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AN IRREGULAR CONJUGATIONPROBLEM FOR THE SYSTEM OF THE PARABOLICE QUATIONS IN THE HOLDER SPACE

Abstract. We consider the conjugation problem for the system of parabolicequations with two small parameters $\kappa > 0$, $\varepsilon > 0$ in the boundary conditions.There are proved the existence, uniqueness and uniform coercive estimates ofthe solution with respect to the small parameters in the Holder space. Thisproblem is linearized one of the nonlinear problem with the free boundary ofFlorin type and it is in the base of the proof of the solidified of this nonlinearproblem in the Holder space.We study the problem with the free boundary of the Florin type in the Holder space $\overset{\circ}{C}{}_{x\ \ t}^{2+l,1+l/2}(\overline{\Omega}_{jT})$, $j=1,2$, where $l-$ is non-integer positive number. Existence, uniqueness, estimates for solution of the problem with constants independent of small parameters in the Holder space areproved. It gives us the opportunity to establish the existence, uniqueness and estimates of the solution of the problem without loss of smoothness of given functions for $\kappa = 0$, $\varepsilon > 0$; $\kappa > 0$; $\varepsilon = 0$ and $\kappa = 0$, $\varepsilon = 0$.

Key words :parabolic equations, existence, uniqueness of the solution, coercive estimates, Holder space.

1 Introduction

In the work the problem of Florin type is studied for the system of parabolicequations in the Holder spaces. This problem is a mathematical model describingfiltration of liquids and gases in the porous medium. Linear problems with smallparameters with time derivatives functions of the free boundary were studied in [1]-[7]. In this article the problem is studied without time derivatives functionsof the free boundaries $\psi(t)$ in the right-hand sides of the conditions (6), (7), whichcorresponds to a degenerate nonlinear the free boundary problem of melting binaryalloys and in which free boundary is set as an implicit function. In contrast fromproblems in $[1]-[7]$, where free border is set explicitly.

Let$\Omega_1 = (0,\rho_0)$, $\Omega_2 = (\rho_0, b), 0 < \rho_0 < b$, $b > 0$, $\Omega_{jT} = \Omega_j \times (0,T)$, $j = 1,2$, $\sigma_T = (0,T)$, $\chi(\lambda)$ be a smooth shear function, equal to one at$|\lambda| \le \delta_0$andzero for$|\lambda| \ge 2\delta_0$and having the rating$|d^m\chi/dx^m| \le C_m\delta_0^{-m}, \delta_0 = const > 0$.

Define second order elliptic operators

$$A_j(x,t;\partial_x)v_j \coloneqq a_j(x,t)\partial_x^2 v_j + \\ +b_j(x,t)\partial_x v_j + d_j(x,t)v_j,$$

$$A_{j+2}(x,t;\partial_x)z_j \coloneqq a_{j+2}(x,t)\partial_x^2 z_j + \\ +b_{j+2}(x,t)\partial_x z_j + d_{j+2}(x,t)z_j,$$

where $a_j(x,t), a_{j+2}(x,t) \ge a_0 = const > 0$ in $\overline{\Omega}_{jT}$, $j = 1,2$.

It is required to find functions $v_j(x,t)$, $z_j(x,t)$, $j = 1,2$, and $\psi(t)$ satisfyingparabolic equations

$$\partial_t v_j - A_j(x,t;\partial_x)v_j - \alpha_j(x,t)\chi(x-\rho_0)D_t\psi = \\ = f_j(x,t)\text{in}\overline{\Omega}_{jT},\ \ j = 1,2, \tag{1}$$

$$\partial_t z_j - A_{j+2}(x,t;\partial_x)z_j - \beta_j(x,t)\chi(x-\rho_0)D_t\psi = \\ = f_{j+2}(x,t)\text{in}\overline{\Omega}_{jT},\ \ j = 1,2, \tag{2}$$

and initial conditions

$$\psi|_{t=0} = 0,\quad v_j|_{t=0} = v_{oj}, \\ z_j|_{t=0} = z_{oj},\ \text{in}\Omega,\ \ j = 1,2, \tag{3}$$

boundary conditions

$$v_1|_{x=0} = p_1(t),\ v_2|_{x=b} = p_2(t),\quad t \in \sigma_T, \tag{4}$$

$$z_1|_{x=0} = q_1(t),\ z_2|_{x=b} = q_2(t),\quad t \in \sigma_T, \tag{5}$$

and conjugation conditions on the border $x = \rho_0$

$$(v_1 - v_2)|_{x=\rho_0} = \eta_0(t), \quad t \in \sigma_T, \tag{6}$$

$$\big(z_j - \gamma_j(x,t)v_j\big)|_{x=\rho_0} = \eta_j(t), \quad j = 1,2, \quad t \in \sigma_T, \tag{7}$$

$$(\lambda_1(x,t)\partial_x v_1 - \kappa\lambda_2(x,t)\partial_x v_2)|_{x=\rho_0} = = g_1(t) + \kappa g_2(t), \quad t \in \sigma_T, \tag{8}$$

$$(k_1(x,t)\partial_x z_1 - \varepsilon k_2(x,t)\partial_x z_2)|_{x=\rho_0} = = g_3(t) + \varepsilon g_4(t), \quad t \in \sigma_T, \tag{9}$$

where $\gamma_j(x,t) \geq d_1 = const > 0,\ \lambda_j(x,t) \geq d_2 = const > 0,\ k_j(x,t) \geq d_3 = const > 0, j = 1,2,$ $\kappa > 0, \varepsilon > 0$ –small parameters, $\partial_t = \partial/\partial_t,\ \partial_x = \partial/\partial_x, D_t = d/dt$.

Problem (1)-(9) is a linearized problem of the Florintype nonlinear problem, which describes the process of filtering liquids and gases in the porous medium.

This problem will be studied in Holder spaces. Let l be a noninteger positivenumber, $\alpha = l - [l] \in (0,1)$.

