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AN IRREGULAR CONJUGATION PROBLEM FOR THE SYSTEM OF THE PARABOLIC EQUATIONS IN THE HOLDER SPACE

Abstract. We consider the conjugation problem for the system of parabolic equations with two small parameters \( \nu > 0, \varepsilon > 0 \) in the boundary conditions. There are proved the existence, uniqueness and uniform coercive estimates of the solution with respect to the small parameters in the Holder space. This problem is linearized one of the nonlinear problem with the free boundary of Florin type and it is in the base of the proof of the solidified of this nonlinear problem in the Holder space. We study the problem with the free boundary of the Florin type in the Holder space \( C_{z_{j},l} (\overline{\Omega}_{jT}) \), \( j = 1,2 \), where \( l \) is non-integer positive number. Existence, uniqueness, estimates for solution of the problem with constants independent of small parameters in the Holder space are proved. It gives us the opportunity to establish the existence, uniqueness and estimates of the solution of the problem without loss of smoothness of given functions for \( \kappa = 0, \nu > 0; \kappa > 0, \varepsilon = 0 \) and \( \kappa = 0, \varepsilon = 0 \).

Key words: parabolic equations, existence, uniqueness of the solution, coercive estimates, Holder space.

1 Introduction

In the work the problem of Florin type is studied for the system of parabolic equations in the Holder spaces. This problem is a mathematical model describing filtration of liquids and gases in the porous medium. Linear problems with small parameters with time derivatives functions of the free boundary were studied in [1]-[7]. In this article the problem is studied without time derivatives functions of the free boundary \( \psi(t) \) in the right-hand sides of the conditions (6), (7), which corresponds to a degenerate nonlinear the free boundary problem of melting binary alloys and in which free boundary is set as an implicit function. In contrast from problems in [1]-[7], where free border is set explicitly.

Let \( \Omega_1 = (0, \rho_0), \Omega_2 = (\rho_0, b), 0 < \rho_0 < b, b > 0, \Omega_{jT} = \Omega_j \times (0, T), j = 1,2, \sigma_T = (0, T), \chi(\lambda) \) be a smooth shear function, equal to one at \( |\lambda| \leq \delta_0 \) and zero for \( |\lambda| \geq 2\delta_0 \) and having the rating \( d^m_\chi / dx^m \leq C_m \delta_0^{-m}, \delta_0 = const > 0 \).

Define second order elliptic operators

\[
A_j(x, t; \partial_x) v_j := a_j(x, t) \partial_x^2 v_j + b_j(x, t) \partial_x v_j + d_j(x, t) v_j,
\]

where \( a_j(x, t), a_{j+2}(x, t) \geq a_0 = const > 0 \) in \( \overline{\Omega}_{jT}, j = 1,2 \).

It is required to find functions \( v_j(x, t), z_j(x, t), j = 1,2, \) and \( \psi(t) \) satisfying parabolic equations

\[
\partial_t v_j - A_j(x, t; \partial_x) v_j = \chi(x - \rho_0) D_\psi = f_j(x, t) \in \overline{\Omega}_{jT}, j = 1,2,
\]

and initial conditions

\[
\psi|_{t=0} = 0, \quad v_j|_{t=0} = v_{oj}, \quad z_j|_{t=0} = z_{o,j}, \quad \text{in} \Omega, j = 1,2,
\]

boundary conditions

\[
v_1|_{x=0} = p_1(t), v_2|_{x=b} = p_2(t), \quad t \in \sigma_T,
\]

\[
z_1|_{x=0} = q_1(t), z_2|_{x=b} = q_2(t), \quad t \in \sigma_T,
\]
and conjugation conditions on the border \( x = \rho_0 \)
\[
(v_1 - v_2)_{x=\rho_0} = \eta_0(t), \quad t \in \sigma_T, \quad (6)
\]
\[
(z_j - \gamma_j(x,t)v_j)_{x=\rho_0} = \eta_j(t), \quad j = 1, 2, \quad t \in \sigma_T, \quad (7)
\]
\[
(\lambda_1(x,t)\partial_x v_1 - \kappa \lambda_2(x,t)\partial_x v_2)_{x=\rho_0} = \gamma_1(t) + \kappa \gamma_2(t), \quad t \in \sigma_T, \quad (8)
\]
\[
(k_1(x,t)\partial_x z_1 - \varepsilon k_2(x,t)\partial_x z_2)_{x=\rho_0} = \gamma_3(t) + \varepsilon \gamma_4(t), \quad t \in \sigma_T, \quad (9)
\]
where \( \gamma_j(x,t) \geq d_1 = \text{const} > 0, \lambda_j(x,t) \geq d_2 = \text{const} > 0, j = 1, 2, \kappa > 0, \varepsilon > 0 \) small parameters, \( \partial_t = \partial / \partial t, \partial_x = \partial / \partial x, D_t = d / dt \).

