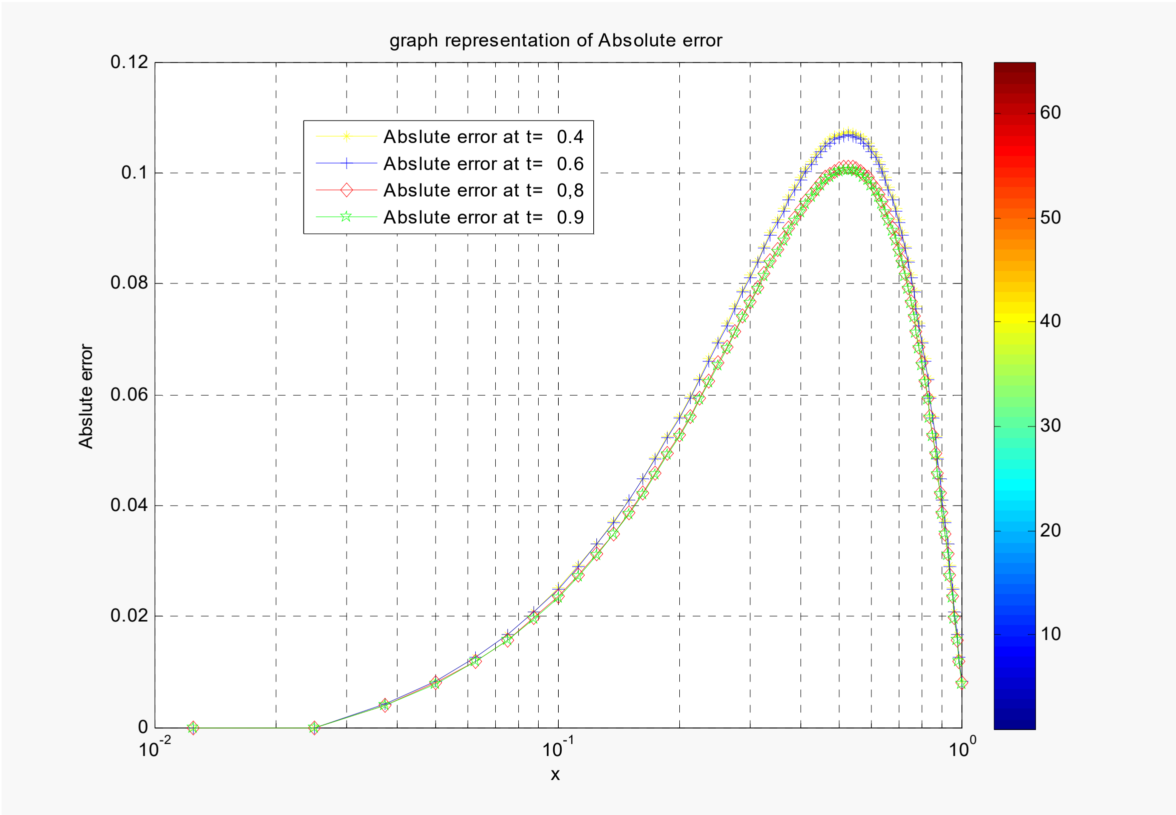

Figure 6 – The physical behavior of Absolute error
for example two when $h = 0.0125$ and $k = 0.01$.

Table 3 – Representation of L_2 and L_∞ norm to show the accuracy of the method for problem give in example three with computations carried out until final time $T = 1$ with mesh size h and k.

i) Representing of performance of the present method with example 3 for decreasing step length $(\boldsymbol{h} \to \mathbf{0})$ with the fixed time step $(\boldsymbol{k} = \mathbf{0.01})$

h	k	L_∞	L_2
0.1	0.01	3.2076E-04	1.0289E-05
0.05	0.01	7.8760E-05	1.2406E-06
0.025	0.01	1.5531E-05	9.6480E-08
0.0125	0.01	1.3273E-06	1.4094E-09

ii) Representing of performance of the present method with example 3 for decreasing time step $\boldsymbol{k} \to \mathbf{0}$ with fixed step length $(\boldsymbol{h} = \mathbf{0.1})$

h	k	L_∞	L_2
0.1	0.0667	1.1049e-04	8.1385e-07
0.1	0.025	1.6940E-04	1.4348E-06
0.1	0.02	2.6235E-04	1.7207E-06
0.1	0.0125	3.1089E-04	1.2081E-06

