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FINITE DOMAIN STRUCTURES IN THE FRAMEWORK OF THE CONCEPT OF A MODEL-THEORETIC PROPERTY

Abstract. In this work, we follow the algebraic approach using definability by formulas presentable in both existential and universal forms. The class of algebraic Cartesian interpretations of theories is studied presenting a foundation of the finitary first-order combinatorics. Common properties of first-order definability in finite models are studied. Some relations are obtained between automorphism groups of finite models and isomorphisms of Cartesian extensions of their theories. A formal definition of the notion of a model-theoretic property is analyzed based on a separate consideration of cases of theories with finite and infinite models. A description of model-theoretic properties defined via finite domains is found. It is established that the class of all finite models with first-order definable elements as well as the corresponding class of theories of such models forms the only model-theoretic property and, therefore, is of little interest as a database with an interface based on the first-order logic language.
Key words: first-order logic, Cartesian extension of a theory, Tarski-Lindenbaum algebra, model-theoretic property, computable isomorphism.

Introduction

We use the radical approach in model theory counting that *model-theoretic properties* are classes of complete theories, cf. [1]. By specification [2], a class $\mathfrak{p}$ of complete theories is a *real model-theoretic property* (corresponding to the common practice of investigations in model theory), if $\mathfrak{p}$ is closed under algebraic isomorphisms of theories as well as under Cartesian extensions and inverse passages in the operation of a Cartesian extension of a theory. A preliminary motivation to the possibility of a formal definition for the concept of a model-theoretic property is considered in [3], while the work [2] describes a final version of this definition. Some applications of the definition of a model-theoretic property are contained in [4].

In this work, structure of real model-theoretic properties is studied based on a separate consideration of the cases of complete theories with finite and infinite models. Based on this, we give an application concerning finite models.

Preliminaries

We consider theories in first-order predicate logic *with equality* and use general concepts of model theory, algorithm theory, constructive models, and Boolean algebras found in [5], [6], and [7]. Special concepts used in the works are defined in [3]. Generally, *incomplete theories* are considered. In the work, the signatures are considered only, which admit Godel's numberings of the formulas. Such a signature is called *enumerable*.

By $L(T)$, we denote the *Tarski-Lindenbaum algebra* of formulas of theory T without free variables, while $\mathcal{L}(T)$ denotes the Tarski-Lindenbaum algebra $L(T)$ considered together with a *Gödel numbering* γ; thereby, the concept of a *computable isomorphism* is applicable to such objects. A finite signature is called *rich*, if it contains at least one n-ary predicate or function symbol for $n \geqslant 2$, or two unary function symbols. By $\mathbb{C}$, we denote the class of all complete theories of enumerable signatures. The record $T \approx S$ means *isomorphism* of theories T and S, while $T \approx_a S$ stands for *algebraic isomorphism* of the theories, cf. [3].

We follow the *algebraic type of definability* using $\exists\cap\forall$-formulas affecting more delicate properties of theories in comparison with the normal approach based on the definability via arbitrary first-order formulas. As an $\exists\cap\forall$-formula $\varphi(\bar{x})$ of signature σ, we mean a pair of formulas $\left(\varphi^e(\bar{x}), \varphi^a(\bar{x})\right)$ together with the *domain sentence* $DomEA\left(\varphi(\bar{x})\right) = (\forall\bar{x})[\varphi^e(\bar{x}) \leftrightarrow \varphi^a(\bar{x})]$, where $\varphi^e(\bar{x})$ is an $\exists$-formula, while $\varphi^a(\bar{x})$ is a $\forall$-formula of signature σ. A formula $\varphi(\bar{x})$ of theory T is said to be $\exists\cap\forall$-*presentable* in T if $T \vdash DomEA\left(\varphi(\bar{x})\right)$. If $\psi(\bar{x})$ is a quantifier-free formula, $DomEA(\psi(\bar{x}))$ is supposed to be a generally true formula. If $\varkappa$ is a finite set (or a sequence) of $\exists\cap\forall$-formulas $\psi_i(\bar{x}_i)$, $i < k$, we denote by $DomEA(\varkappa)$ the conjunction $\wedge_{i<k} DomEA(\psi_i(\bar{x}_i))$.

We formulate a technical statement.

