IRSTI 27.35.63 https://doi.org/10.26577/ijmph.2020.v11.i2.01

S.N. Kharin[1,3*], T.A.Nauryz[1,2,3,4], B. Miedzinski[5]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[3]Kazakh-British Technical University, Almaty, Kazakhstam
[4]Satbayev University, Almaty, Kazakhstan
[5]Wroclaw University, Wroclaw, Poland
*e-mail: staskharin@yahoo.com

TWO PHASE SPHERICAL STEFAN INVERSE PROBLEM SOLUTION WITH LINEAR COMBINATION OF RADIAL HEAT POLYNOMIALS AND INTEGRAL ERROR FUNCTIONS IN ELECTRICAL CONTACT PROCESS

Abstract. In this research the inverse Stefan problem in spherical model where heat flux has to be determined is considered. This work is continuing of our research in electrical engineering that when heat flux passes through one material to the another material at the point where they contact heat distribution process takes the place. At free boundary $\alpha(t)$ contact spot starts to boiling and at $\beta(t)$ stars to melting and there appear two phase: liquid phase and solid phase. Our aim to calculate temperature in liquid and solid phase, then find heat flux entering into contact spot. The exact solution of problem represented in linear combination of series for radial heat polynomials and integral error functions. The recurrent formulas for determine unknown coefficients are represented. The effectiveness of method is checked by test problem and approximate problem in which exact solution and approximate solution of heat flux is compared. The coefficients of heat at liquid and solid phases and heat flux are found. The heat flux equation is checked by testing problem by using Mathcad program.

Key words: Stefan problem, radial heat polynomials, Faa-di Bruno, collocation method.

Introduction

Heat flux entering in electrical contact materials from electrical arc distributes radially and axially. Spherical model is most convenient, introduced by Holm R. [1], in the problem of heat distribution in electrical materials. In this problem generalized heat equation can be used. The generalized heat equation of the form

$$\frac{\partial\theta}{\partial t} = a_1^2 \frac{1}{r^v}\frac{\partial}{\partial x}\left(r^v \frac{\partial\theta}{\partial x}\right)$$

have the fundamental solution with delta-function containing initial condition by using Laplace transform can be represented as

$$G(x,y,t) = \frac{C_v}{2t}(xy)^{-\beta} e^{\frac{x^2+y^2}{4t}} I_\beta\left(\frac{xy}{2t}\right),$$

where

$$\beta = \frac{v-1}{2}, \quad C_v = 2^{-\beta}\Gamma(\beta+1)$$

We can consider the heat potentials related to this solution in form [2]

$$Q_{n,v}(x,t) = 2^{-\beta}\Gamma(\beta+1)^{-1}\int_0^\infty G(x,y,t)y^{2n+v}dy$$

and by using integration by parts method we have the following explicit formula of heat polynomials

$$Q_{n,v}(x,t) = \sum_{k=0}^{n} 2^{2k}\frac{n!\Gamma(\beta+1)x^{2(n-k)}t^k}{k!(n-k)!\Gamma(\beta+1+n-k)}$$

For applications it is convenient to multiply both sides of this equation by $\frac{\Gamma(\beta+1+n)}{\Gamma(\beta+1)}$ and we get the following solution

$$Q_{n,v}(x,t) = \sum_{k=0}^{n} 2^{2k}\frac{n!\Gamma(\beta+1+n)x^{2(n-k)}t^k}{k!(n-k)!\Gamma(\beta+1+n-k)}$$

which satisfy the generalized heat equation.

In this research we consider $v=2$ which allow to transform to generalized heat equation to spherical heat equation [3]. The similar problems are considered in [4]-[7].

Mathematical model

Let us consider the liquid phase described in domain $\alpha(t)<r<\beta(t),\ t>0$ and solid phase in $\beta(t)<r<\infty,\ t>0$ with spherical heat equations

$$\frac{\partial\theta_i}{\partial t}=a_i^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta_i}{\partial r}\right),\ \ i=1,2 \tag{1}$$

and each phase has initial condition as follows

$$\theta_1(\alpha(t),0)=0,\quad \alpha(0)=\beta(t)=0, \tag{2}$$

$$\theta_2(r,0)=f(r),\quad f(0)=\theta_m. \tag{3}$$

Heat flux entering $P(t)$ into spherical domain from electrical arc with radius r_0 in process pf heat transfer within electrical contact materials can be determined from condition

$$-\lambda_1\frac{\partial\theta_1}{\partial r}\bigg|_{r=r_0}=P(t). \tag{4}$$

Temperatures in liquid and solid phase at free boundary $\alpha(t)$ is equal to melting temperature

$$\theta_i(\beta(t),t)=\theta_m,\ \ i=1,2. \tag{5}$$

Motion of the free boundary can be calculated at Stefan's condition

$$-\lambda_1\frac{\partial\theta_1}{\partial r}\bigg|_{r=\beta(t)}=-\lambda_2\frac{\partial\theta_2}{\partial r}\bigg|_{r=\beta(t)}+L\gamma\frac{d\beta}{dt} \tag{6}$$

and temperature of solid zone at infinity turns to zero

$$\theta_2\big|_{r=\infty}=0. \tag{7}$$

Problem solution

The solution of problem (1)-(7) we represent as linear combination of series for radial heat equation and integral error functions