Problem (1)-(9) is a linearized problem of the Florintype nonlinear problem, which describes the process of filtering liquids and gases in the porous medium.

This problem will be studied in Holder spaces. Let \( l \) be a noninteger positive number, \( \alpha = l - [l] \in (0,1) \).

Under \( C_x^{1/2}((\Omega_T), C_t^{12}(\sigma_T)) \), we will understand Banach spaces of functions \( u(x,t) \) and \( \psi(t) \) with the norms

\[
|u|_{\sigma_T}^{(l)} := \sum_{m_0+m=0}^{[l]} |\partial_t^{m_0} \partial_x^m u|_{\Omega_T} + \sum_{m_0+m=[l]} \left( |\partial_t^{m_0} \partial_x^m u|_{x,\Omega_T}^{(a)} + |\partial_t^{m_0} \partial_x^m u|_{t,\Omega_T}^{(a/2)} \right) + \sum_{m_0+m=[l]} |\partial_t^{m_0} \partial_x^m u|_{t,\Omega_T}^{((1+a)/2)},
\]

\[
|\psi|_{\sigma_T}^{(l/2)} := \sum_{m_0=0}^{[l/2]} |D_t^{m_0} \psi|_{\sigma_T} + |D_t^{(l/2)} \psi|_{\sigma_T}^{(l/2-[l/2])}, \quad (10)
\]

\[
\Omega_T = \Omega \times (0, T), \quad [v]_{\sigma_T} = \sup_{(x,t) \in \Omega_T} |v|,
\]

\[
[v]_{\sigma_T}^{(\alpha)} = \sup_{(x,t),(x,t) \in \Omega_T} \frac{|v(x,t) - v(x,t)|}{|x - z|^\alpha}, \quad [v]_{\sigma_T}^{(\alpha)} = \sup_{(x,t),(x,t) \in \Omega_T} \frac{|v(x,t) - v(x,t_1)|}{|t - t_1|^\alpha}.
\]

Through \( C_x^{1/2}(\Omega_T) \), we denote the subspaces \( C_x^{l/2}(\Omega_T) \) of functions \( u(x,t) \) belonging to \( C_x^{l}((\Omega_T)) \) and satisfying the conditions

\[
\partial_t^k u|_{t=0^+}, \quad k = 0, ..., [l/2].
\]

The following lemma holds.

**Lemma 1.** In the space \( C_x^{2+l/2}(\sigma_T) \), \( l \) is a non-integer positive number, the norm \( |\psi|_{\sigma_T}^{(1+l/2)} \), defined by the formula (10), is equivalent to the norm

\[
\|\psi\|_{\sigma_T}^{(1+l/2)} = \sup_{t \in \sigma_T} t^{1+l/2} |\psi|_{\sigma_T} + \int \left| D_t^{(1+l/2)} \psi \right|^{1+l/2}_{\sigma_T}.
\]

We define function of Banach spaces for solving the problem. Let

\[
B(\Omega_T) := C_x^{2+l/2}(\Omega_T) \times C_x^{l/2}(\Omega_T) \times C_x^{2+l/2}(\Omega_T) \times C_x^{l/2}(\Omega_T) \times C_x^{2+l/2}(\Omega_T) \times C_x^{l/2}(\Omega_T).
\]
be the space of vector-functions $w = (v_1, v_2, z_1, z_2, \psi)$ with the norm
\[
\|w\|_{H(\Omega_T)} = \sum_{i=1}^{2} \left( \|v_i\|_{\Omega_T}^{(2+i)} + |\psi|_{\Omega_T}^{(1+i/2)} \right),
\]
\[
C_{x,t} (\Omega_{1T}) \times C_{x,t} (\Omega_{1T}) \times C_{x,t} (\Omega_{1T}) \times C_{x,t} (\Omega_{1T}) \times C_{x,t} (\Omega_{1T})
\]
\[
\|h\|_{H(\Omega_T)} = \sum_{j=1}^{2} \left( |f_j|_{\sigma_T}^{(l)} + |f_{j+2}|_{\sigma_T}^{(l)} + |p_j|_{\sigma_T}^{(1+i/2)} + |q_j|_{\sigma_T}^{(1+i/2)} \right) + 
\sum_{k=0}^{2} \eta_k |g_k|_{\sigma_T}^{(1+i)} + \kappa |g_2|_{\sigma_T}^{(1+i/2)} + |g_3|_{\sigma_T}^{(1+i/2)} + \varepsilon |g_4|_{\sigma_T}^{(1+i/2)}.
\]

It is required fulfillment of the conditions for matching the initial and boundary data for solving boundary value problems for parabolic equations in a Holder space.

We define these conditions for the problem (1)–(9) [8]. They are found from the boundary conditions (4)–(9) by differentiating them by $t$, excluding the derivatives $\partial_x^p v_j, \partial_x^p z_j, j = 1, 2, p = 0, 1, \ldots$, found from the equations (1), (2), and using the initial conditions (3). Find them.