Lemma 0.1. [8, Lemma 2.4.2] *Let $\mathfrak{M}$ be a finite model of an enumerable signature σ. Then, any formula $\varphi(x_1, \dots, x_n)$ of signature σ is equivalent in theory $T = Th(\mathfrak{M})$ to an $\exists\cap\forall$-formula of signature σ.*

Proof. By condition, theory T has a unique up to an isomorphism model $\mathfrak{M}$; moreover, $\mathfrak{M}$ is finite. Therefore, any isomorphic embedding of models of theory T is elementary. By Robinson's Criterion, [9], we obtain that theory T is model complete. Hence, we have the $\exists$-reducibility as well as $\forall$-reducibility of any formula in theory T. □

Cartesian-type interpretations

We use a standard concept of an *interpretation* of a theory T_0 in the region $U(x)$ of a theory T_1, [10, Section 4.7]. An interpretation is called *effective* if it is defined by a computable function. Classes of *isostone* and *model-bijective* interpretations are introduced in [11]. In this section, we introduce a technical class of interpretations presenting finitary methods in first-order logic.

Given a signature σ and a finite sequence of formulas of this signature of either of the following forms:

$$\text{(a)}\ \varkappa = \langle \varphi_1^{m_1} / \varepsilon_1, \varphi_2^{m_2} / \varepsilon_2, \dots, \varphi_s^{m_s} / \varepsilon_s \rangle, \quad (1.1)$$

$$\text{(b)}\ \varkappa = \langle \varphi_1^{m_1}, \varphi_2^{m_2}, \dots, \varphi_s^{m_s} \rangle,$$

where $\varphi_k(\bar{x}_k)$ is a formula with m_k free variables, $\varepsilon_k(\bar{y}_k, \bar{z}_k)$ is a formula with $2m_k$ free variables such that $Len(\bar{y}_k) = Len(\bar{z}_k) = m_k$; moreover, (1.1)(b) is a simplified notation instead of the common entry (1.1)(a) in the case when $\varepsilon_k(\bar{y}_k, \bar{z}_k)$ coincides with $\bar{y}_k = \bar{z}_k$ for all $k \leqslant s$.

Starting from a model $\mathfrak{M}$ of signature σ together with a tuple $\varkappa$ of any of the forms (1.1)(a,b), we are going to construct a new model $\mathfrak{M}_1$ of signature

$$\sigma_1 = \sigma \cup \{U^1, U_1^1, U_2^1, \dots, U_s^1\} \cup$$
$$\{K_1^{m_1+1}, K_2^{m_2+1}, \dots, K_s^{m_s+1}\} \quad (1.2)$$

as follows. As the universe, we take $|\mathfrak{M}_1| = |\mathfrak{M}| \cup A_1 \cup A_2 \cup \dots \cup A_s$, where all specified parts are pairwise disjoint sets. On the set $|\mathfrak{M}|$, all symbols of signature σ are defined exactly as they were defined in $\mathfrak{M}$; in the remainder, they are defined trivially; predicate $U(x)$ distinguishes $|\mathfrak{M}|$; predicate $U_k(x)$ distinguishes A_k; the other predicates are defined by specific rules depending on the case. In the case (1.1)(b), each predicate $K_k(\bar{x}_k, u)$ in (1.2) should be defined so that it would represent a one-to-one correspondence between the set of tuples $\{\bar{a} \mid \mathfrak{M} \vDash \varphi_k(\bar{a})\}$ and the set $A_k = U_k(\mathfrak{M}_1)$. Turn to the most common case (1.1)(a). Denote by $Equiv(\varepsilon_k, \varphi_k)$ a sentence stating that ε_k is an equivalence relation on the set of tuples distinguished by the formula $\varphi_k(\bar{x}_k)$ in $\mathfrak{M}$. In this case, $(m_k + 1)$-ary predicate $K_k(\bar{x}_k, u)$ should be defined so that it would represent a one-to-one correspondence between the quotient set $\{\bar{a} \mid \mathfrak{M} \vDash \varphi_k(\bar{a})\} / \varepsilon'_k$ and the set $U_k(\mathfrak{M}_1)$, where

$$\varepsilon'_k(\bar{y}_k, \bar{z}_k) = \varepsilon_k(\bar{y}_k, \bar{z}_k) \vee \neg Equiv(\varepsilon_k, \varphi_k). \quad (1.3)$$

The model $\mathfrak{M}_1$ obtained from $\mathfrak{M}$ and $\varkappa$ as explained above is denoted by $\mathfrak{M}\langle\varkappa\rangle$.