Under $C_{x\ t}^{l,l/2}(\overline{\Omega}_T)$, $C_t^{l/2}(\overline{\sigma}_T)$, we will understand Banach spaces of functions$u(x,t)$and $\psi(t)$,with thenorms

$$|u|_{D_T}^{(l)} := \sum_{2m_0+m=0}^{[l]} |\partial_t^{m_0}\partial_x^m u|_{\Omega_T} + \sum_{2m_0+m=[l]} \left([\partial_t^{m_0}\partial_x^m u]_{x,\Omega_T}^{(\alpha)} + [\partial_t^{m_0}\partial_x^m u]_{t,\Omega_T}^{(\alpha/2)}\right) + + \sum_{2m_0+m=[l]} [\partial_t^{m_0}\partial_x^m u]_{t,\Omega_T}^{((1+\alpha)/2)},$$

$$|\psi|_{\sigma_T}^{(l/2)} := \sum_{m_0=0}^{[l/2]} |D_t^{m_0}\psi|_{\sigma_T} + [D_t^{[l/2]}\psi]_{\sigma_T}^{(l/2-[l/2])}, \tag{10}$$

$$\Omega_T = \Omega \times (0,T), \qquad |v|_{\Omega_T} = \sup_{(x,t)\in\Omega_T} |v|,$$

$$[v]_{x,\Omega_T}^{(\alpha)} = \sup_{(x,t),(z,t)\in\Omega_T} \frac{|v(x,t) - v(z,t)|}{|x-z|^\alpha},\ [v]_{t,\Omega_T}^{(\alpha)} = \sup_{(x,t),(x,t_1)\in\Omega_T} \frac{|v(x,t) - v(x,t_1)|}{|t-t_1|^\alpha}.$$

Through $\overset{\circ}{C}{}_{x\,t}^{l,l/2}(\overline{\Omega}_T)$, we denote the subspaces of functions $u(x,t)$ belonging to $C_{x\,t}^{l,l/2}(\overline{\Omega}_T)$ and satisfying the conditions

$$\partial_t^k u|_{t=0},\ k = 0, \dots, [l/2].$$

The following lemma holds.

Lemma 1. In the space $\overset{\circ}{C}{}_t^{\frac{(1+l)}{2}}(\overline{\sigma}_T)$, $l-$ is a non-integer positive number, the norm $|\psi|_{\overline{\sigma}_T}^{(1+l/2)}$, defined by the formula (10), is equivalent to the norm

$$\|\psi\|_{\overline{\sigma}_T}^{\left(\frac{1+l}{2}\right)} = \sup_{t\in\sigma_T} t^{-\frac{1+l}{2}}|\psi|_{\overline{\sigma}_T} + + \left[D_t^{[(1+l)/2]}\psi\right]_{\overline{\sigma}_T}^{\left(\frac{1+l}{2}-\left[\frac{1+l}{2}\right]\right)}.$$

We define function of Banach spaces for solving the problem. Let

$$B(\Omega_T) := \overset{\circ}{C}{}_{x\ \ t}^{2+l,1+l/2}(\overline{\Omega}_{1T}) \times \times \overset{\circ}{C}{}_{x\ \ t}^{2+l,1+l/2}(\overline{\Omega}_{2T}) \times \overset{\circ}{C}{}_{x\ \ t}^{2+l,1+l/2}(\overline{\Omega}_{1T}) \times \times \overset{\circ}{C}{}_{x\ \ t}^{2+l,1+l/2}(\overline{\Omega}_{2T}) \times \overset{\circ}{C}{}_t^{1+l/2}(\overline{\sigma}_T)$$

be the space of vector-functions $w = (v_1, v_2, z_1, z_2, \psi)$ with the norm

$$\|w\|_{B(\Omega_T)} = = \sum_{j=1}^{2}\left(|v_j|_{\Omega_{jT}}^{(2+l)} + |z_j|_{\Omega_{jT}}^{(2+l)}\right) + |\psi|_{\sigma_T}^{(1+l/2)}, \quad (11)$$

$$:= \overset{\circ}{C}{}_{x\,t}^{\,l,l/2}(\overline{\Omega_{1T}}) \times \overset{\circ}{C}{}_{x\,t}^{\,l,l/2}(\overline{\Omega_{1T}}) \times \overset{\circ}{C}{}_{x\,t}^{\,l,l/2}(\overline{\Omega_{1T}}) \times \overset{\circ}{C}{}_{x\,t}^{\,l,l/2}(\overline{\Omega_{1T}}) \times \overset{\circ}{C}{}_{t}^{\,1+l/2}(\overline{\sigma_T}) \times \overset{\circ}{C}{}_{t}^{\,1+l/2}(\overline{\sigma_T}) \times$$

$$\times \overset{\circ}{C}{}_{t}^{\,1+l/2}(\overline{\sigma_T}) \times \overset{\circ}{C}{}_{t}^{\,1+l/2}(\overline{\sigma_T}) \times \overset{\circ}{C}{}_{t}^{\,1+l/2}(\overline{\sigma_T}) \times$$

$$\times \overset{\circ}{C}{}_{t}^{\,1+l/2}(\overline{\sigma_T}) \times \overset{\circ}{C}{}_{t}^{\,1+l/2}(\overline{\sigma_T}) \times \overset{\circ}{C}{}_{t}^{\,\frac{(1+l)}{2}}(\overline{\sigma_T}) \times$$

$$\times \overset{\circ}{C}{}_{t}^{\,\frac{(1+l)}{2}}(\overline{\sigma_T}) \times \overset{\circ}{C}{}_{t}^{\,\frac{(1+l)}{2}}(\overline{\sigma_T}) \times \overset{\circ}{C}{}_{t}^{\,\frac{(1+l)}{2}}(\overline{\sigma_T})$$

$$H(\Omega_T)$$

is the space of vector-functions $h = (f_1, f_2, f_3, f_4, p_1, p_2, q_1, q_2, \eta_0, \eta_1, \eta_2, g_1, \kappa g_2, g_3, \varepsilon g_4)$with the norm

$$\|h\|_{H(\Omega_T)} = \sum_{j=1}^{2}\left(|f_j|_{\Omega_{jT}}^{(l)} + |f_{j+2}|_{\Omega_{jT}}^{(l)} + |p_j|_{\sigma_T}^{\left(1+\frac{l}{2}\right)} + |q_j|_{\sigma_T}^{\left(1+\frac{l}{2}\right)}\right) +$$
$$+ \sum_{k=0}^{2}|\eta_k|_{\sigma_T}^{\left(1+\frac{l}{2}\right)} + |g_1|_{\sigma_T}^{\frac{1+l}{2}} + \kappa|g_2|_{\sigma_T}^{\frac{1+l}{2}} + |g_3|_{\sigma_T}^{\frac{1+l}{2}} + \varepsilon|g_4|_{\sigma_T}^{(1+l)/2}. \quad (12)$$

It is required fulfillment of the conditions for matching the initial and boundarydata for solving boundary value problems for parabolic equations in a Holder space.