$$\theta_1(r,t)=\sum_{n=0}^{\infty}A_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)r^{2(n-k)}t^k}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}+ \\ +\sum_{n=0}^{\infty}B_n\frac{(2a_1\sqrt{t})^{2n+1}}{r}\left(i^{2n+1}erfc\frac{-r}{2a_1\sqrt{t}}-i^{2n+1}erfc\frac{r}{2a_1\sqrt{t}}\right), \tag{8}$$

$$\theta_1(r,t)=\sum_{n=0}^{\infty}C_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)r^{2(n-k)}t^k}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}+ \\ +\sum_{n=0}^{\infty}D_n\frac{(2a_2\sqrt{t})^{2n+1}}{r}\left(i^{2n+1}erfc\frac{-r}{2a_2\sqrt{t}}-i^{2n+1}erfc\frac{r}{2a_2\sqrt{t}}\right). \tag{9}$$

The equations (8) and (9) satisfy heat equation (1) and undetermined coefficients A_n, B_n, C_n and D_n have to be founded to determine temperatures in phases. The function at initial condition for $\theta_2(r,t)$ is represented in expansion by Maclaurin series $f(r)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}r^n$ and free boundaries can be considered in power series $\alpha(t)=\sum_{n=0}^{\infty}\alpha_n t^{n/2+1}$ and $\beta(t)=\sum_{n=0}^{\infty}\beta_n t^{n/2+1}$. Heat flux which have to be determined from condition (4) can be written in

$$P(t)=p_0+p_1t^{1/2}+p_2t+p_3t^{3/2}...=\sum_{n=0}^{\infty}p_nt^{n/2}.$$

At first, we must find temperatures in liquid and solid zones, then by using property of integral error function to condition (3) we get

$$\sum_{n=0}^{\infty}C_nr^{2n}+\sum_{n=0}^{\infty}D_n\frac{2}{(2n+1)!}r^{2n}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}r^n \tag{10}$$

By comparing the power of r in both sides (10) we obtain the following form

$$C_n+D_n\frac{2}{(2n+1)!}=\frac{f^{(2n)}(0)}{(2n)!} \tag{11}$$

and from conditions (5) we have

$$\sum_{n=0}^{\infty} A_n \sum_{k=0}^{n} \frac{2^{2k} n! \Gamma\left(\frac{3}{2}+n\right) \beta(\tau)^{2(n-k)} \tau^{2k}}{k!(n-k)! \Gamma\left(\frac{3}{2}+n-k\right)} + \sum_{n=0}^{\infty} B_n \frac{(2a_1\tau)^{2n+1}}{\beta(\tau)} \left(i^{2n+1} erfc(-\upsilon(\tau)) - i^{2n+1} erfc(\upsilon(\tau))\right) = \theta_m, \quad (12)$$

$$\sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \frac{2^{2k} n! \Gamma\left(\frac{3}{2}+n\right) \beta(\tau)^{2(n-k)} \tau^{2k}}{k!(n-k)! \Gamma\left(\frac{3}{2}+n-k\right)} + \sum_{n=0}^{\infty} D_n \frac{(2a_2\tau)^{2n+1}}{\beta(\tau)} \left(i^{2n+1} erfc(-\upsilon(\tau)) - i^{2n+1} erfc(\upsilon(\tau))\right) = \theta_m, \quad (13)$$

and from Stefan's condition we obtain

$$-\lambda_1 \left[\sum_{n=0}^{\infty} A_n \sum_{k=0}^{n} \frac{2^{2k} n! \Gamma\left(\frac{3}{2}+n\right) 2(n-k) \beta(\tau)^{2(n-k)-1} \tau^{2k}}{k!(n-k)! \Gamma\left(\frac{3}{2}+n-k\right)} + \sum_{n=0}^{\infty} B_n \left(-\frac{(2a_1\tau)^{2n+1}}{\beta^2(\tau)} \left(i^{2n+1} erfc(-\upsilon(\tau)) - i^{2n+1} erfc(\upsilon(\tau))\right) \right.\right.$$
$$\left.\left. -\frac{(2a_1\tau)^{2n}}{\alpha(\tau)} \left(i^{2n} erfc(-\upsilon(\tau)) + i^{2n} erfc(\upsilon(\tau))\right) \right) \right] = -\lambda_2 \left[\sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \frac{2^{2k} n! \Gamma\left(\frac{3}{2}+n\right) 2(n-k) \beta(\tau)^{2(n-k)-1} \tau^{2k}}{k!(n-k)! \Gamma\left(\frac{3}{2}+n-k\right)} + \right. \quad (14)$$
$$\left. + \sum_{n=0}^{\infty} D_n \left(-\frac{(2a_2\tau)^{2n+1}}{\beta^2(\tau)} \left(i^{2n+1} erfc(-\upsilon(\tau)) - i^{2n+1} erfc(\upsilon(\tau))\right) - \frac{(2a_2\tau)^{2n}}{\beta(\tau)} \left(i^{2n} erfc(-\upsilon(\tau)) + i^{2n} erfc(\upsilon(\tau))\right) \right) \right] + L\gamma \frac{d\beta}{d\tau},$$

where $\sqrt{t} = \tau$ and

$$\upsilon(\tau) = \frac{\beta_0 + \beta_1\tau + \beta_2\tau^2 + ...}{2a_1} = \frac{1}{2a_1} \sum_{n=0}^{\infty} \upsilon_n \tau^n .$$