The aim of replacement of ε_k by ε'_k using $Equiv(\varepsilon_k, \varphi_k)$ is to provide the total definiteness of the operation $(\mathfrak{M}, \varkappa) \mapsto \mathfrak{M}\langle\varkappa\rangle$ independently of whether the formulas ε_k, $k = 1,2, \dots, s$, represent equivalence relations in corresponding domains or not. In the case (1.1)(a), $\mathfrak{M}\langle\varkappa\rangle$ is said to be a *Cartesian-quotient extension* of $\mathfrak{M}$, while in the case (1.1)(b), the model $\mathfrak{M}\langle\varkappa\rangle$ is said to be a *Cartesian extension* of $\mathfrak{M}$ *by a sequence of formulas* $\varkappa$.

Mention some kind of determinism for the operation under consideration.

Lemma 1.1. *Given a signature σ and a tuple $\varkappa$ of the form* (1.1)(a)*. For a fixed choice of signature* (1.2)*, Cartesian-quotient extension $\mathfrak{M}_1 = \mathfrak{M}\langle\varkappa\rangle$ of the model $\mathfrak{M}$ is defined uniquely, up to an isomorphism over $\mathfrak{M}$. Moreover, we have $|\mathfrak{M}_1| = acl(U(\mathfrak{M}_1))$. Thus, any automorphism $\lambda: \mathfrak{M} \to \mathfrak{M}$ can be extended, by a unique way, up to an automorphism $\lambda^*: \mathfrak{M}\langle\varkappa\rangle \to \mathfrak{M}\langle\varkappa\rangle$.*

Proof. This statement is an immediate consequence of the construction. □

Expand the operation of an extension (initially defined for models) on theories. Given a theory T and a tuple $\varkappa$ of the form (1.1). Using a fixed signature (1.2) for extensions of models, we define a new theory $T' = T\langle\varkappa\rangle$ as follows: $T' = Th(K)$, $K = \{\mathfrak{M}\langle\varkappa\rangle \mid \mathfrak{M} \in Mod(T)\}$. In the case (1.1)(a) it is called a *Cartesian-quotient extension* of T, while in the case (1.1)(b) it is called a *Cartesian extension* of T *by a sequence* $\varkappa$.

We study simple properties of Cartesian-type extensions.

Lemma 1.2. *For any model* $\mathfrak{M}$ *of theory* $T\langle\varkappa\rangle$*, there is a model* $\mathfrak{N}$ *of theory* T *such that an isomorphism* $\mathfrak{M} \cong \mathfrak{N}\langle\varkappa\rangle$ *takes place.*

Proof. Immediately, from definition of the operation $T \mapsto T\langle\varkappa\rangle$. □

In theory $T\langle\varkappa\rangle$, the region $U(x)$ represents a model of theory T. Particularly, the transformation $T \mapsto T\langle\varkappa\rangle$ defines a natural interpretation $I_{T,\varkappa}$ of T in $T\langle\varkappa\rangle$. It is called a *plain Cartesian-quotient interpretation*. Similar definition applies to the other case of the tuple $\varkappa$; thereby, the concept of a *plain Cartesian interpretation* is also defined. Considering theories up to an algebraic isomorphism, we may use shorter terms *Cartesian-quotient* or, respectively, *Cartesian* interpretation, for details, cf. [12].