We define these conditions for the problem (1)−(9) [8]. They are found from the boundary conditions (4)−(9) by differentiating them by t, excluding thederivatives $\partial_t^p v_j, \partial_t^p z_j, j = 1,2, p = 0,1, \dots,$ found from the equations (1), (2), and using the initial conditions (3). Find them.

From the equations (1), (2) we find the time derivatives

$$\partial_t v_j = A_j(x,t;\partial_x)v_j + \alpha_j(x,t)\chi(x-\rho_0)D_t\psi + +f_j(x,t), \; j = 1,2, \quad (13)$$

$$\partial_t z_j = A_{j+2}(x,t;\partial_x)z_j + \beta_j(x,t)\chi(x-\rho_0)D_t\psi + +f_{j+2}(x,t), \; j = 1,2, \quad (14)$$

we substitute them into the boundary and conjugation conditions (4)−(7).

By virtue of the initial conditions (3) we have

$$\partial_x^2 v(x,t)|_{t=0} = v_0''(x),$$
$$\partial_x v(x,t)|_{t=0} = v_0'(x),$$
$$v(x,t) = |_{t=0} = v_0(x).$$

The zero order matching conditions will be

$$v_{01}(0,0) = p_1(0), \qquad z_{01}(0,0) = q_1(0),$$
$$v_{02}(b,0) = p_2(0), \qquad z_{02}(b,0) = q_2(0),$$

for$x = 0, \; x = b$and

$$v_{01}(\rho_0,0) - v_{02}(\rho_0,0) = \eta_0(0),$$

$$z_{0j}(\rho_0,0) - \gamma_{0j}(\rho_0,0)(\rho_0,0)v_{0j}(\rho_0,0) = = \eta_j(0), \; j = 1,2,$$

$$\begin{pmatrix}\lambda_1(\rho_0,0)\partial_x v_{01}(x,0) - \\ -\kappa\lambda_2(\rho_0,0)\partial_x v_{02}(x,0)\end{pmatrix}|_{x=\rho_0} = = g_1(0) + \kappa g_2(0),$$

$$\begin{pmatrix}k_1(\rho_0,0)\partial_x z_{01}(x,0) - \\ \backslash -\varepsilon k_2(\rho_0,0)\partial_x z_{02}(x,0)\end{pmatrix}|_{x=\rho_0} = = g_3(0) + \varepsilon g_4(0)$$

for$x = \rho_0$.

To obtain the first order matching condition, let us differentiate the boundary and conjugation conditions (4)−(7) with respect to the variable t, taking intoaccount (13), (14).

The first order matching conditions for $x = \rho_0$ are

$$\big(A_1 v_{01}(x,0) - A_2 v_{02}(x,0)\big)|_{x=\rho_0} + +\big(\alpha_1(\rho_0,0) - \alpha_2(\rho_0,0)\big)D_t\psi|_{t=0} + +f_1(\rho_0,0) - f_2(\rho_0,0) = D_t\eta_0(0), \quad (15)$$

$$A_{j+2}z_{0j}(x,0)|_{x=\rho_0} + \beta_j(\rho_0,0)D_t\psi|_{t=0} + +f_{j+2}(\rho_0,0) - \gamma_j(\rho_0,0)(A_j v_{0j}(x,0)|_{x=\rho_0} + +\alpha_j(\rho_0,0)D_t\psi|_{t=0} + f_j(\rho_0,0)) = = D_t\eta_j(0), \; j = 1,2. \quad (16)$$

Find the $D_t\psi|_{t=0}$from formulas (15) and (16)

$$D_t\psi|_{t=0} := \frac{D_t\eta_0(0) - \big(A_1 v_{01}(x,0) - A_2 v_{02}(x,0)\big)|_{x=\rho_0} - f_1(\rho_0,0) + f_2(\rho_0,0)}{\alpha_1(\rho_0,0) - \alpha_2(\rho_0,0)}, \tag{17}$$

and

$$D_t\psi|_{t=0} := \frac{D_t\eta_j(0) - A_{j+2}z_{0j}(x,0)|_{x=\rho_0} - f_{j+2}(\rho_0,0) + \gamma_j(\rho_0,0)(A_j v_{0j}(x,0)|_{x=\rho_0} - f_j(\rho_0,0)}{\beta_j(\rho_0,0) - \gamma_j(\rho_0,0)\alpha_j(\rho_0,0)}, \tag{18}$$

where $|\alpha_1(\rho_0,0) - \alpha_2(\rho_0,0)| > 0, |\beta_j(\rho_0,0) - \gamma_j(\rho_0,0)\alpha_j(\rho_0,0)| > 0,\ j = 1,2.$

Equating the found derivatives (17), (18) , we get the matching condition

$$D_t\psi|_{t=0} \equiv \frac{D_t\eta_0(0) - \big(A_1 v_{01}(x,0) - A_2 v_{02}(x,0)\big)|_{x=\rho_0} - f_1(\rho_0,0) + f_2(\rho_0,0)}{\alpha_1(\rho_0,0) - \alpha_2(\rho_0,0)} =$$

$$= \frac{D_t\eta_j(0) - A_{j+2}z_{0j}(x,0)|_{x=\rho_0} - f_{j+2}(\rho_0,0) + \gamma_j(\rho_0,0)(A_j v_{0j}(x,0)|_{x=\rho_0} - f_j(\rho_0,0)}{\beta_j(\rho_0,0) - \gamma_j(\rho_0,0)\alpha_j(\rho_0,0)}, j = 1,2.$$

We say that for the problem (4)−(7) the conditions for matching the order of$p$ are satisfied if the equalities

$$\partial_t^p v_1(x,t)|_{x=0,t=0} = D_t^{(p)} p_1(0),$$
$$\partial_t^p z_1(x,t)|_{x=0,t=0} = D_t^{(p)} q_1(0),$$

$$\partial_t^p v_2(x,t)|_{x=b,t=0} = D_t^{(p)} p_2(0),$$
$$\partial_t^p z_2(x,t)|_{x=b,t=0} = D_t^{(p)} q_2(0),$$

$$\left(\partial_t^p v_1(x,t) - \partial_t^p v_2(x,t)\right)|_{x=\rho_0,t=0} = D_t^{(p)}\eta_0(0),$$
$$p = 0, \dots ,1 + [l/2],$$

$$\left(\partial_t^p z_j(x,t) - \gamma_j(x,t)\partial_t^p v_j(x,t)\right)|_{x=\rho_0,t=0}$$
$$= D_t^{(p)}\eta_j(0), j = 1,2, p = 0, \dots ,1 + [l/2]$$

take place.