From the equations (1), (2) we find the time derivatives
\[
\partial_t v_j = A_j(x, t; \partial_x) v_j + \alpha_j(x, t) \chi(x - \rho_0) D_t \psi + 
+f_j(x, t), \quad j = 1, 2,
\]
\[
\partial_t z_j = A_{j+2}(x, t; \partial_x) z_j + \beta_j(x, t) \chi(x - \rho_0) D_t \psi + 
+f_{j+2}(x, t), \quad j = 1, 2,
\]
we substitute them into the boundary and conjugation conditions (4)–(7).

By virtue of the initial conditions (3) we have
\[
\partial_x^2 v(x, t)|_{t=0} = v_0''(x),
\partial_x v(x, t)|_{t=0} = v_0'(x),
\]
\[
v(x, t)|_{t=0} = v_0(x).
\]

The zero order matching conditions will be
\[
v_{01}(0, 0) = \rho_1(0), \quad z_{01}(0, 0) = q_1(0),
\]
\[
v_{02}(0, 0) = \rho_2(0), \quad z_{02}(0, 0) = q_2(0),
\]
\[
\text{for } \rho_0 = 0, \quad x = \text{band}
\]
\[
v_{01}(\rho_0, 0) - v_{02}(\rho_0, 0) = \eta_0(0),
\]
\[
z_{01}(\rho_0, 0) - z_{02}(\rho_0, 0) = \eta_0(0),
\]
\[
\text{with the norm}
\]
\[
\|h\|_{H(\Omega_T)} = \sum_{j=1}^{2} \left( |f_j|_{\sigma_T}^{(l)} + |f_{j+2}|_{\sigma_T}^{(l)} + |p_j|_{\sigma_T}^{(1+i/2)} + |q_j|_{\sigma_T}^{(1+i/2)} \right) + 
\sum_{k=0}^{2} \eta_k |g_k|_{\sigma_T}^{(1+i)} + \kappa |g_2|_{\sigma_T}^{(1+i/2)} + |g_3|_{\sigma_T}^{(1+i/2)} + \varepsilon |g_4|_{\sigma_T}^{(1+i/2)}.
\]
and
\[
D_t\psi|_{t=0} = \frac{D_t\eta_0(0) - (A_1 v_{01}(x, 0) - A_2 v_{02}(x, 0))|_{x=\rho_0} - f_1(\rho_0, 0) + f_2(\rho_0, 0)}{\alpha_1(\rho_0, 0) - \alpha_2(\rho_0, 0)},
\]
where \(|\alpha_1(\rho_0, 0) - \alpha_2(\rho_0, 0)| > 0, |\beta_j(\rho_0, 0) - \gamma_j(\rho_0, 0)\alpha_j(\rho_0, 0)| > 0, j = 1, 2.

Equating the found derivatives (17), (18), we get the matching condition
\[
D_t\psi|_{t=0} = \frac{D_t\eta_0(0) - (A_1 v_{01}(x, 0) - A_2 v_{02}(x, 0))|_{x=\rho_0} - f_1(\rho_0, 0) + f_2(\rho_0, 0)}{\alpha_1(\rho_0, 0) - \alpha_2(\rho_0, 0)} = \frac{D_t\eta_j(0) - A_{j+2} z_0(x, 0)|_{x=\rho_0} - f_{j+2}(\rho_0, 0) + \gamma_j(\rho_0, 0)(A_1 v_{01}(x, 0)|_{x=\rho_0} - f_j(\rho_0, 0))}{\beta_j(\rho_0, 0) - \gamma_j(\rho_0, 0)\alpha_j(\rho_0, 0)}, j = 1, 2.
\]

We say that for the problem (4)–(7) the conditions for matching the order of the equalities
\[
\partial_t^p v_1(x, t)|_{x=0, t=0} = D_t^{(p)} p_1(0), \\
\partial_t^p z_1(x, t)|_{x=0, t=0} = D_t^{(p)} q_1(0), \\
\partial_t^p v_2(x, t)|_{x=b, t=0} = D_t^{(p)} p_2(0), \\
\partial_t^p z_2(x, t)|_{x=b, t=0} = D_t^{(p)} q_2(0), \\
(\partial_t^p v_1(x, t) - \partial_t^p v_2(x, t))|_{x=\rho_0, t=0} = D_t^{(p)} \eta_0(0), \\
\partial_t^p z_1(x, t)|_{x=0, t=0} = D_t^{(p)} \eta_j(0), j = 1, 2, p = 0, \ldots, 1 + \lceil l/2 \rceil,
\]
take place.