Firstly, we take l-th derivative both sides of (13) when $\tau = 0$ using Leibniz rule for first and second term of (13)

$$\left. \frac{\partial^l \left[\tau^{2k} \beta(\tau)^{2(n-k)} \right]}{\partial \tau^l} \right|_{\tau=0} = \frac{l!}{(l-2k)!} [\beta(\tau)]^{(2n-4k-l)}, \quad (15)$$

$$\left. \frac{\partial^l \left[\tau^{2k+1} \left[i^{2n+1} erfc(-\upsilon(\tau)) - i^{2n+1} erfc(\upsilon(\tau)) \right] \right]}{\partial \tau^l} \right|_{\tau=0} = \quad (16)$$
$$= \frac{l!}{(l-2k-1)!} \left[i^{2n+1} erfc(-\upsilon(\tau)) - i^{2n+1} erfc(\upsilon(\tau)) \right]^{l-2k-1}.$$

Using Faa-di Bruno for (15) and (16) we get

$$\left. \frac{l!}{(l-2k)!} [\beta(\tau)]^{(l-2n)} \right|_{\tau=0} = \quad (17)$$
$$= \frac{l!}{(l-2k)!} \sum_{m=1}^{l-2n} \beta_0^{(m)} \sum_{b_i} \frac{(l-2k-1)! \beta_1^{b_1} \beta_2^{b_2} ... \beta_{l-2n+1}^{b_{l-2n+1}}}{b_1! b_2! ... b_{l-2n+1}!},$$

$$\left. \frac{l!}{(l-2k-1)!} \left[i^{2n+1} erfc(-\upsilon(\tau)) - i^{2n+1} erfc(\upsilon(\tau)) \right]^{l-2k-1} \right|_{\tau=0} =$$
$$= \frac{l!}{(l-2k-1)!} \sum_{m=1}^{l-2k-1} \left[i^{2n+1} erfc(-\upsilon_0) - i^{2n+1} erfc(\upsilon_0) \right]^{(m)} \times \quad (18)$$
$$\times \sum_{b_i} \frac{(l-2k-1)! \upsilon_1^{b_1} \upsilon_2^{b_2} ... \upsilon_{l-2n}^{b_{l-2n}}}{b_1! b_2! ... b_{l-2k}!}.$$

From system of equations (11) and (13) we determine the coefficients C_n, D_n. Multiplying both sides of (13) by $\beta(\tau)$ we have

$$\sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \frac{2^{2k} n! \Gamma\left(\frac{3}{2}+n\right) \beta(\tau)^{2(n-k)+1} \tau^{2k}}{k!(n-k)! \Gamma\left(\frac{3}{2}+n-k\right)} +$$
$$+ \sum_{n=0}^{\infty} D_n (2a_2\tau)^{2n+1} \left(i^{2n+1} erfc(-\upsilon(\tau)) - i^{2n+1} erfc(\upsilon(\tau))\right) =$$
$$= \theta_m \beta(\tau),$$

Taking l-th derivative both sides of this expression and using (10) we have

$$D_n = \frac{(2n+1)! \left[\theta_m \beta_l l! (2n)! - f^{(2n)}(0) \delta_{n,l} \right]}{2(2n)! \xi_{n,l}}, \quad (19)$$

$$C_n = \frac{f^{(2n)}(0)}{(2n)!} - \\ -\frac{2}{(2n+1)!}\frac{(2n+1)!\left[\theta_m \beta_l l!(2n)! - f^{(2n)}(0)\delta_{n,l}\right]}{2(2n)!\xi_{n,l}}, \quad (20)$$

where

$$\delta_{n,l} = \frac{2^{2k} n!\Gamma\left(\frac{3}{2}+n\right)}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}\frac{l!}{(l-2k)!}\times \\ \times\sum_{m=1}^{l-2n}\beta_0^{(m)}\sum_{b_i}\frac{(l-2k-1)!\beta_1^{b_1}\beta_2^{b_2}...\beta_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!},$$

$$\xi_{n,l} = \frac{(2a_1)^{2n+1} l!}{(l-2k-1)!}\sum_{m=1}^{l-2k-1}\left[i^{2n+1}erfc(-\upsilon_0) - i^{2n+1}erfc(\upsilon_0)\right]^{(m)}\times \\ \times\sum_{b_i}\frac{(l-2k-1)!\upsilon_1^{b_1}\upsilon_2^{b_2}...\upsilon_{l-2n}^{b_{l-2n}}}{b_1!b_2!...b_{l-2k}!}.$$