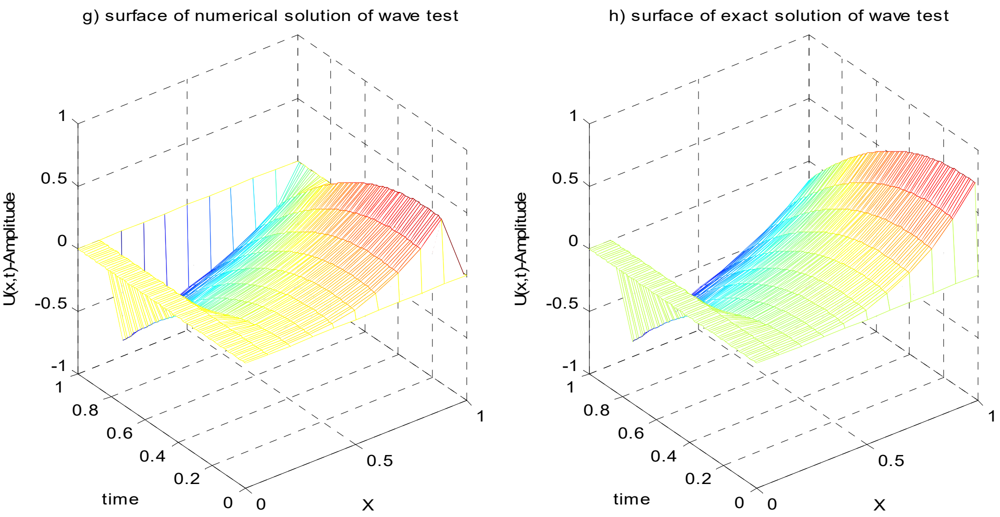

Figure 7 – Surface graphs of example 3 showing the physical behavior of the one-dimensional wave equation when $h = 0.1$ and $k = 0.01$

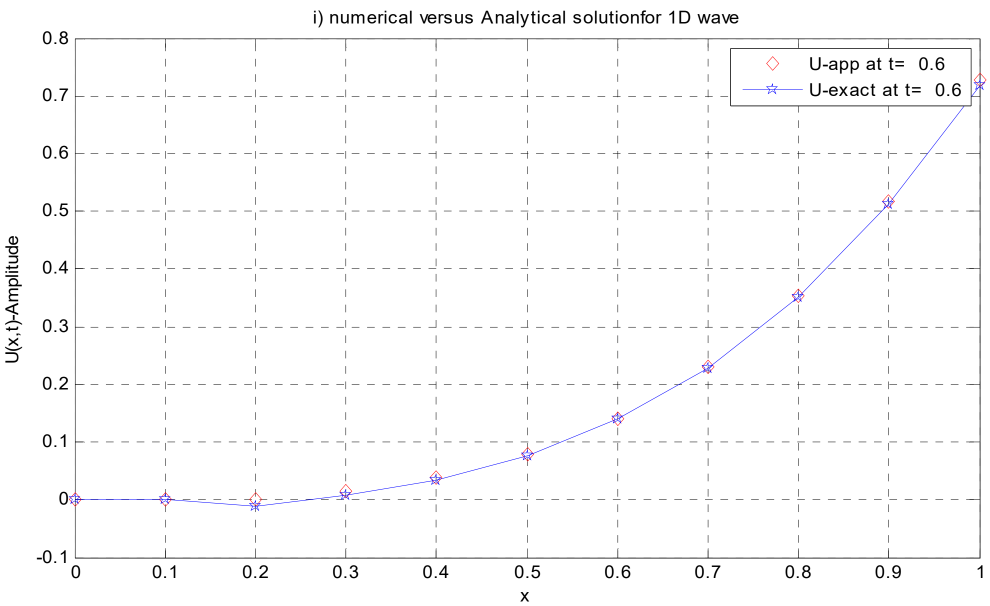

Figure 8 – The physical behavior of the solution for example 3 for Comparison of the Approximate and Exact solution for step length $h = 0.1$ and time step $k = 0.01$