Here the derivatives \(\partial_t^p v_j, \partial_t^p z_j, j = 1, 2\), are determined by the recurrence formulas:
\[
\partial_t v_j = A_j(x, t; \partial_x) v_j + \alpha_j(x, t)\chi(x - \rho_0)D_t \psi + f_j(x, t), \\
\partial_t^2 v_j = \partial_t(\partial_t v_j) = \partial_t A_j v_j + \alpha_j(t)\chi(x - \rho_0)D_t \psi(t) + \\
+ \alpha_j(x, t)\chi(x - \rho_0)D_t^2 \psi(t) + \partial_t f_j(x, t), \\
\partial_t^p v_j = \partial_t^p(\partial_t^{p-1} v_j) = \partial_t v_j + \\
+(p-1)\partial_t^{p-2} A_j \partial_t v_j + \cdots + A_j \partial_t^{p-2} v_j + \\
+ \partial_t^{p-1} \alpha_j(x, t)\chi(x - \rho_0)D_t \psi + \\
+(p-1)\partial_t^{p-2} \alpha_j(x, t)\chi(x - \rho_0)D_t \psi(t) + \cdots + \\
+ \alpha_j(x, t)\chi(x - \rho_0)D_t^p \psi(t) + \partial_t^{p-1} f_j(x, t),
\]
where the operator \(A\) is defined by the expressions in the left parts of the equations and conjugation conditions problem (1)–(9).

We will assume that the following conditions:
\[
\begin{align*}
\alpha_j(x, t), a_{j+2}(x, t), b_j(x, t), b_{j+2}(x, t), \\
d_j(x, t), d_{j+2}(x, t), \alpha_j(x, t), \beta_j(x, t) \\
&\in C_{x \times \left(\Omega_{j\tau}, \Omega_{(j+1)\tau}\right)}, \\
&\in C_{x \times \left(\Omega_{j\tau}, \Omega_{(j+1)\tau}\right)}, \\
\lambda_j(x, t), k_j(x, t) &\in C_{x \times \left(\Omega_{j\tau}, \Omega_{(j+1)\tau}\right)}, \quad j = 1, 2, \alpha = l - \lfloor l \rfloor \in (0, 1); \\
\end{align*}
\]
b) \((\alpha_1(x,t) - \alpha_2(x,t))|_{x=\rho_0} \geq d_4 = \text{const} > 0, t \in \bar{\Omega}, (\beta_j(x,t) - \gamma_j(x,t)\alpha_j(x,t))|_{x=\rho_0} \geq d_5 = \text{const} > 0, j = 1, 2, t \in \bar{\Omega}, (\beta_j(x,t) - \gamma_j(x,t)\alpha_j(x,t))|_{x=\rho_0} \geq d_6 = \text{const} > 0, j = 1, 2, t \in \bar{\Omega}\) are fulfilled.

2 Main results

Theorem 1. Let \(0 \leq \kappa \leq \kappa_0, 0 < \varepsilon \leq \varepsilon_0\) or \(0 < \kappa \leq \kappa_0, 0 \leq \varepsilon \leq \varepsilon_0\) and conditions a), b) are fulfilled.

For any functions \(f_j(x,t) \in C_{x,t}(\bar{\Omega}_j),\)

\[f_{j+2}(x,t) \in C_{x,t}(\bar{\Omega}_j), \quad v_{0j}(x) \in C_{x}(\bar{\Omega}_j),\]

\[\psi(t) \in C_{t}(\bar{\Omega}_j)\] and following estimate is true

\[
\|w\|_{\mathcal{B}(\Omega)} = \sum_{j=1}^{2} \left[ \|v[j]_{\Omega_{jT}}^{(2+1)} \|^{2} + \|z[j]_{\Omega_{jT}}^{(2+1)} \|^{2} + \|\psi[j]_{\sigma_T}^{(1+1)} \|^{2} \right] \leq C_1 \left[ \sum_{j=1}^{2} \left( \|f[j]_{\Omega_{jT}}^{(1)} \| + \|f_{j+2}[j]_{\Omega_{jT}}^{(1)} \| + \|v[0j]_{\Omega_{jT}}^{(2+1)} \| + \|z[0j]_{\Omega_{jT}}^{(2+1)} \| + \|p[j]_{\sigma_T}^{(1+1)} \| + \|q[j]_{\sigma_T}^{(1+1)} \| \right) \right] + \sum_{k=0}^{2} \left[ \eta[k]_{\sigma_T}^{(1+1/2)} + \|g[1+2][j]_{\sigma_T}^{(1+1/2)} \| + \|g[2+3][j]_{\sigma_T}^{(1+1/2)} \| + \|g[3+4][j]_{\sigma_T}^{(1+1/2)} \| \right], \tag{20}\]

where \(C_1\) does not depend on \(\kappa\) and \(\varepsilon\).

Proof. The existence of the solution is proved by constructing the regularizer \([8]\), and the estimate (20) is proved by the Schauder method.