Multiplying $\beta(\tau)$ both sides of (12) and (14) we have

$$\sum_{n=0}^{\infty}A_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)\beta(\tau)^{2(n-k)+1}\tau^{2k}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)} + \\ +\sum_{n=0}^{\infty}B_n(2a_1\tau)^{2n+1}\left(i^{2n+1}erfc(-\upsilon(\tau)) - i^{2n+1}erfc(\upsilon(\tau))\right) = \\ = \theta_m\beta(\tau), \quad (21)$$

$$-\lambda_1\left[\sum_{n=0}^{\infty}A_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)\beta(\tau)^{2(n-k)}\tau^{2k}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)} - \sum_{n=0}^{\infty}B_n(2a_1\tau)^{2n}\left(i^{2n}erfc(-\upsilon(\tau)) + i^{2n}erfc\upsilon(\tau)\right)\right] = \\ = -\lambda_2\left[\sum_{n=0}^{\infty}C_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)\beta(\tau)^{2(n-k)}\tau^{2k}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)} - \sum_{n=0}^{\infty}D_n(2a_2\tau)^{2n}\left(i^{2n}erfc(-\upsilon(\tau)) + i^{2n}erfc(\upsilon(\tau))\right)\right] \\ +(\lambda_2 - \lambda_1)\theta_m + \frac{L\gamma}{2}\nu(\beta)_m, \quad (22)$$

where $\beta'(\tau)\beta(\tau) = \frac{1}{2}\frac{d}{dr}\beta^2(\tau)$ and

$$\beta^2(\tau) = \sum_{n=0}^{\infty}\nu(\beta)_m\tau^n, \quad \nu(\beta)_0 = \beta_0^{\ 2},$$

$$\nu(\beta)_m = \frac{1}{m\beta_0}\sum_{k=1}^{m}(3k-m)\beta_k\nu(\beta)_{m-k}, \quad m \ge 1$$

Taking l-th derivative both sides of equations (21) and (22) at $\tau = 0$ we get

$$\sum_{n=0}^{l}A_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}\frac{l!}{(l-2k)!}\sum_{m=1}^{l-2n}\beta_0^{(m)}\sum_{b_i}\frac{(l-2k-1)!\beta_1^{b_1}\beta_2^{b_2}...\beta_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!} \\ +\sum_{n=0}^{l}B_n\frac{(2a_1)^{2n+1}l!}{(l-2k-1)!}\sum_{m=1}^{l-2k-1}\left[i^{2n+1}erfc(-\upsilon_0) - i^{2n+1}erfc(\upsilon_0)\right]^{(m)}\sum_{b_i}\frac{(l-2k-1)!\upsilon_1^{b_1}\upsilon_2^{b_2}...\upsilon_{l-2n}^{b_{l-2n}}}{b_1!b_2!...b_{l-2k}!} = \theta_m\beta_l l! \quad (23)$$

and

$$-\lambda_1\left[2\sum_{n=0}^{l}A_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}\frac{l!}{(l-2k)!}\sum_{m=1}^{l-2n}\beta_0^{(m)}\sum_{b_i}\frac{(l-2k)!\beta_1^{b_1}\beta_2^{b_2}...\beta_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!}-\right.$$
$$\left.-\sum_{n=0}^{l}B_n\frac{(2a_1)^{2n+1}l!}{(l-2n)!}\sum_{m=1}^{l-2n}\left[(-1)^m i^{2n-m}erfc(-\upsilon_0)-i^{2n}erfc(\upsilon_0)\right]^{(m)}\sum_{b_i}\frac{(l-2n)!\upsilon_1^{b_1}\upsilon_2^{b_2}...\upsilon_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!}\right]=$$
$$=-\lambda_2\left[2\sum_{n=0}^{l}C_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}\frac{l!}{(l-2k)!}\sum_{m=1}^{l-2n}\beta_0^{(m)}\sum_{b_i}\frac{(l-2k)!\beta_1^{b_1}\beta_2^{b_2}...\beta_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!}-\right.$$
$$\left.-\sum_{n=0}^{l}D_n\frac{(2a_2)^{2n+1}l!}{(l-2n)!}\sum_{m=1}^{l-2n}\left[(-1)^m i^{2n-m}erfc(-\upsilon_0)-i^{2n}erfc(\upsilon_0)\right]^{(m)}\sum_{b_i}\frac{(l-2n)!\upsilon_1^{b_1}\upsilon_2^{b_2}...\upsilon_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!}\right]+\frac{L\gamma}{2}l!\nu(\beta)_{l+1} \quad (24)$$

From recurrent equations (23) and (24) we can determined the coefficients A_n and B_n as free boundary is known.

$$A_n=\frac{\theta_m\beta_l l!-B_n\eta_{n,l}}{\omega_{n,l}},\ B_n=\frac{\chi_{n,l}+2\lambda_1\frac{\theta_m\beta_l l!\vartheta_{n,l}}{\omega_{n,l}}}{\lambda_1\left[2\frac{\eta_{n,l}}{\omega_{n,l}}\vartheta_{n,l}+\zeta_{n,l}\right]} \quad (25)$$

where

$$\eta_{n,l}=\frac{(2a_1)^{2n+1}l!}{(l-2k-1)!}\sum_{m=1}^{l-2k-1}\left[i^{2n+1}erfc(-\upsilon_0)-i^{2n+1}erfc(\upsilon_0)\right]^{(m)}\times\sum_{b_i}\frac{(l-2k-1)!\upsilon_1^{b_1}\upsilon_2^{b_2}...\upsilon_{l-2n}^{b_{l-2n}}}{b_1!b_2!...b_{l-2k}!},$$

$$\omega_{n,l}=\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}\frac{l!}{(l-2k)!}\times\sum_{m=1}^{l-2n}\beta_0^{(m)}\sum_{b_i}\frac{(l-2k-1)!\beta_1^{b_1}\beta_2^{b_2}...\beta_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!},$$