Lemma 1.3. *Given a theory* T *of an enumerable signature* σ *and a sequence of formulas* $\varkappa$*. The plain Cartesian-quotient interpretation* $I_{T,\varkappa}: T \rightarrowtail T\langle\varkappa\rangle$ *is effective, model-bijective, and isostone. In particular, the interpretation* $I_{T,\varkappa}$ *determines a computable isomorphism* $\mu_{T,\varkappa}$: $\mathcal{L}(T) \longrightarrow \mathcal{L}(T\langle\varkappa\rangle)$ *between the Tarski-Lindenbaum algebras.*

Proof. Immediately. □

Normally, we consider passages $T \mapsto T\langle\varkappa\rangle$ with a sequence (1.1) satisfying the following technical condition:

$$\begin{array}{c} formulas\ \varphi_k(\bar{x}_k)\ and\ \varepsilon_k(\bar{y}_k, \bar{z}_k)\ are \\ \exists\cap\forall\text{-}presentable,\ for\ all\ k \leqslant s. \end{array} \tag{1.4}$$

Denote by $KD(\sigma)$ and $KC(\sigma)$ the sets of tuples of formulas of signature σ of the forms, respectively, (1.1)(a) and (1.1)(b), while KD and KC are unions of these sets for all possible enumerable signatures σ. We denote by $KC_{\exists\cap\forall}$ the set of all tuples (1.1)(b) satisfying (1.4), while $KD^{\varepsilon}_{\exists\cap\forall}$ is the set of all tuples (1.1)(a) satisfying (1.4). When using an entry $T\langle\varkappa\rangle$, we always suppose that theory T is applicable to the tuple $\varkappa$; moreover, it is supposed that $T \vdash DomEA(\varkappa)$ whenever $\varkappa \in KC_{\exists\cap\forall}$ or $\varkappa \in KD^{\varepsilon}_{\exists\cap\forall}$.

By applying an extra term *algebraic*, we explicitly indicate that the algebraic approach is accepted. For instance, a passage $T \mapsto T\langle\varkappa\rangle$ is said to be an *algebraic Cartesian-quotient extension* whenever $\varkappa \in KD^{\varepsilon}_{\exists\cap\forall}$, an interpretation $I_{T,\varkappa}$ is said to be a *plain algebraic Cartesian interpretation* if $\varkappa \in KC_{\exists\cap\forall}$, etc.

We consider combinatorial properties of Cartesian-type extensions.

Lemma 1.4. *Given a theory* T *of an enumerable signature* σ *together with a sequence of formulas* $\varkappa$*. The following statements are satisfied, where all indicated passages are supposed to be effective with respect to Gödel's numbers of tuples of formulas; moreover, the choice of tuples is limited by the condition of applicability to corresponding theories:*

(a) *Suppose that* $\varkappa \in KD$*. For any* $\varkappa'$ *in* KD*, there is a tuple* $\varkappa''$ *in* KD *such that an isomorphism*

$$T\langle\varkappa\hat{\ }\varkappa'\rangle \approx (T\langle\varkappa\rangle)\langle\varkappa''\rangle \tag{1.5}$$

takes place; and vice versa, for any $\varkappa''$ *in* KD*, there is a tuple* $\varkappa'$ *in* KD *such that an isomorphism* (1.5) *takes place.*

(b) *Suppose that* $\varkappa \in KC$*. For any* $\varkappa'$ *in* KC*, there is a tuple* $\varkappa''$ *in* KC *such that an isomorphism* (1.5) *takes place; and vice versa, for any* $\varkappa''$ *in* KC*, there is a tuple* $\varkappa'$ *in* KC *such that an isomorphism* (1.5) *takes place.*

(c) *Suppose that* $\varkappa \in KC_{\exists\cap\forall}$*. For any* $\varkappa'$ *in* $KC_{\exists\cap\forall}$*, there is a tuple* $\varkappa''$ *in* $KC_{\exists\cap\forall}$ *such that an isomorphism*

$$T\langle\varkappa\hat{\ }\varkappa'\rangle \approx_a (T\langle\varkappa\rangle)\langle\varkappa''\rangle \tag{1.6}$$

takes place; and vice versa, for any $\varkappa''$ *in* $KC_{\exists\cap\forall}$*, there is a tuple* $\varkappa'$ *in* $KC_{\exists\cap\forall}$ *such that an isomorphism* (1.6) *takes place.*

Proof. Validity of these statements can be checked by applying a routine construction based on expressive possibilities of first-order logic. □