We cover the domain \(\Omega\) with intervals \(K_{\delta_{1}}, K_{\delta_{2}}\) with common center \(\xi_i\). Let \(\zeta_i(x), \mu_i(x)\) be smooth shear functions subject to the covering domains \(\Omega\). Such that \(\zeta_i(x) = 1, \quad |x - \xi_i| \leq \delta\) and \(\zeta_i(x) = 0, \quad |x - \xi_i| \geq 2\delta\), and with properties \(\sum_{i} \zeta_i(x) \mu_i(x) = 1\) and \(\sum_{i} \zeta_i(x) \mu_i(x) = 1\) and 

\[\|D_m \zeta_i|, |D_m \mu_i| \leq C_{m,i} \delta^{-m}.\]

Denote the intervals as follows: \(\forall \rho \in \mathcal{K}_1\) intervals \(K_{\delta_{1}}^{(1)}\) contain a point \(\rho\), \(\forall \rho \in \mathcal{K}_2\) and \(i \in \mathcal{K}_3\) intervals \(K_{\delta_{2}}^{(1)}\) adjoin the boundary of the domain \(x = 0\) and \(x = l\) accordingly, with \(i \in \mathcal{K}_4\) intervals \(K_{\delta_{2}}^{(1)}\) are entirely contained in \(\Omega_1 \cup \Omega_2\).

Note that for the equations (1), (2) \(\chi(x - \rho_0) = 1, \quad i \in \mathcal{K}_1\) and \(\chi(x - \rho_0) = 0\) at \(i \in \mathcal{K}_2 \cup \mathcal{K}_3\), \(\delta_0 \leq \delta\).

We define the regularizer \(\mathcal{R}\) by formula

\[\mathcal{R} = \{\mathcal{R}_1, h, \mathcal{R}_2, h, \mathcal{R}_3, h, \mathcal{R}_4, h, \mathcal{R}_5, h\} = \left\{ \sum_{i \in \mathcal{K}_1} \mu_i(x)v_{1,i}(x,t), \sum_{i \in \mathcal{K}_2} \mu_i(x)v_{2,i}(x,t), \sum_{i \in \mathcal{K}_3} \mu_i(x)z_{1,i}(x,t), \sum_{i \in \mathcal{K}_4} \mu_i(x)z_{2,i}(x,t), \sum_{i \in \mathcal{K}_5} \mu_i(x) \psi_i(t) \right\},\]

where \(\mathcal{R} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_4\) are functions \(v_{j,i}(x,t), z_{j,i}(x,t), j = 1, 2, \psi_i(t)\) satisfy zero initial data and are defined as solutions model conjugation problem for \(i \in \mathcal{K}_1\), the first boundary value problem for \(i \in \mathcal{K}_2 \cup \mathcal{K}_3\) and Cauchy problem with \(i \in \mathcal{K}_4\).
Let  
\[ D^{(i)}_{j^2} = D^{(i)}_j \times (0,T), \quad K^{(i)}_{2j^2} = \left( K^{(i)}_{2j^2} \cap (\Omega_1 \cup \Omega_2) \right) \times (0,T). \]

1. Let  \( i \in \mathbb{K}_1 \). Perform a coordinate transformation  \( x = Y_i^{-1}(y); x = y + \rho_0 \). This conversion translates areas  \( \{ y \mid y < 0 \} \) and  \( x > \rho_0 \) into  \( \{ y \mid y > 0 \} \).

Set  \( \zeta_i(x) f_j(x, t), \zeta_i(x) g_j(x, t) |_{x = Y_i^{-1}(y)} = f_j(y, t), f_{j+2,i}(y, t), \zeta_i(x) p_j(t) |_{x = 0}, \)
\( \zeta_i(x) q_j(t) |_{x = 0} = p_{j,i}(t), q_{j,i}(t), \)
\( \zeta_i(x) \eta_j(t) |_{x = \rho_0} = \eta_{j,i}(t), \)
\( \zeta_i(x) \kappa g_j(t), \zeta_i(x) g_j(t), \zeta_i(x) \epsilon g_4(t) = g_{j,i}(t), \kappa g_{j,i}(t), g_{j,i}(t), \epsilon g_{4,i}(t) \)
and continue the functions  \( f_{j,i}, f_{j+2,i} \) zero in  \( D^{(i)}_{j^2} \).

We define the functions  \( v_{j,i}(y, t), z_{j,i}(y, t), j = 1,2, \psi_i(t) \) as a solution to the following conjugation problem:

\[ \frac{\partial}{\partial x} v_{j,i}(t) - a_j(\xi_i, 0) \frac{\partial^2}{\partial x^2} v_{j,i}(t) - a_j(\xi_i, 0) \chi(\xi_i) D \psi_i = f_{j,i}(y, t) \text{in} D^{(i)}_j, \]
\[ j = 1, 2, \]
\[ \frac{\partial}{\partial x} z_{j,i}(t) - a_j(\xi_i, 0) \frac{\partial^2}{\partial x^2} z_{j,i}(t) - \beta_j(\xi_i, 0) \chi(\xi_i) D \psi_i = f_{j+2,i}(y, t) \text{in} D^{(i)}_j, \]
\[ j = 1, 2, \]
(21)
\( (z_{j,i} - y_j(\xi_i, 0)v_{j,i}(t)) |_{y = 0} = \eta_{j,i}(t), j = 1, 2, \)
\[ \sum_{j=1}^2 \left[ |v_{j,i}(t)|_{D^{(i)}_j}^{(2)} + |z_{j,i}(t)|_{D^{(i)}_j}^{(2)} \right] + |\psi_i(t)|_{\sigma_j}^{(1+i/2)} \leq C_2 \sum_{j=1}^2 \left[ |f_{j,i}(t)|_{D^{(i)}_j}^{(2)} + |f_{j+2,i}(t)|_{D^{(i)}_j}^{(2)} + |\eta_{j,i}(t)|_{\sigma_j}^{(1+i/2)} + |g_{j,i}(t)|_{\sigma_j}^{(1+i/2)} \right] \]
\[ + \kappa |g_{j,i}(t)|_{\sigma_j}^{(1+i/2)} + |g_{3,i}(t)|_{\sigma_j}^{(1+i/2)} + \epsilon |g_{4,i}(t)|_{\sigma_j}^{(1+i/2)}, \]
(26)

where  \( C_2 \) does not depend on  \( \kappa \) and  \( \epsilon \).

The functions  \( v_{j,i}(y, t), z_{j,i}(y, t), j = 1, 2, \) are defined as a solution of the first boundary-value problems  \( i \in \mathbb{K}_2 \cup \mathbb{K}_3 \) and functions
\( v_{j,i}(y, t), z_{j,i}(y, t), j = 1, 2 \) with  \( i \in \mathbb{K}_4 \) are solution of Cauchy problem. Each of these problems under the conditions of the Theorem 2 has a unique solution and it is subject to estimates (27), (28) [8]
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\[ |v_{i,j}|^{(2i+1)}_{2\partial T} \leq C_{5} \xi_{i}(x) f_{j}^{(i)}_{\partial T}, \]
\[ |z_{i,j}|^{(2i+1)}_{2\partial T} \leq C_{6} \xi_{i}(x) f_{j}^{(i)}_{\partial T}, \]
for \( i, j \in \mathbb{N}_{4}, j = 1, 2. \) (28)

We introduce the norm \([1]\)
\[ \|w\|_{B(\Omega_{T})} = \max_{i} \|w_{i}\|_{B(\bar{\Omega}_{T})}, \]
\[ \|h\|_{H(\Omega_{T})} = \max_{j} \|h_{j}\|_{H(\bar{\Omega}_{T})}, \] (29)
where \( K_{2\partial T}^{(i)} = (K_{2\partial T}^{(i)} \cap (\Omega_{1} \cup \Omega_{2})) \times (0, T). \) The norms of \( \{w\}_{B(\Omega_{T})}, \{h\}_{H(\Omega_{T})} \) are defined by formulas (11), (12). Note that the norms (29) are equivalent to the norms \( \|w\|_{B(\Omega_{T})}, \|h\|_{H(\Omega_{T})} \).

**Lemma 2.** The operator \( \mathfrak{R} : H(\Omega_{T}) \rightarrow B(\Omega_{T}) \) is bounded: \( \|\mathfrak{R}h\|_{B(\Omega_{T})} \leq C_{7} \|h\|_{H(\Omega_{T})}. \)

Let us turn to the problem (1)-(9), which we recorded in operator form (19) \( Aw = h. \) Obviously, \( A : B(\Omega_{T}) \rightarrow H(\Omega_{T}). \)

**Lemma 3.** For any \( h \in H(\Omega_{T}), \) the equality \( A\mathfrak{R}h = h + Ph, \) takes place. Here \( Ph = \{P_{1}h, P_{2}h, P_{3}h, P_{4}h, 0, 0, 0, 0, P_{5}h, P_{6}h, P_{7}h, P_{8}h\}, \)
\( P_{j}h, P_{j+2}h, P_{4+j}h, j = 1, 2, 3, h = 1, 2, 3, \) contain lower terms or higher terms with small coefficients.

**Lemma 4.** Under the conditions of the Theorem 1 for \( t \leq T_{0} \) estimate
\[ \|Ph\|_{H(\Omega_{T})} \leq \alpha \|h\|_{H(\Omega_{T})} \] (30)
is fulfilled, where \( \alpha \in (0, 1). \)

**Lemma 5.** Under the conditions of the Theorem 2 there exists for \( t \leq T_{0} \) bounded right inverse operator \( A_{r}^{-1} = \mathfrak{R}(E + P)^{-1} : H(\Omega_{T}) \rightarrow B(\Omega_{T}), \)
where \( E \) is the unit operator.