$$\chi_{n,l}=-\lambda_2\left[2\sum_{n=0}^{l}C_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}\frac{l!}{(l-2k)!}\sum_{m=1}^{l-2n}\beta_0^{(m)}\sum_{b_i}\frac{(l-2k)!\beta_1^{b_1}\beta_2^{b_2}...\beta_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!}-\right.$$
$$\left.-\sum_{n=0}^{l}D_n\frac{(2a_2)^{2n+1}l!}{(l-2n)!}\sum_{m=1}^{l-2n}\left[(-1)^m i^{2n-m}erfc(-\upsilon_0)-i^{2n}erfc(\upsilon_0)\right]^{(m)}\sum_{b_i}\frac{(l-2n)!\upsilon_1^{b_1}\upsilon_2^{b_2}...\upsilon_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!}\right]+\frac{L\gamma}{2}l!\nu(\beta)_{l+1},$$

$$\zeta_{n,l}=\frac{(2a_1)^{2n+1}l!}{(l-2n)!}\sum_{m=1}^{l-2n}\left[(-1)^m i^{2n-m}erfc(-\upsilon_0)-i^{2n}erfc(\upsilon_0)\right]^{(m)}\times\sum_{b_i}\frac{(l-2n)!\upsilon_1^{b_1}\upsilon_2^{b_2}...\upsilon_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!}.$$

From condition at heat flux entering we have the following equation

$$-\lambda_1\left[\sum_{n=0}^{\infty}B_n\left(-\frac{(2a_1\tau)^{2n+1}}{\alpha^2(\tau)}\left(i^{2n+1}erfc(-\varphi(\tau))-i^{2n+1}erfc(\varphi(\tau))\right)-\frac{(2a_1\tau)^{2n}}{\alpha(\tau)}\left(i^{2n}erfc(-\varphi(\tau)+i^{2n}erfc(\varphi(\tau))\right)\right)+\right.$$
$$\left.+\sum_{n=0}^{l}A_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}\alpha(\tau)^{2(n-k)-1}\tau^{2n-1}\right]=\sum_{n=0}^{\infty}p_n\tau^n \tag{26}$$

Multiplying both sides by $\alpha^2(\tau)$ we obtain the next equation

$$-\lambda_1\left[\sum_{n=0}^{\infty}B_n\left(-(2a_1\tau)^{2n+1}\left(i^{2n+1}erfc(-\varphi(\tau))-i^{2n+1}erfc(\varphi(\tau))\right)-(2a_1\tau)^{2n}\alpha(\tau)\left(i^{2n}erfc(-\varphi(\tau)+i^{2n}erfc(\varphi(\tau))\right)\right)+\right.$$
$$\left.+\sum_{n=0}^{l}A_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}\alpha(\tau)^{2(n-k)+1}\tau^{2n-1}\right]=\sum_{n=0}^{\infty}p_n\tau^n u^2(\tau) \tag{27}$$

where $\varphi(\tau)=\dfrac{\alpha_0+\alpha_1\tau+\alpha_2\tau^2+...}{2a_1}=\dfrac{1}{2a_1}\sum_{n=0}^{\infty}\varphi_n\tau^n$ and

$$\alpha^2(\tau)=\sum_{n=0}^{\infty}u(\alpha)_m\tau^n,\quad u(\alpha)_0=\alpha_0^{\ 2},\quad u(\alpha)_m=\frac{1}{m\alpha_0}\sum_{k=1}^{m}(3k-m)\alpha_k u(\alpha)_{m-k},\ \ m\geq 1\,.$$

Analogously, taking l-th derivative of both sides of equation (27) we have

$$-\lambda_1\left[\sum_{n=0}^{l}B_n\left\{-\frac{(2a_1)^{2n+1}l!}{(l-2n-1)!}\sum_{m=1}^{l-2n-1}\left((-1)^m i^{2n+1-m}erfc(-\varphi_0)-i^{2n+1-m}erfc(\varphi_0)\right)\sum_{b_i}\frac{(l-2n)!\varphi_1^{b_1}\varphi_2^{b_2}...\varphi_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!}-\right.\right.$$
$$-\frac{(2a_1)^{2n}l!}{(l-2n)!}\sum_{m=1}^{l-2n}\binom{l-2n}{m}[\alpha_0]^{(m)}\sum_{b_i}\frac{(l-2n)!\alpha_1^{b_1}\alpha_2^{b_2}...\alpha_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!}\sum_{p=1}^{l-2n-m}(-1)^p(i^{2n-p}erfc(-\varphi_0)+$$
$$\left.+i^{2n-p}erfc(-\varphi_0))\sum_{b_i}\frac{(l-2n)!\varphi_1^{b_1}\varphi_2^{b_2}...\varphi_{l-2n+1}^{b_{l-2n+1}}}{b_1!b_2!...b_{l-2n+1}!}\right\}-\sum_{n=0}^{l}A_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}\frac{l!}{(l-2n+1)!} \tag{28}$$
$$\left.\cdot(2n-2k+1)!\sum_{m=1}^{l-2n+1}\alpha_0^{4n-2k-l}\sum_{b_i}\frac{(l-2n+1)!\varphi_1^{b_1}\varphi_2^{b_2}...\varphi_{l-2n+2}^{b_{l-2n+2}}}{b_1!b_2!...b_{l-2n+2}!}\right]=\sum_{n=0}^{l}p_n\frac{l!}{2(l-n)!}l!u(\alpha_0)_{l+1}$$

From recurrent equation (28) we can determine the coefficients of heat flux in process of electrical contact materials.