Introduce notations for two following relations on the class of arbitrary theories including both complete and incomplete ones:

$$\begin{array}{c} \text{(a)}\ \ T \cong_a S \Leftrightarrow_{dfn} \\ (\exists\varkappa'\varkappa'' \in KC_{\exists\cap\forall})[T\langle\varkappa'\rangle \approx_a S\langle\varkappa''\rangle], \end{array} \tag{1.7}$$

$$\begin{array}{c} \text{(b)}\ \ T \cong_a^{\circ} S \Leftrightarrow_{dfn} \\ \big(\exists\ computable\ isomorphism\ \mu: \mathcal{L}(T) \longrightarrow \mathcal{L}(S)\big) \\ \\ (\forall\ complete\ extension\ T' \supseteq T) \\ (\forall\ complete\ extension\ S' \supseteq S) \\ \\ [S' = \mu(T') \Rightarrow (\exists\varkappa'\varkappa'' \in \\ KC_{\exists\cap\forall})(T'\langle\varkappa'\rangle \approx_a S'\langle\varkappa''\rangle)]. \end{array}$$

Lemma 1.5. *The relation* (1.7)(a) *on the class of theories of enumerable signatures is reflexive, symmetric, and transitive (that is, this is an* *equivalence relation). Besides,* (1.7)(b) *is also an equivalence relation on the class of all theories. Moreover, we have* $T \approxeq_a S \Rightarrow T \approxeq_a^\circ S$ *for all theories* T *and* S, *and* $T_1 \approxeq_a T_2 \Leftrightarrow T_1 \approxeq_a^\circ T_2$ *for all complete theories* T_1 *and* T_2.

Proof. Obviously, $\approxeq_a$ is reflexive and symmetric. Now, suppose that $T \approxeq_a H$ and $H \approxeq_a S$ is satisfied. By definition, there are tuples $\xi_i \in KC_{\exists\cap\forall}$, $i = 1,2,3,4$, such that $T\langle\xi_1\rangle \approx_a H\langle\xi_2\rangle$ and $H\langle\xi_3\rangle \approx_a S\langle\xi_4\rangle$. By applying Lemma 1.4(c), we can find tuples ξ'_2 and ξ'_3 in $KC_{\exists\cap\forall}$ such that the following algebraic isomorphisms take place: $T\langle\xi_1{}^\wedge\xi'_3\rangle \approx_a H\langle\xi_2{}^\wedge\xi_3\rangle \approx_a H\langle\xi_3{}^\wedge\xi_2\rangle \approx_a S\langle\xi_4{}^\wedge\xi'_2\rangle$. Thus, we obtain $T \approxeq_a S$ ensuring the transitivity property. The fact that relation (1.7)(b) is reflexive, symmetric, and transitive on the class of all theories is checked immediately. As for the pointed out links between the relations $\approxeq_a$ and $\approxeq_a^\circ$, they are derived based on definitions (1.7)(a) and (1.7)(b) together with properties of the computable isomorphisms μ in Lemma 1.3. □

There are model-type versions $\approxeq$ and $\approxeq^\circ$ of the relations without index a, thus, discarding the algebraic mode of definability. For this, we have to use common class KC instead of specialized one $KC_{\exists\cap\forall}$ in the rules (1.7)(a) and (1.7)(b).

Formal specification for a model-theoretic property

We use a general specification to the concept of a *real model-theoretic property*, [2]. By accepting the *pragmatic* approach, cf. Definition 4 and Definition 6 in [2], we have for all complete theories T and S:

$$T \text{ and } S \text{ have identical real model-theoretic properties} \Leftrightarrow T \approxeq_a S. \quad (2.1)$$

As for the common rule (1.7)(b), it represents the relation of coincidence of real model-theoretic properties for arbitrary first-order theories (including incomplete ones).

Virtual isomorphisms for finite models

We prove the following fact of a technical character.

Lemma 3.1. [8, Theorem 2.4.4] *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be finite models of enumerable signatures such that an isomorphism* $Aut(\mathfrak{M}) \cong Aut(\mathfrak{N})$ *takes place. Then, we have* $Th(\mathfrak{M}) \approxeq Th(\mathfrak{N})$, *i.e., the following relation is satisfied:* $(\exists\varkappa\varkappa' \in KC)[Th(\mathfrak{M})\langle\varkappa\rangle \approx Th(\mathfrak{N})\langle\varkappa'\rangle]$.