Here the derivatives $\partial_t^p v_j,\ \partial_t^p z_j, j = 1,2$, are determined by the recurrence formulas:

$$\partial_t v_j = A_j(x,t;\partial_x)v_j$$
$$+\alpha_j(x,t)\chi(x-\rho_0)D_t\psi + f_j(x,t),$$

$$\partial_t^2 v_j = \partial_t(\partial_t v_j) = \partial_t A_j v_j + A_j\partial_t v_j +$$
$$+\ \partial_t\alpha_j(x,t)\chi(x-\rho_0)D_t\psi(t) +$$

$$+\alpha_j(x,t)\chi(x-\rho_0)D_t^2\psi(t) + \partial_t f_j(x,t),$$
$$\dots$$

$$\partial_t^p v_j = \partial_t(\partial_t^{p-1}v_j) = \partial_t^{p-1}A_j v_j +$$
$$+(p-1)\partial_t^{p-2}A_j\partial_t v_j + \dots + A_j\partial_t^{p-1}v_j +$$
$$+\partial_t^{p-1}\alpha_j(x,t)\chi(x-\rho_0)D_t\psi +$$
$$+(p-1)\partial_t^{p-2}\alpha_j(x,t)\chi(x-\rho_0)D_t^2\psi(t) + \dots +$$
$$+\alpha_j(x,t)\chi(x-\rho_0)D_t^p\psi(t) + \partial_t^{p-1}f_j(x,t),$$

$$\partial_t^p z_j = \partial_t(\partial_t^{p-1}z_j) = \partial_t^{p-1}A_{j+2}z_j +$$
$$+(p-1)\partial_t^{p-2}A_{j+2}\partial_t z_j + \dots + A_{j+2}\partial_t^{p-1}z_j +$$
$$+\partial_t^{p-1}\beta_j(x,t)\chi(x-\rho_0)D_t\psi +$$
$$+(p-1)\partial_t^{p-2}\beta_j(x,t)\chi(x-\rho_0)D_t^2\psi(t) + \dots +$$
$$+\beta_j(x,t)\chi(x-\rho_0)D_t^p\psi(t) + \partial_t^{p-1}f_{j+2}(x,t).$$

We rewrite the problem (1)−(9) in the operator form

$$Aw = h, \tag{19}$$

where the operator A is defined by the expressions in the left parts of the equationsand conjugation conditions problem (1)−(9).

We will assume that the following conditions:

$$a) a_j(x,t), a_{j+2}(x,t), b_j(x,t), b_{j+2}(x,t),$$
$$d_j(x,t), d_{j+2}(x,t), \alpha_j(x,t), \beta_j(x,t)$$
$$\in C_{x\,t}^{l,l/2}(\overline{\Omega}_{jT}), \gamma_j(x,t) \in C_{x\,t}^{1+l/2}(\overline{\sigma_T}),$$
$$\lambda_j(x,t), k_j(x,t) \in C_{x\,t}^{(1+l)/2}(\overline{\sigma_T}),$$
$$j = 1,2, \alpha = l - [l] \in (0,1);$$

$b)$ $(\alpha_1(x,t)-\alpha_2(x,t))|_{x=\rho_0} \geq d_4 = const > 0, t \in \bar{\sigma}_T, (\beta_j(x,t)-\gamma_j(x,t)\alpha_j(x,t))|_{x=\rho_0} \geq d_5 = const > 0, j=1,2, t \in \bar{\sigma}_T, (\beta_{3-j}(x,t) - \gamma_{3-j}(x,t)\alpha_j(x,t))|_{x=\rho_0} \geq d_6 = const > 0, j=1,2, t \in \bar{\sigma}_T$ are fulfilled.

2 Main results

Theorem 1. Let $0 \leq \kappa \leq \kappa_0, 0 < \varepsilon \leq \varepsilon_0$ or $0 < \kappa \leq \kappa_0, 0 \leq \varepsilon \leq \varepsilon_0$ and conditions $a)$, $b)$ are fulfilled.

For any functions $f_j(x,t) \in \overset{\circ}{C}{}^{l,l/2}_{x\,t}(\overline{\Omega}_{jT})$, $f_{j+2}(x,t) \in \overset{\circ}{C}{}^{l,l/2}_{x\,t}(\overline{\Omega}_{jT})$, $v_{0j}(x) \in \overset{\circ}{C}{}^{2+l}_{x}(\overline{\Omega}_j)$, $z_{0j}(x) \in \overset{\circ}{C}{}^{2+l}_{x}(\overline{\Omega}_j)$, $p_j(t)$, $q_j(t) \in \overset{\circ}{C}{}^{1+l/2}_{t}(\overline{\sigma}_T)$, $j=1,2$, $\eta_k(t) \in \overset{\circ}{C}{}^{1+l/2}_{t}(\overline{\sigma}_T)$, $k=0,1,2; g_1(t)$, $\kappa g_2(t), g_3(t), \varepsilon g_4(t) \in C^{(1+l)/2}_{x\,t}(\overline{\sigma}_T)$, problem (1)-(9) has a unique solution $v_j(x,t) \in \overset{\circ}{C}{}^{2+l,1+l/2}_{x\;\;t}(\overline{\Omega}_{jT})$, $z_j(x,t) \in \overset{\circ}{C}{}^{2+l,1+l/2}_{x\;\;t}(\overline{\Omega}_{jT})$, $\psi(t) \in \overset{\circ}{C}{}^{1+l/2}_{t}(\overline{\sigma}_T)$ and following estimate is true