Exact solution of test problem

In this section we consider test problem to check effectiveness of method of radial heat polynomials and integral error functions for inverse problem of spherical Stefan problem (1)-(7). The free boundaries are given in the form $\alpha(t)=\alpha_0\sqrt{t}$ and $\beta(t)=\beta_0\sqrt{t}$, then from the initial condition (3) and boundary condition (6) we have

$$C_n + D_n \frac{2}{(2n+1)!} = \frac{f^{(2n)}(0)}{(2n)!} \tag{29}$$

$$\sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \frac{2^{2k} n!\Gamma\left(\frac{3}{2}+n\right)\beta_0^{2(n-k)} t^n}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)} + \sum_{n=0}^{\infty} D_n \frac{(2a_1)^{2n+1}}{\beta_9}\left(i^{2n+1}erfc\frac{-\beta_0}{2a_1} - i^{2n+1}erfc\frac{\beta_0}{2a_1}\right) = \theta_m, \tag{30}$$

$$\sum_{n=0}^{\infty} A_n \sum_{k=0}^{n} \frac{2^{2k} n!\Gamma\left(\frac{3}{2}+n\right)\beta_0^{2(n-k)} t^n}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)} + \sum_{n=0}^{\infty} B_n \frac{(2a_2)^{2n+1}}{\beta_9}\left(i^{2n+1}erfc\frac{-\beta_0}{2a_2} - i^{2n+1}erfc\frac{\beta_0}{2a_2}\right) = \theta_m, \tag{31}$$

and from Stefan's condition at free boundary $\beta(t)$ we obtain

$$-\lambda_1\left[2\sum_{n=0}^{\infty} A_n \sum_{k=0}^{n} \frac{2^{2k} n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)\beta_0^{2(n-k)}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)} t^n - \sum_{n=0}^{\infty} B_n (2a_1\sqrt{t})^n \left(i^{2n}erfc\frac{-\beta_0}{2a_1} + i^{2n}erfc\frac{\beta_0}{2a_1}\right)\right] =$$
$$= -\lambda_1\left[2\sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \frac{2^{2k} n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)\beta_0^{2(n-k)}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)} t^n - \sum_{n=0}^{\infty} D_n (2a_2\sqrt{t})^n \left(i^{2n}erfc\frac{-\beta_0}{2a_2} + i^{2n}erfc\frac{\beta_0}{2a_2}\right)\right] + \tag{32}$$
$$+(\lambda_2-\lambda_1)\theta_m + \frac{L\gamma}{2}\beta_0^2$$

For $n=0$ from system of equations (28)-(29) we have

$$C_0 = f(0) - \frac{\theta_m - f(0)}{\frac{a_2}{\beta_0}\left(i^1 erfc\frac{-\beta_0}{2a_2} - i^1 erfc\frac{\beta_0}{2a_2}\right) - 1} \tag{33}$$

$$D_0 = \frac{\theta_m - f(0)}{\frac{2a_2}{\beta_0}\left(i^1 erfc\frac{-\beta_0}{2a_2} - i^1 erfc\frac{\beta_0}{2a_2}\right) - 2} \tag{34}$$

and from system of equations (30)-(31) we obtain

$$B_0 = \frac{\lambda_1 D_0\left(i^0 erfc\frac{-\beta_0}{2a_2} + i^0 erfc\frac{\beta_0}{2a_2}\right) + (\lambda_2-\lambda_1)\theta_m + \frac{L\gamma}{2}\beta_0^2}{\lambda_1\left(i^0 erfc\frac{-\beta_0}{2a_1} + i^0 erfc\frac{\beta_0}{2a_1}\right)} \tag{35}$$

$$A_0 = \theta_m - B_0\frac{2a_1}{\beta_0}\left(i^0 erfc\frac{-\beta_0}{2a_2} + i^0 erfc\frac{\beta_0}{2a_2}\right). \tag{36}$$

For $n \geq 1$ we have the following results

$$D_n = \frac{-\dfrac{f^{(2n)}(0)}{(2n)!}\displaystyle\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)\beta_0^{2(n-k)}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}}{\dfrac{(2a_2)^{2n+1}}{\beta_0}\left(i^{2n+1}erfc\dfrac{-\beta_0}{2a_2}-i^{2n+1}erfc\dfrac{\beta_0}{2a_2}\right)-\dfrac{2}{(2n+1)!}\displaystyle\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)\beta_0^{2(n-k)}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}} \quad (37)$$