Proof. Consider two finite models $\mathfrak{M}$ and $\mathfrak{N}$ whose automorphism groups are isomorphic. Let $T = Th(\mathfrak{M})$ and $S = Th(\mathfrak{N})$. We assume that the universe sets of the models $|\mathfrak{M}| = \{a_1, \dots, a_m\}$, $|\mathfrak{N}| = \{b_1, \dots, b_n\}$ as well as their signatures τ and σ are disjoint. Fix an isomorphism $q: Aut(\mathfrak{M}) \to Aut(\mathfrak{N})$ and construct a new model $\mathfrak{P}$ of signature $\tau \cup \sigma \cup \{U^1, V^1, R^{m+n}\}$ as follows. We put

$$|\mathfrak{P}| = |\mathfrak{M}| \cup |\mathfrak{N}|,$$

$$U(x) \Leftrightarrow x \in |\mathfrak{M}|, \;\; V(x) \Leftrightarrow x \in |\mathfrak{N}|,$$

$$\tau\text{-}relations\ on\ |\mathfrak{M}|\ are\ the\ same\ as\ in\ \mathfrak{M},\ they\ are\ trivial\ in\ remains,$$

$$\sigma\text{-}relations\ on\ |\mathfrak{N}|\ are\ the\ same\ as\ in\ \mathfrak{N},\ they\ are\ trivial\ in\ remains,$$

$$R = \{\langle\mu(a_1), \dots, \mu(a_m), q\mu(b_1), \dots, q\mu(b_n)\rangle \mid \mu \in Aut(\mathfrak{M})\}.$$

Due to connections via predicate R, any automorphism of the model $\mathfrak{P}$ acts in coordination on both models $\mathfrak{M}$ and $\mathfrak{N}$. In particular, we have $Aut(\mathfrak{M}) \cong Aut(\mathfrak{P}) \cong Aut(\mathfrak{N})$. Moreover, any automorphism λ of $\mathfrak{P}$ is an identical mapping on the whole model $\mathfrak{P}$ whenever it is identical on $|\mathfrak{M}|$. By Beth's Definability Theorem, [5], all elements in $\mathfrak{P}$ are first-order definable over its domain $U(\mathfrak{P})$. Therefore, the natural interpretation of T in $Th(\mathfrak{P})$ is exact. By Lemma 3.2 in [12], the theory $Th(\mathfrak{P})$ is isomorphic to the theory $T\langle\varkappa'\rangle$ for a sequence $\varkappa' \in KD$. Moreover, Lemma 3.3 in [12] is applicable. Thus, we have $\varkappa' \in KC$. A similar reasoning shows that theory $Th(\mathfrak{P})$ is isomorphic to theory $S\langle\varkappa''\rangle$ for a sequence $\varkappa'' \in KC$. □

Model-theoretic properties versus finite/infinite models

In this paragraph, we establish how finite models are related with the concept of a real model-theoretic property introduced in [2].

From the rule (2.1) we obtain that the set of all real model-theoretic properties has the form of a complete Boolean algebra of subsets $\mathcal{P}(\mathbb{C} / \approxeq_a)$. Moreover, separate classes $[T]_{\approxeq_a}$, $T \in \mathbb{C}$, are *atoms* of this Boolean algebra. They are said to be *atomic model-theoretic properties*.

The following presentation takes place.

Lemma 4.1. *An arbitrary class* $\mathfrak{p}$ *of complete theories is a real model-theoretic property if and only if* $\mathfrak{p}$ *is the union of a family of atomic model-theoretic properties.*

Proof. Immediately. □

Lemma 4.2. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be abritrary models of enumerable signatures such that* $(\exists \varkappa\varkappa' \in KD)[Th(\mathfrak{M})\langle\varkappa\rangle \approx Th(\mathfrak{N})\langle\varkappa'\rangle]$. *The following assertions are satisfied :*

(a) $\mathfrak{M}$ *is finite if and only if* $\mathfrak{N}$ *is finite,*

(b) $Aut(\mathfrak{M}) \cong Aut(\mathfrak{N})$.