$$\|w\|_{B(\Omega_T)} \equiv \sum_{j=1}^{2}\left(|v_j|^{(2+l)}_{\Omega_{jT}} + |z_j|^{(2+l)}_{\Omega_{jT}}\right) + |\psi|^{\left(1+\frac{l}{2}\right)}_{\sigma_T} \leq$$

$$\leq C_1\left(\sum_{j=1}^{2}\left(|f_j|^{(l)}_{\Omega_{jT}} + |f_{j+2}|^{(l)}_{\Omega_{jT}} + |v_{0j}|^{(2+l)}_{\Omega_j} + |z_{0j}|^{(2+l)}_{\Omega_j} + |p_j|^{(1+l/2)}_{\sigma_T} + |q_j|^{(1+l/2)}_{\sigma_T}\right) + \right.$$

$$\left. + \sum_{k=0}^{2}|\eta_k|^{(1+l/2)}_{\sigma_T} + |g_1|^{(1+l)/2}_{\sigma_T} + \kappa|g_2|^{(1+l)/2}_{\sigma_T} + |g_3|^{(1+l)/2}_{\sigma_T} + \varepsilon|g_4|^{(1+l)/2}_{\sigma_T}\right), \qquad (20)$$

where C_1 does not depend on κ and ε.

Proof. The existence of the solution is proved by constructing the regularizer [8], and the estimate (20) is proved by the Schauder method.

We cover the domain Ω with intervals $K^{(i)}_{\delta}, K^{(i)}_{2\delta}$ with common center ξ_i. Let $\zeta_i(x)$, $\mu_i(x)$ be smooth shear functions subject to the covering domains Ω such that $\zeta_i(x)=1$, if $|x-\xi_i| \leq \delta$ and $\zeta_i(x)=0$, if $|x-\xi_i| \geq 2\delta$, and with properties $\sum_i \zeta_i(x)\,\mu_i(x) = 1$ and $|D_m\zeta_i|, |D_m\mu_i| \leq C_{m,i}\delta^{-m}$. Denote the intervals asfollows: for $i \in \aleph_1$ intervals $K^{(i)}_{\delta}$ contain a point ρ_0, for $i \in \aleph_2$ and $i \in \aleph_3$ intervals $K^{(i)}_{2\delta}$ adjoin the boundary of the domain $x=0$ and $x=l$ accordingly, with $i \in \aleph_4$ intervals $K^{(i)}_{2\delta}$ are entirely contained in $\Omega_1 \cup \Omega_2$.

Note that for the equations (1), (2) $\chi(x-\rho_0)=1$, $i \in \aleph_1$ and $\chi(x-\rho_0)=0$ at $i \in \aleph_2 \cup \aleph_3$, $\delta_0 \leq \delta$.

We define the regularizer $\mathfrak{R}$ by formula

$$\mathfrak{R}h = \{\mathfrak{R}_1 h, \mathfrak{R}_2 h, \mathfrak{R}_3 h, \mathfrak{R}_4 h, \mathfrak{R}_5 h\} =$$

$$= \left\{\sum_{i\in\aleph}\mu_i(x)v_{1,i}(x,t), \sum_{i\in\aleph}\mu_i(x)v_{2,i}(x,t), \sum_{i\in\aleph}\mu_i(x)z_{1,i}(x,t), \sum_{i\in\aleph}\mu_i(x)z_{2,i}(x,t), \sum_{i\in\aleph_1}\mu_i(x)\psi_i(t)\right\},$$

where $\aleph := \aleph_1 \cup \aleph_2 \cup \aleph_3 \cup \aleph_4$, are functions $v_{j,i}(x,t), z_{j,i}(x,t), j=1,2, \psi_i(t)$ satisfy zero initial data and are defined as solutions model conjugation problem for $i \in \aleph_1$, the first boundary value problem for $i \in \aleph_2 \cup \aleph_3$ and Cauchy problem with $i \in \aleph_4$.

Let

$$D_{jT}^{(i)} = D_j^{(i)} \times (0,T), K_{2\delta T}^{(i)} = \\ = \left(K_{2\delta}^{(i)} \cap (\Omega_1 \cup \Omega_2)\right) \times (0,T).$$

1. Let $i \in \aleph_1$. Perform a coordinate transformation $x = Y_i^{-1}(y): x = y + \rho_0$. This conversion translates areas $x < \rho_0, D_1^{(i)} = \{y \mid y < 0\}$ and $x > \rho_0$ into $D_2^{(i)} = \{y \mid y > 0\}$.

Set $\zeta_i(x) f_j(x,t),\ \zeta_i(x) g_j(x,t)|_{x=Y_i^{-1}(y)} =$

$$= f_{j,i}(y,t), f_{j+2,i}(y,t);\ \zeta_i(x)p_1(t)|_{x=0},\\ \zeta_i(x)p_2(t)|_{x=l}, \zeta_i(x)q_1(t)|_{x=0},\\ \zeta_i(x)q_2(t)|_{x=l} = p_{j,i}(t), q_{j,i}(t),\\ \zeta_i(x)\eta_j(t)|_{x=\rho_0} = \eta_{j,i}(t), j = 1,2, \zeta_i(x)g_1(t),\\ \zeta_i(x)\kappa g_2(t), \zeta_i(x)g_3(t), \zeta_i(x)\varepsilon g_4(t) =\\ = g_{1,i}(t), \kappa g_{2,i}(t), g_{3,i}(t), \varepsilon g_{4,i}(t)$$

and continue the functions$f_{j,i}, f_{j+2,i}$zero in$D_j^{(i)}$.