Using this result and put in (29) we can find coefficient C_n directly. And for other coefficients we get

$$B_n = \frac{\lambda_2\left[D_n(2a_2)^{2n}\left(i^{2n}erfc\dfrac{-\beta_0}{2a_2}+i^{2n}erfc\dfrac{-\beta_0}{2a_2}\right)-C_n\displaystyle\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)\beta_0^{2(n-k)}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}\right]}{\lambda_1\left[\dfrac{\dfrac{2(2a_1)^{2n+1}}{\beta_0}\left(i^{2n+1}erfc\dfrac{-\beta_0}{2a_1}-i^{2n}erfc\dfrac{\beta_0}{2a_1}\right)}{\displaystyle\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)\beta_0^{2(n-k)}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}}\displaystyle\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)\beta_0^{2(n-k)}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}-(2a_1)^{2n}\left(i^{2n}erfc\dfrac{-\beta_0}{2a_1}+i^{2n}erfc\dfrac{-\beta_0}{2a_1}\right)\right]} \quad (38)$$

and

$$A_n = -B_n\frac{\dfrac{(2a_1)^{2n+1}}{\beta_0}\left(i^{2n+1}erfc\dfrac{-\beta_0}{2a_1}-i^{2n+1}erfc\dfrac{-\beta_0}{2a_1}\right)}{\displaystyle\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)\beta_0^{2(n-k)}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}} \quad (39)$$

Heat flux can be determined form condition (3) which takes the form

$$-\lambda_1\left[\sum_{n=0}^{\infty}A_n\sum_{k=0}^{n}\frac{2^{2k}n!\Gamma\left(\frac{3}{2}+n\right)2(n-k)\alpha_0^{2(n-k)-1}t^{n-\frac{1}{2}}}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}-\sum_{n=0}^{\infty}B_n\left(\frac{(2a_1)^{2n+1}t^{n-\frac{1}{2}}}{\alpha_0^{2}}\left(i^{2n+1}erfc\frac{-\alpha_0}{2a_1}-i^{2n+1}erfc\frac{\alpha_0}{2a_1}\right)-\right.\right.$$
$$\left.\left.+\frac{(2a_1)^{2n}t^{n-\frac{1}{2}}}{\alpha_{00}}\left(i^{2n}erfc\frac{-\alpha_0}{2a_1}+i^{2n}erfc\frac{\alpha_0}{2a_1}\right)\right)\right]=\sum_{n=0}^{\infty}p_n t^{\frac{n}{2}} \quad (40)$$

Then from expression (40) we obtain the coefficients of heat flux passes through liquid and solid phases

$$\begin{cases} p_1 = -\lambda_1\left[A_1 2\alpha_0 - B_1\left(\frac{(2a_1)^3}{\alpha_0^2}\left(i^3 erfc\frac{-\alpha_0}{2a_1} - i^3 erfc\frac{\alpha_0}{2a_1}\right) + \frac{(2a_1)^2}{\alpha_0}\left(i^2 erfc\frac{-\alpha_0}{2a_1} + i^2 erfc\frac{\alpha_0}{2a_1}\right)\right)\right] \\ p_3 = -\lambda_1\left[A_2(4\alpha_0^3 + 40\alpha_0) - B_2\left(\frac{(2a_1)^5}{\alpha_0^2}\left(i^5 erfc\frac{-\alpha_0}{2a_1} - i^5 erfc\frac{\alpha_0}{2a_1}\right) + \frac{(2a_1)^4}{\alpha_0}\left(i^4 erfc\frac{-\alpha_0}{2a_1} + i^4 erfc\frac{\alpha_0}{2a_1}\right)\right)\right] \\ p_5 = -\lambda_1\left[A_3(6\alpha_0^5 + 168\alpha_0^3 + 840\alpha_0) - B_3\left(\frac{(2a_1)^7}{\alpha_0^2}\left(i^7 erfc\frac{-\alpha_0}{2a_1} - i^7 erfc\frac{\alpha_0}{2a_1}\right) + \frac{(2a_1)^6}{\alpha_0}\left(i^6 erfc\frac{-\alpha_0}{2a_1} + i^6 erfc\frac{\alpha_0}{2a_1}\right)\right)\right] \\ \vdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_{2n+1} = -\lambda_1\left[A_{n+1}\sum_{k=0}^{n+1}\frac{2^{2k}(n+1)!\Gamma\left(\frac{5}{2}+n\right)2(n-k+1)\alpha_0^{2(n-k)+1}}{k!(n-k+1)!\Gamma\left(\frac{5}{2}+n-k\right)} - B_{n+1}\left(\frac{(2a_1)^{2n+3}}{\alpha_0^2}\left(i^{2n+3} erfc\frac{-\alpha_0}{2a_1} - i^{2n+3} erfc\frac{\alpha_0}{2a_1}\right) + \right.\right. \\ \left.\left. + \frac{(2a_1)^{2n+2}}{\alpha_0}\left(i^{2n+2} erfc\frac{-\alpha_0}{2a_1} + i^{2n+2} erfc\frac{\alpha_0}{2a_1}\right)\right)\right] \end{cases} \quad (41)$$

and even indexed coefficients of heat flux $p_{2n} = 0$. By using Mathcad 15 and taking $a_1 = a_2 = L = \gamma = \alpha_0 = \beta_0 = \lambda_1 = \lambda_2 = 1$ and melting temperature θ_m we get exact values of first three coefficients of temperature in two phase coefficients $A_1 = B_1 = C_1 = D_1 = C_0 = D_0 = 0$ and $A_2 = C_2 = -1.574\times10^{-4}$, $B_2 = D_2 = 9.442\times10^{-3}$ are calculated from system of equations (33)-(39). Then first three coefficients of heat flux is $p_0 = p_1 = 0$ and $p_2 = 0.057$ which can be found from (41).