Proof. These statements are provided by construction of a Cartesian-quotient extension of a model, cf. Lemma 1.1 together with Lemma 1.2. □

Let us present the set $\mathbb{C}$ of all complete theories of enumerable signatures in the form $\mathbb{C} = \mathbb{C}_{\infty} \cup \mathbb{C}_{0}$, where

$$\mathbb{C}_{\infty} = \{T \in \mathbb{C} \mid T \text{ has an infinite model}\},$$

$$\mathbb{C}_{0} = \{T \in \mathbb{C} \mid T \text{ has a finite model}\}.$$

By definition, we have $KC_{\exists\cap\forall} \subseteq KC \subseteq KD$. Therefore, by Lemma 4.2, each of the sets $\mathbb{C}_{\infty}$ and $\mathbb{C}_0$ is closed under the equivalence relation $\approxeq_a$. Thus, any real model-theoretic property $\mathfrak{p} \subseteq \mathbb{C}/\approxeq_a$ can be decomposed into two parts as follows:

$$\mathfrak{p} = \mathfrak{p}' \cup \mathfrak{p}'', \quad \text{where } \mathfrak{p}'' \subseteq \mathbb{C}_{\infty}/_{\approxeq_a} \text{ and } \mathfrak{p}' \subseteq \mathbb{C}_0/\approxeq_a. \tag{4.1}$$

Moreover, decomposition (4.1) is defined uniquely for any given property $\mathfrak{p}$.

A model-theoretic property $\mathfrak{p}$ is said to be *purely infinite* if the part $\mathfrak{p}'$ in decomposition (4.1) is empty. The property $\mathfrak{p}$ is said to be *purely finite* if the part $\mathfrak{p}''$ in (4.1) is empty. Obviously, there are properties $\mathfrak{p}$ for which both parts $\mathfrak{p}'$ and $\mathfrak{p}''$ in (4.1) are nonempty. Purely infinite model-theoretic properties are normally considered in traditional model theory. As for the purely finite model-theoretic properties, no regular view on this concept had been available before the definition of a model-theoretic property in the work [2] was appeared.

Lemma 4.3. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be finite models of enumerable signatures such that* $Aut(\mathfrak{M}) \cong Aut(\mathfrak{N})$. *Then, we have* $(\exists \varkappa\varkappa' \in KC_{\exists\cap\forall})[Th(\mathfrak{M})\langle\varkappa\rangle \approx_a Th(\mathfrak{N})\langle\varkappa'\rangle]$.

Proof. By applying Lemma 3.1 together with Lemma 0.1. □

Theorem 4.4. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be finite models. The theories* $Th(\mathfrak{M})$ *and* $Th(\mathfrak{N})$ *have identical real model-theoretic properties if and only if their automorphism groups* $Aut(\mathfrak{M})$ *and* $Aut(\mathfrak{N})$ *are isomorphic.*

Proof. Part ⇒ is provided by relations (2.1) and (1.7)(a) together with an inclusion $KC_{\exists\cap\forall} \subseteq KD$ and Lemma 4.2. The back implication ⇐ is proved from Lemma 4.3 together with relations (1.7)(a) and (2.1).

The following statement characterizes atomic purely finite model-theoretic properties.

Theorem 4.5. *An arbitrary class* $\mathfrak{p}$ *of complete theories is an atomic purely finite model-theoretic property if and only if the following is satisfied for a finite group* G*:*

$$p = p_G =_{dfn} \{Th(\mathfrak{M}) \mid \mathfrak{M} \in FinMod \;\&\; Aut(\mathfrak{M}) \cong G\},$$

where $FinMod$ *is the class of all finite models of enumerable signatures.*

Proof. By applying Theorem 4.4. □

Conclusion

We used a general specification of the concept of a model-theoretic property introduced in [2]. Based on separate analysis of cases for finite and infinite models, we characterize the structure of real model-theoretic properties.

Statements of Theorem 4.4 and Theorem 4.5 fully characterize the case of model-theoretic properties for complete theories with finite models. It is a simple fact that elements in a finite model with the trivial automorphism group are uniquely defined. Thus, such models as well as their theories can be considered as a basis for constructing abstract databases in applied logic. By Theorem 4.5, all models of this class form the only model-theoretic property; i.e., they are not distinguishable from the point of view of model theory. Thereby, it is possible to conclude that the class of all finite models with unique elements as well as the corresponding class of complete theories is not of interest as a database with an interface based on the first-order logic language.

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