We define the functions $v'_{j,i}(y,t), z'_{j,i}(y,t)$, $j = 1,2$, $\psi_i(t)$ as a solution to thefollowing conjugation problem:

$$\partial_t v'_{j,i} - a_j(\xi_i,0)\partial_y^2 v'_{j,i} -\\ -\alpha_j(\xi_i,0)\chi(\xi_i)D_t\psi_i == f_{j,i}(y,t) \text{in} D_{jT}^{(i)},\\ j = 1,2, \tag{21}$$

$$\partial_t z'_{j,i} - a_{j+2}(\xi_i,0)\partial_y^2 z'_{j,i} -\\ -\beta_j(\xi_i,0)\chi(\xi_i)D_t\psi_i = f_{j+2,i}(y,t) \text{in} D_{jT}^{(i)},\\ j = 1,2, \tag{22}$$

$$\left(z'_{j,i} - \gamma_j(\xi_i,0)v'_{j,i}\right)|_{y=0} = \eta_{j,i}(t),\ j = 1,2, \tag{23}$$

$$\left(\lambda_1(\xi_i,0)\partial_x v'_{1,i} - \kappa\lambda_2(\xi_i,0)\partial_x v'_{2,i}\right)|_{y=0} =\\ = g_{1,i}(t) + \kappa g_{2,i}(t), \tag{24}$$

$$\left(k_1(\xi_i,0)\partial_x z'_{1,i} - \varepsilon k_2(\xi_i,0)\partial_x z'_{2,i}\right)|_{y=0} =\\ = g_{3,i}(t) + \varepsilon g_{4,i}(t). \tag{25}$$

In [1], Theorem was proved.

Theorem 2. Let$0 < \kappa \le \kappa_0, 0 < \varepsilon \le \varepsilon_0$ and be executed conditions $a), b)$.

For any functions $f_{j,i}(y,t) \in \overset{\circ}{C}{}_{x\,t}^{l,l/2}(\overline{D}_{jT}^{(i)})$,

$$f_{j+2,i}(x,t) \in \overset{\circ}{C}{}_{x\,t}^{l,l/2}(\overline{D}_{jT}^{(i)}),\\ \eta_{j,i}(t) \in \overset{\circ}{C}{}_t^{1+l/2}(\overline{\sigma}_T),\\ j = 1,2, g_{1,i}(t) \in \overset{\circ}{C}{}_t^{(1+l)/2}(\overline{\sigma}_T),\\ \kappa g_2(t) \in \overset{\circ}{C}{}_t^{(1+l)/2}(\overline{\sigma}_T), g_3(t) \in \overset{\circ}{C}{}_t^{(1+l)/2}(\overline{\sigma}_T),\\ \varepsilon g_4(t) \in \overset{\circ}{C}{}_t^{(1+l)/2}(\overline{\sigma}_T),$$

problem (21)-(25) has a unique solution $v'_{j,i} \in \overset{\circ}{C}{}_{x\ \ t}^{2+l,1+l/2}(\overline{D}_{jT}^{(i)}), z'_{j,i} \in \overset{\circ}{C}{}_{x\ \ t}^{2+l,1+l/2}(\overline{D}_{jT}^{(i)})$, $\psi_i(t) \in \overset{\circ}{C}{}_t^{1+l/2}(\overline{\sigma}_T)$, and following estimate is true

$$\sum_{j=1}^{2}\left(\left|v'_{j,i}\right|_{D_{jT}^{(i)}}^{(2+l)} + \left|z'_{j,i}\right|_{D_{jT}^{(i)}}^{(2+l)}\right) + |\psi_i|_{\sigma_T}^{\left(1+\frac{l}{2}\right)} \le C_2\sum_{j=1}^{2}\left(\left|f_{j,i}\right|_{D_{jT}^{(i)}}^{(l)} + \left|f_{j+2,i}\right|_{D_{jT}^{(i)}}^{(l)} + \left|\eta_{j,i}\right|_{\sigma_T}^{\left(1+\frac{l}{2}\right)} + \left|g_{1,i}\right|_{\sigma_T}^{\frac{1+l}{2}} + \\ +\kappa\left|g_{2,i}\right|_{\sigma_T}^{(1+l)/2} + \left|g_{3,i}\right|_{\sigma_T}^{(1+l)/2} + \varepsilon\left|g_{4,i}\right|_{\sigma_T}^{(1+l)/2}\right), \tag{26}$$

where C_2 does not depend on κ and ε.

The functions $v'_{j,i}(y,t), z'_{j,i}(y,t), j = 1,2$, are defined as asolution of the first boundaryvalue problems $i \in \aleph_2 \cup \aleph_3$, and functions $v_{j,i}(y,t), z_{j,i}(y,t), j = 1,2$ with $i \in \aleph_4$ are solution of Cauchy problem. Each of these problems under the conditions of the Theorem 2has a unique solution and it is subject to estimates (27), (28) [8]

$$\left|v_{j,i}\right|_{K_{2\delta T}^{(i)}}^{(2+l)} \le C_3\left(\left|\zeta_i(x)f_j\right|_{K_{2\delta T}^{(i)}}^{(l)} + \left|\zeta_i(x)p_j\right|_{\sigma_T}^{\left(1+\frac{l}{2}\right)}\right),\\ i \in \aleph_2 \cup \aleph_3, j = 1,2,$$

$$\left|z_{j,i}\right|_{K_{2\delta T}^{(i)}}^{(2+l)} \le C_4\left(\left|\zeta_i(x)g_j\right|_{K_{2\delta T}^{(i)}}^{(l)} + \left|\zeta_i(x)q_j\right|_{\sigma_T}^{\left(1+\frac{l}{2}\right)}\right),\\ i \in \aleph_2 \cup \aleph_3, j = 1,2, \tag{27}$$

$$\left|v_{j,i}\right|^{(2+l)}_{K^{(i)}_{2\delta T}} \le C_5\left|\zeta_i(x) f_j\right|^{(l)}_{K^{(i)}_{2\delta T}},$$
$$\left|z_{j,i}\right|^{(2+l)}_{K^{(i)}_{2\delta T}} \le C_6\left|\zeta_i(x) f_{j+2}\right|^{(l)}_{K^{(i)}_{2\delta T}},$$
$$i \in \aleph_4,\ j = 1,2. \tag{28}$$

We introduce the norm [1]

$$\{w\}_{B(\Omega_T)} = \sup_i \|w\|_{B\left(K^{(i)}_{2\delta T}\right)},$$
$$\{h\}_{H(\Omega_T)} = \sup_i \|h\|_{H(K^{(i)}_{2\delta T})}, \tag{29}$$

where $K^{(i)}_{2\delta T} = (K^{(i)}_{2\delta} \cap (\Omega_1 \cup \Omega_2)) \times (0,T)$. The norms of $\{w\}_{B(\Omega_T)}, \{h\}_{H(\Omega_T)}$ are defined by formulas (11), (12). Note that the norms (29) are equivalent to the norms $\|w\|_{B(\Omega_T)}, \|h\|_{H(\Omega_T)}$[8].