**Proof.** We have the problem \( Aw = h. \) Substituting \( \mathfrak{R}h \) instead of \( w, \) we get \( A\mathfrak{R}h = h + Ph \equiv (E + P)h. \) Let \( (E + P)h = h_{1}, \) where \( h_{1} \in H(\Omega_{T}). \) According to the assessment (30) this equation has a unique solution \( h \in H(\Omega_{T}), \)
which is subject to estimate \( \|h\|_{H(\Omega_{T})} \leq \frac{1}{1-\alpha} \|h_{1}\|_{H(\Omega_{T})} \)
for anyone vector \( h_{1} \in H(\Omega_{T}). \) Then there is a limited the inverse operator \( (E + P)^{-1} \) in the space \( H(\Omega_{T}). \) Substituting \( h = (E + P)^{-1}h_{1} \) into the equation \( A\mathfrak{R}h = h_{1} \equiv h + Ph, \) we get the identity \( A\mathfrak{R}(E + P)^{-1}h_{1} = h_{1} \) for any \( h_{1} \in H(\Omega_{T}) \) or \( A\mathfrak{R}(E + P)^{-1} = E. \) According to the definition implies that the operator \( A \) has the right inverse of the bounded operator \( A_{r}^{-1} = \mathfrak{R}(E + P)^{-1}, \) and the problem \( Aw = h \) has the solution \( w = (v_{1}, v_{2}, z_{1}, z_{2}, \psi) \in B(\Omega_{T}) \) for any vector \( h \in H(\Omega_{T}). \)

We obtain an estimate for the solution of the problem (1)-(9) using the Schauder method. Consider the functions of \( v_{j}(x, t) = \mu_{j}(x) v_{j}(x, t), j, k, i, \) \( j = 1, 2, \psi_{i}(t) = \mu_{i}(x) \psi(t), \) which are defined in \( K_{2\partial T}^{(i)} \times (0, T_{0}) \) and extend them by zero outside this area. Depending on the location of the interval \( K_{2\partial T}^{(i)} \) in \( \Omega \) for functions \( v_{j}(x, t), \) \( j, k, i, j = 1, 2, \psi_{i}(t) \) from the problem (1)-(9) the model pairing problem, the first boundary value problem, and the Cauchy problem can be obtained.

Multiply parabolic equations and problem conditions (1)-(9) on the cutting function \( \mu_{i}(x). \) In the equations and conditions we will make transformations coordinates \( x = Y_{i}^{-1}(y); x = y + \rho_{0} \) as \( i \in \mathbb{N}_{4} \) for functions \( v_{j}(x, t) = v_{j}(x, t), \) \( z_{j}(x, t) = z_{j}(x, t), \) \( j = 1, 2, \psi_{i}(t) \)
we get the conjugation problem with zero initial data in \( D_{j}^{(i)}_{g}, j = 1, 2, \)
\[ \partial_t v'_{j,l} - a_j(\xi_t, 0) \partial_x^2 v'_{j,l} - a_j(\xi_t, 0) \chi(\xi_t) D_t \psi_{l} = f'_{j,l}(y, t) + Y_t^{-1}(y) F_{j,l}(x, t), \]

\[ \partial_t z'_{j,l} - a_{j+2}(\xi_t, 0) \partial_x^2 z'_{j,l} - b_j(\xi_t, 0) \chi(\xi_t) D_t \psi_{l} = f'_{j+2,l}(y, t) + Y_t^{-1}(y) F_{j,l}(x, t), \]

\[ (z'_{j,l} - y_j(\xi_t, 0) v'_{j,l})|_{y=0} = \eta_{j,l}(t) + Y_t^{-1}(y) H_{j,l}(x, t), \quad j = 1, 2, \] (31)

\[ (\lambda_1(\xi_t, 0) \partial_y v'_{j,l} - \kappa \lambda_2(\xi_t, 0) \partial_y z'_{j,l})|_{y=0} = g_{1,l}(t) + \kappa g_{2,l}(t) + Y_t^{-1}(y) H_{3,l}(x, t), \]

\[ (k_1(\xi_t, 0) \partial_x z'_{j,l} - \varepsilon k_2(\xi_t, 0) \partial_x z'_{j,l})|_{y=0} = g_{3,l}(t) + \varepsilon g_{4,l}(t) + Y_t^{-1}(y) H_{4,l}(x, t), \]

where

\[ f'_{j,l}(y, t) = f_{j,l}(x, t)|_{x=\gamma_t^{-1}(y)}, \]

\[ f'_{j+2,l}(y, t) = f_{j+2,l}(x, t)|_{x=\gamma_t^{-1}(y)}, \]

\[ F_{j,l}(x, t), \quad j = 1, 2, H_{j,l}(x, t), \]

\[ H_{3,l}(x, t), \quad H_{4,l}(x, t), \]

contain smallest coefficients or leading coefficients with small coefficients.