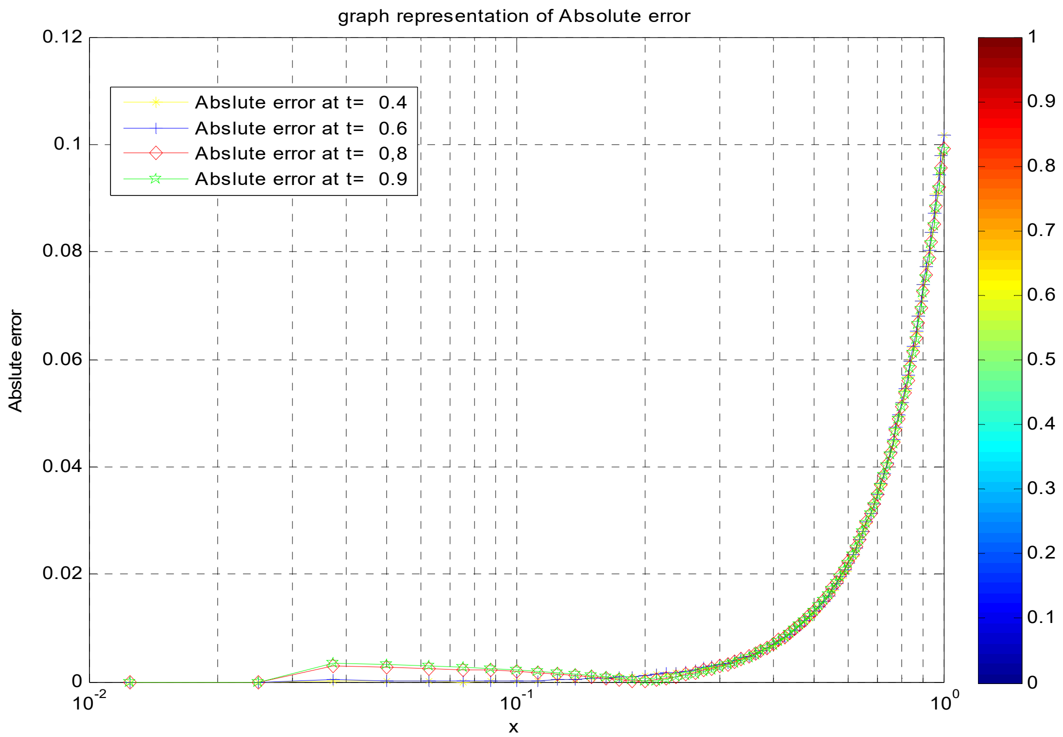

Figure 9 – Graph representation for Absolute error of example three when $h = 0.0125$ and $k = 0.01$

Desiccations

In this paper, the Lax-Wendorff difference scheme with the Richardson extrapolation method is presented for solving a one-dimensional wave equation subjected to integral conditions. To demonstrate the competence of the method, three model examples are solved by taking different values for step size h, and time step k. Numerical results obtained by the present method have been associated with numerical results obtained by the method in [2, 5, 10, 16] and the results are summarized in Tables and graph. Moreover, the maximum absolute errors decrease rapidly as the number of mesh points M and N increases. Further, as shown in Figs. 2, 5, and 8 the proposed method approximates the exact solution very well for different values of step length h and time step k as given above for which most of the current methods fail to give good results. To further verify the applicability of the planned method, graphs were plotted aimed at Examples 1, 2, and 3 for exact solutions versus the numerical solutions obtained. As Figs. 1, 2, 4, 5, 7, and 8 indicate good agreement of the results, presenting exact as well as numerical solutions, which proves the reliability of the method. Also, Figs. 3, 6, and 9 specify the absolute error of obtained numerical solution by the effects of mesh sizes on the solution domain. Further, the numerical results presented in this paper validate the improvement of the proposed method over some of the existing methods described in the literature. Both the theoretical and numerical error bounds have been established. Hence, the Richardson extrapolation method accelerates second-order into fourth-order convergent in spatial variable and third-order into sixth-order convergent in the temporal variable. The results in Tables 1, 2, and 3 further confirmed that the computational rate of convergence and theoretical estimates are in agreement.

As it can also be seen from table 3 when the value of h is decreased with fixed time step k, the maximum absolute error and L_2-norm also decreased the accuracy of the proposed method increase. But for $k \rightarrow 0$ with fixed step size h both maximum absolute error and L_2-norm also increases. So that accuracy of the proposed method decreases. It concludes that the smaller value of h and with fixed time step k *gives* a better approximation to the exact solution. However decreasing both the value of step size h and time step k, affects the accuracy of the present method that developed this paper. Comparison among Table 1-Table 3 and the graphs of the numerical and exact solution of 1D wave equation shows that the present method generates a more accurate result and it is superior to the method developed in [5 ,10, 16] and It is approximate the exact solution very well.

Conclusion

A new approach, lax-Wendorff difference scheme with Richardson extrapolation method is using to solve 1D wave equation numerically is presented in this study. The comparison of the results obtained by the present method with other methods reveals that the present method is more convenient, reliable, and effective. An error analysis based on the Fourier series is also developed in this study. As it can be seen that, the accuracy improves when M and N are increased. Tables and figures indicate that as h decreases with a fixed value of k , the errors decrease more rapidly. Another considerable advantage of the method is that the Richardson extrapolation techniques are well approximate and give better accuracy of the numerical solution using bisected grid point in the domain and searching accurate solution and improve the performance of methods. In a summary, the lax-Wendorff difference scheme with the Richardson extrapolation method is a reliable method that is capable to solve the one-dimensional wave equation. Based on the findings, this method is well approximate and gives better accuracy of the numerical solution with a fixed time step, k, and smaller step size h.

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