Lemma 2. The operator $\Re := H(\Omega_T) \to B(\Omega_T)$ is bounded: $\{\Re h\}_{B(\Omega_T)} \le C_7\{h\}_{H(\Omega_T)}$.

Let us turn to the problem (1)-(9), which we recorded in operator form (19) $Aw = h$. Obviously, $A: B(\Omega_T) \to H(\Omega_T)$.

Lemma 3. For any $h \in H(\Omega_T)$, the equality $A\Re h = h + Ph$, takes place. Here $Ph = \{P_1h, P_2h, P_3h, P_4h, 0,0,0,0,0, P_5h, P_6h, P_7h, P_8h\}$, $P_jh, P_{j+2}h, P_{4+j}h, j = 1,2, P_7h, P_8h$ contain lower terms or higher terms with small coefficients.

Forexample,

$$P_jh = -\sum_{i\in\aleph} \mu_i(x)\big(b_j(x,t)\partial_x v_j + d_j(x,t)v_j\big) - a_j(x,t)\sum_{i\in\aleph}\Big(\partial_x^2\, \mu_i(x) v_{j,i}(x,t) + 2\partial_x\mu_i(x)\partial_x v_{j,i}(x,t)\Big) -$$
$$-\sum_{i\in\aleph}\mu_i(x)\big[a_j(x,t) - a_j(\xi_i,0)\big]\,\partial_x^2 v'_{j,i}(x,t) - \sum_{i\in\aleph_1}\mu_i(x)\big[a_j(x,t)\chi(x-\rho_0) - a_j(\xi_i,0)\chi(\xi_i-\rho_0)\big]\,D_t\psi_i,$$
$$j = 1,2.$$

Lemma 4. Under the conditions of the Theorem 1 for $t \le T_0$estimate

$$\{Ph\}_{H(\Omega_T)} \le æ\{h\}_{H(\Omega_T)} \tag{30}$$

is fulfilled, where $æ \in (0,1)$.

Lemma 5. Under the conditions of the Theorem 2 there exists for $t \le T_0$ bounded right inverse operator $A_r^{-1} = \Re(E+P)^{-1}: H(\Omega_{T_0}) \to B(\Omega_{T_0})$, where E is the unit operator.

Proof. We have the problem $Aw = h$. Substituting $\Re h$ instead of w, we get $A\Re h = h + Ph \equiv (E+P)h$. Let $(E+P)h = h_1$, where $h_1 \in H(\Omega_{T_0})$. Accordingto the assessment (30) this equation has a unique solution $h \in H(\Omega_{T_0})$, which is subject to estimate $\{h\}_{H(\Omega_{T_0})} \le \frac{1}{1-æ}\{h_1\}_{H(\Omega_{T_0})}$ for anyone vector $h_1 \in H(\Omega_{T_0})$. Then there is a limited the inverse operator $(E+P)^{-1}$ in the space $H(\Omega_{T_0})$. Substituting $h = (E+P)^{-1}h_1$ into the equation $A\Re h = h_1 \equiv h + Ph$, we get the identity $A\Re(E+P)^{-1}h_1 = h_1$ for any $h_1 \in H(\Omega_{T_0})$ or $A\Re(E+P)^{-1} = E$. According to the definition implies that the operator A has the right inverse of the bounded operator $A_r^{-1} = \Re(E+P)^{-1}$, and the problem $Aw = h$ has the solution $w = (v_1, v_2, z_1, z_2, \psi) \in B(\Omega_{T_0})$ for any vector $h \in H(\Omega_{T_0})$.

We obtain an estimate for the solution of the problem (1)-(9) using the Schauder method. Consider the functions of $v_{j,i}(x,t) = \mu_i(x)v_j(x,t), z_{j,i}(x,t) = \mu_i(x)z_j(x,t)$, $j = 1,2, \psi_i(t) = \mu_i(x)\,\psi(t)$, which are defined in $K^{(i)}_{2\delta} \times (0,T_0)$ and extend them by zero outside this area. Depending on the location of the interval $K^{(i)}_{2\delta}$ in Ω for functions $v_{j,i}(x,t), z_j(x,t), j = 1,2, \psi_i(t)$ from the problem (1)−(9) the model pairing problem, the first boundary value problem, and the Cauchyproblem can be obtained.

Multiply parabolic equations and problem conditions (1)−(9) on the cutting function $\mu_i(x)$. In the equations and conditions we will make transformations coordinates $x = Y_i^{-1}(y): x = y + \rho_0$ as $i \in \aleph_1$ for functions $v'_{j,i}(y,t) = v_{j,i}(x,t)|_{x=Y_i^{-1}(y)}, z'_{j,i}(y,t) = z_{j,i}(x,t)|_{x=Y_i^{-1}(y)}$, $j = 1,2, \psi_i(t)$ we get the conjugation problem with zero initial data in $D^{(i)}_{jT}, j = 1,2,$

$$\partial_t v'_{j,i} - a_j(\xi_i,0)\partial_y^2 v'_{j,i} - \alpha_j(\xi_i,0)\chi(\xi_i)D_t\psi_i = f'_{j,i}(y,t) + Y_i^{-1}(y)F_{j,i}(x,t),$$

$$\partial_t z'_{j,i} - a_{j+2}(\xi_i,0)\partial_y^2 z'_{j,i} - \beta_j(\xi_i,0)\chi(\xi_i)D_t\psi_i = f'_{j+2,i}(y,t) + Y_i^{-1}(y)F_{j,i}(x,t),$$

$$\left(z'_{j,i} - \gamma_j(\xi_i,0)v'_{j,i}\right)|_{y=0} = \eta_{j,i}(t) + Y_i^{-1}(y)H_{j,i}(x,t),\ j=1,2, \tag{31}$$

$$\left(\lambda_1(\xi_i,0)\partial_y v'_{1,i} - \kappa\lambda_2(\xi_i,0)\partial_y v'_{2,i}\right)|_{y=0} = g_{1,i}(t) + \kappa g_{2,i}(t) + Y_i^{-1}(y)H_{3,i}(x,t),$$

$$\left(k_1(\xi_i,0)\partial_x z'_{1,i} - \varepsilon k_2(\xi_i,0)\partial_x z'_{2,i}\right)|_{y=0} = g_{3,i}(t) + \varepsilon g_{4,i}(t) + Y_i^{-1}(y)H_{4,i}(x,t),$$

where $f'_{j,i}(y,t) = f_{j,i}(x,t)|_{x=Y_i^{-1}(y)}$, $f'_{j+2,i}(y,t) = f_{j+2,i}(x,t)|_{x=Y_i^{-1}(y)}$, $F_{j,i}(x,t)$, $F_{j+2,i}(x,t)$, $j=1,2, H_{j,i}(x,t)$, $H_{3,i}(x,t)$, $H_{4,i}(x,t)$, contain smallest coefficients or leading coefficients with small coefficients.