The conjugation problem (31), according to Theorem 2, uniquely solvable and for it's solution estimate (26) holds. Therefore, solution of the problem (31) obeys the inequality

\[ \sum_{j=1}^{2} \left( |v'_{j,l}|^{(2+\alpha)}_{K_{2\delta}^{(i)}} + |z'_{j,l}|^{(2+\alpha)}_{K_{2\delta}^{(i)}} \right) + |\psi_{l}|^{(1+\alpha)}_{\sigma T} \leq C_\beta \sum_{j=1}^{2} \left( |f'_{j,l}|^{(i)}_{K_{2\delta}^{(i)}} + |f'_{j+2,l}|^{(i)}_{K_{2\delta}^{(i)}} + |\eta_{j,l}|^{(1+\alpha)}_{\sigma T} + |g_{1,l}|^{(1+\alpha)}_{\sigma T} + \right. \]

\[ + \kappa |g_{2,l}|^{(1+\alpha)}_{\sigma T} + |g_{3,l}|^{(1+\alpha)}_{\sigma T} + \kappa |g_{4,l}|^{(1+\alpha)}_{\sigma T} + |F_{j,l}|^{(i)}_{K_{2\delta}^{(i)}} + |F_{j+2,l}|^{(i)}_{K_{2\delta}^{(i)}} + \]

\[ + |H_{j,l}(x, t)|^{(1+\alpha)}_{\sigma T} + |H_{j+2,l}(x, t)|^{(1+\alpha)}_{\sigma T} \right), \quad j = 1, 2, \] (32)

where \( C_\beta \) does not depend on \( \kappa \) and \( \varepsilon \).

We also get the functions \( v'_{j,l}(y, t), \)

\( z'_{j,l}(y, t), \quad j = 1, 2 \) as solutions of the first boundary value problems for \( i \in \Phi_2 \cup \Phi_4 \), and functions \( v_{j,l}(x, t), z_{j,l}(x, t), j = 1, 2 \) for \( i \in \Phi_4 \) as the is solution of the Cauchy problems. Based on [1], the first boundary problem and the Cauchy problem are uniquely solvable. To solve the first boundary problem and the problem Cauchy valid estimates similar to those estimated, (27), (28) and (32), as well as similar functions arising in right part of the equations of the first boundary value problems and the Cauchy problem.

The norms of the functions \( F_{j,l}(x, t), \)

\( F_{j+2,l}(x, t), \)

\( H_{j,l}(x, t), \)

\( H_{j+2,l}(x, t), \quad j = 1, 2, \) are estimated the same way as with the proof of Lemma 3 norms of operators \( P_i h, P_{j+2} h, P_{4+j} h, j = 1, 2, P_7 h, P_8 h. \) Moreover, if note that in the interval \( K_8^{(i)} \) the cutoff function \( \mu_i(x) = 1 \), then in \( K_8^{(i)} \times (0, T) \),

\[ v_{j,l} := \mu_i v_j = v_j, z_{j,l} := \mu_i z_j = z_j, \psi_{l} := \mu_i \psi = \psi. \]

Using estimates of solutions of the conjugation problem (32), the first boundary problems and Cauchy problems we will have inequality

\[ \| \mu_i w \|_{B(K_{2\delta}^{(i)})} \leq C_9 \| \mu_i h \|_{H(K_{2\delta}^{(i)})} + \right. \]

\[ + \alpha_1 \| \mu_i w \|_{B(K_{2\delta}^{(i)})}, \]

\[ K_{2\delta}^{(i)} = \left( K_2^{(i)} \cap \Omega_1 \cap \Omega_2 \right) \times (0, t), \alpha_1 \in (0, 1). \]

Hence, we get

\[ \| \mu_i w \|_{B(K_{2\delta}^{(i)})} \leq \frac{C_9}{1 - \alpha_1} \| \mu_i h \|_{H(K_{2\delta}^{(i)})}, \]

\[ \alpha_1 \in (0, 1), \] (33)

where \( K_{2\delta}^{(i)} = \left( K_2^{(i)} \cap \Omega_4 \cap \Omega_2 \right) \times (0, t). \)

We proceed in the inequality (33) to the supremum on \( i \), taking into account the definitions of the norms \( \{ w \}_{i}^{H(\alpha_T)} \) and \( \{ h \}_{H(\alpha_T)} \) in (29), as a result get anestimate
\{w\}_{B(\alpha_{T_1})} \leq C_{10}(h)_{H(\alpha_{T_1})}.

From this inequality, by virtue of the equivalence of norms \{w\}_{B(\alpha_{T_1})}, (h)_{H(\alpha_{T_1})} and \|w\|_{B(\alpha_{T_1})}, \|h\|_{H(\alpha_{T_1})}, follows the evaluation of the solution to the problem (1)–(9)

\|w\|_{B(\alpha_{T_1})} \leq C_{11} \|h\|_{H(\alpha_{T_1})}. \quad (34)

The problem (1)–(9) is linear problem. The uniqueness of the solution follows from the evaluation (34). We proved the existence and uniqueness of the solution of the problem (1)–(9) for \(t \leq \min(T_0, T_1)\). Continuing the solution by \(t\) as in [8], we obtain Theorem 1 for \(T > 0\).

References

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