Approximate solution of test problem

In this section we consider collocation method that useful to engineers for testing and we try to show that by using three points $t = 0$, $t = 0.5$ and $t = 1$ we can obtain no error estimates. Let $a_1 = a_2 = L = \gamma = 1$ and $\theta_m = 0$, then for calculation Mathcad 15 is used and we get the next approximate coefficients for temperature in liquid and solid zones $A_0 = -0.25$, $B_0 = 0.125$, $A_1 = B_1 = C_1 = D_1 = C_0 = D_0 = 0$ and $A_2 = C_2 = -1.574 \times 10^{-4}$, $B_2 = D_2 = 9.442 \times 10^{-3}$. Then approximate values of first three heat flux is similar to exact values. The Fig.1 shows the graphs of approximate heat flux (approx_P(t)) and exact heat flux (exact_P(t)).

By calculating relative error with Mathcad 15 we get Fig.2 in which we can see that that at each point $t = 0$, $t = 0.5$, $t = 1$ we have zero error estimate function (Err(t))

Then we can summarize that method radial heat polynomials and integral error functions is the most effective in the heat transfer problem appearing in electrical contact process.

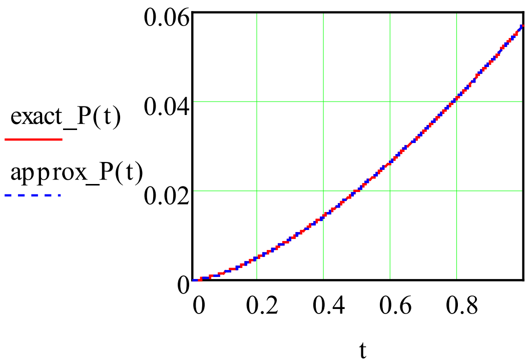

Figure 1 – Graphs of approximate and exact heat flux functions

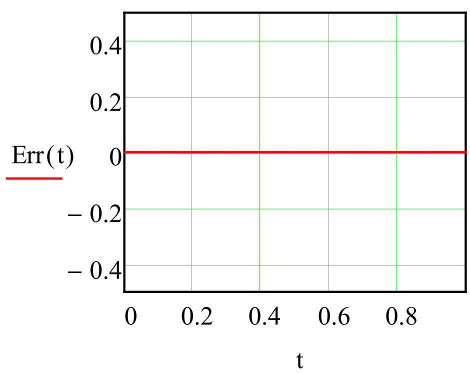

Figure 2 – Graph of relative error function

Conclusion

The new method radial heat polynomials is introduced and is used for testing heat process in two phases when heat flux passes through these two zones. The coefficients of temperatures $\theta_1(r,t)$ and $\theta_2(r,t)$ are determined from recurrent formulas (19), (20) and (25), then by using these coefficients and comparing degree of time from condition (3) heat flux is described. To testing effectiveness of radial heat polynomials and integral error function test problem is considered in which free boundaries are represented in self-similar form $\alpha(t)=\alpha_0\sqrt{t}$ and $\beta(t)=\beta_0\sqrt{t}$ which are convenient for testing and with approximation method (collocation method) checked the error estimates between exact solution and approximate solution of this inverse problem.

Acknowledgement

The research work is supported by the grant project AP05133919 (2018-2020) from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1 R. Holm, Electrical Contacts, Fourth Edition, Springer Verlag (198): 482.

2 V.I. Kudrya, A.E. Pudy, S.N. Kharin. Fundamental solution and heat potentials of heat equation for a rod with a variable cross-section. Equations with discontinuous coefficients and their applications, Nauka (1985): 76-81.

3 S.N.Kharin, M.M.Sarsengeldin, H.Nouri. Analytical solution of two phase spherical Stefan problem by heat polynomials and integral error functions. AIP Conference Proceedings 1759, 020031, (2016), doi: http://dx.doi.org/10.1063/1.4959645

4 Alexey A. Kavokin, Targyn A. Nauryz, Nazerke T. Bizhigitova, "Exact solution of two phase spherical Stefan problem with free boundaries", AIP Conference Proceedings 1759, 020117 (2016), https://doi.org/10.1063/1.495931

5 M. M. Sarsengeldin, A. S. Erdogan, T.A. Nauryz, H. Nouri, "An approach for solving an inverse spherical Stefan problem arising in modeling of electrical contact phenomena", *Mathematical Methods in the Applied Sciences* vol. 41, No. 2 (2017): https://doi.org/10.1007/978-3-319-67053-937.

6 M. Sarsengeldin, S. N. Kharin, "Method of the Integral Error Functions for the solution of one- and two-phase Stefan problems and its application", Filomat 31:4 (2017): 1017-1029. DOI 10.2298/FIL1704017S. Available at: http://www.pmf.ni.ac.rs/filomat.

7 Merey M. Sarsengeldin, Targyn A. Nauryz, Nazerke T. Bizhigitova, Alibek M. Orinbasar. Solution of an inverse two phase spherical Stefan test problem. Accepted for publication. Proceedings in Mathematics and Statistics, Springer book series, volume: Functional Analysis in Interdisciplinary Applications (2017).