The conjugation problem (31), according to Theorem 2, uniquely solvable and for it's solution estimate (26) holds. Therefore, solutionof the problem (31)obeys the inequality

$$\sum_{j=1}^{2}\left(\left|v'_{j,i}\right|_{K_{2\delta T}^{(i)}}^{(2+l)} + \left|z'_{j,i}\right|_{K_{2\delta T}^{(i)}}^{(2+l)}\right) + |\psi_i|_{\sigma_T}^{\left(1+\frac{l}{2}\right)} \le C_8\sum_{j=1}^{2}\left(\left|f_{j,i}\right|_{K_{2\delta T}^{(i)}}^{(l)} + \left|f_{j+2,i}\right|_{K_{2\delta T}^{(i)}}^{(l)} + \left|\eta_{j,i}\right|_{\sigma_T}^{\left(1+\frac{l}{2}\right)} + \left|g_{1,i}\right|_{\sigma_T}^{\left(\frac{1+l}{2}\right)} + \right.$$

$$+\kappa|g_{2,i}|_{\sigma_T}^{(1+l)/2} + |g_{3,i}|_{\sigma_T}^{(1+l)/2} + \varepsilon|g_{4,i}|_{\sigma_T}^{(1+l)/2} + |F_{j,i}|_{K_{2\delta T}^{(i)}}^{(l)} + |F_{j+2,i}|_{K_{2\delta T}^{(i)}}^{(l)} +$$

$$\left. +|H_{j,i}(x,t)|_{\sigma_T}^{\left(1+\frac{l}{2}\right)} + |H_{j+2,i}(x,t)|_{\sigma_T}^{\left(1+\frac{l}{2}\right)}\right), j=1,2, \tag{32}$$

where C_8 does not depend on κ and ε.

We also get the functions $v'_{j,i}(y,t)$, $z'_{j,i}(y,t), j=1,2$ as solutions of the firstboundary value problems for $i \in \aleph_2 \cup \aleph_3$, and functions $v_{j,i}(x,t), z_j(x,t), j=1,2$ for $i \in \aleph_4$ as the is solution of the Cauchy problems. Based on [1], the firstboundary problem and the Cauchy problem are uniquely solvable. To solve the first boundary problem and the problem Cauchy valid estimates similar to those

estimated. (27), (28) and (32), as well as similar functions arising in right partsof the equations of the first boundary value problems and the Cauchy problem.

The norms of the functions $F_{j,i}(x,t)$, $F_{j+2,i}(x,t)$, $H_{j,i}(x,t)$, $H_{j+2,i}(x,t)$, $j=1,2$, are estimated the same way as with the proof of Lemma 3 norms of operators $P_j h, P_{j+2}h, P_{4+j}h, j=1,2, P_7 h, P_8 h$. Moreover, if note that in the interval $K_\delta^{(i)}$ the cutoff function $\mu_i(x)=1$, then in $K_\delta^{(i)} \times (0,T), v_{j,i} := \mu_i v_j = v_j, z_{j,i} := \mu_i z_j = z_j, \psi_i := \mu_i\psi = \psi$.

Using estimates of solutions of the conjugation problem (32), the first boundaryproblems and Cauchy problems we will have inequality

$$\|\mu_i w\|_{B\left(K_{2\delta t}^{(i)}\right)} \le C_9\|\mu_i h\|_{H\left(K_{2\delta t}^{(i)}\right)} + $$
$$+æ_1\|\mu_i w\|_{B\left(K_{2\delta t}^{(i)}\right)},$$

$$K_{2\delta t}^{(i)} = \left(K_{2\delta}^{(i)} \cap (\Omega_1 \cap \Omega_2)\right) \times (0,t), æ_1 \in (0,1).$$

Hence, we get

$$\|\mu_i w\|_{B\left(K_{2\delta t}^{(i)}\right)} \le \frac{C_9}{1-æ_1}\|\mu_i h\|_{H\left(K_{2\delta t}^{(i)}\right)},$$
$$æ_1 \in (0,1), \tag{33}$$

where $K_{\delta t}^{(i)} = \left(K_\delta^{(i)} \cap (\Omega_1 \cap \Omega_2)\right) \times (0,t)$.

We proceed in the inequality (33) to the supremum on i, taking into accountthe definitions of the norms $\{w\}_{B(\Omega_{T_1})}$ and $\{h\}_{H(\Omega_{T_1})}$ in (29), as a result get anestimate

$$\{w\}_{B(\Omega_{T_1})} \leq C_{10}\{h\}_{H(\Omega_{T_1})}.$$

From this inequality, by virtue of the equivalence of norms $\{w\}_{B(\Omega_{T_1})}, \{h\}_{H(\Omega_{T_1})}$ and $\|w\|_{B(\Omega_{T_1})}, \|h\|_{H(\Omega_{T_1})}$, follows the evaluation of the solution to the problem (1)−(9)

$$\|w\|_{B(\Omega_{T_1})} \leq C_{11}\|h\|_{H(\Omega_{T_1})}. \qquad (34)$$

The problem (1)−(9) is linear problem. The uniqueness of the solution followsfrom the evaluation (34). We proved the existence and uniqueness of the solutionof the problem (1)−(9) for $t \leq \min(T_0, T_1)$. Continuing the solution by t as in [8], we obtain Theorem 1 for $T > 0$.

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