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EXPLICIT MODEL FOR SURFACE WAVES IN A PRE-STRESSED, COMPRESSIBLE ELASTIC HALF-SPACE

Abstract. The paper is concerned with the derivation of the hyperbolic-elliptic asymptotic model for surface wave in a pre-stressed, compressible, elastic half-space, within the framework of plane-strain assumption. The consideration extends the existing methodology of asymptotic theories for Rayleigh and Rayleigh-type waves induced by surface/edge loading, and oriented to extraction of the contribution of studied waves to the overall dynamic response. The methodology relies on the slow-time perturbation around the eigensolution, or, equivalently, accounting for the contribution of the poles of the studied wave. As a result, the vector problem of elasticity is reduced to a scalar one for the scaled Laplace equation in terms of the auxiliary function, with the boundary condition is formulated as a hyperbolic equation with the forcing terms. Moreover, hyperbolic equations for surface displacements are also presented. Scalar hyperbolic equations for surface displacements could potentially be beneficial for further development of methods of non-destructive evaluation.

Key words: surface wave; pre-stressed, compressible, elastic half-space; Rayleigh and Rayleigh-type waves.

Introduction and Literature Review

Mathematical modelling of dynamic problems of elasticity related to propagation of surface waves is an important problem having various applications in modern engineering and technology, including in particular areas of seismic protection, non-destructive testing, development of high-speed railway transport, etc., see e.g. [1-3] and references therein.

Studies of elastic surface waves originate from the classical work of Lord Rayleigh [4], followed by numerous contributions to the subject, see for example [5-8] to name a few. One of the important sub-areas is associated with propagation of surface waves in pre-stressed media [9,10], which becomes especially relevant for more accurate modelling of seismic vibrations in the near-surface domain. Some more recent advances in the area of Rayleigh wave include waves with transverse structure [11], representation through quasi-particles [12], as well as reciprocity approach [13].

A prospective methodology of hyperbolic-elliptic asymptotic models for surface waves (induced by prescribed surface loading) oriented to surface waves only has been developed in the last decade, see e.g. [14, 15] and references therein. This formulation relies on the representation of a surface wave field in terms of a single harmonic function [5, 16], with the decay over the interior governed by the Laplace equation. At the same time, the wave propagation is described by a hyperbolic equation on the surface, with the loading terms appearing in the right hand side. The results of the reduced model prove to be especially relevant in dynamic problems of elasticity, when the Rayleigh wave dominates, in particular, in the near-resonant regimes of moving loads, see e.g. [17]. The approach has also been extended to a special case of anisotropy associated with attenuation without oscillations in [18], as well as pre-stressed incompressible half-space [19]. A parallel parabolic-elliptic formulation for dispersive bending edge Rayleigh-type waves has been presented in [20]. Other recent developments include composite models for dispersion of waves in an elastic layer [21], application of the formulation to seismic meta-surfaces [22], as well as coated half-space with clamped surface and relatively soft coating layer [23].

In this paper, we extend the previous considerations to a pre-stressed compressible elastic half-space within the plane strain assumption, complementing the results in [19]. It is known that incorporation of compressibility may enrich the dynamics behaviour of pre-stressed material, see e.g. [24]. First, a slow-time perturbation procedure is constructed, revealing at leading order the eigensolution for surface waves, including the scaled Laplace equation for the auxiliary function following from the fourth order elliptic equation for one of the displacements. Then, at next order correction, a hyperbolic equation on the surface is established, implying hyperbolic equations for surface displacements. It is also noticed that in absence of the pre-stress, the results reduce to a known formulation for an isotropic elastic half-space. The results are also discussed within the framework of the known asymptotic models for Rayleigh and Rayleigh-type waves for other material properties.

Materials and Methods

Statement of the problem

Consider an elastic, isotropic, compressible body in three-dimensional Cartesian coordinate system in its natural unstressed state $\mathcal{B}_u$. The body is then subjected to a homogeneous static deformation $x_i\left(X_A\right)$, thus transforming to a finitely deformed equilibrium configuration $\mathcal{B}_e$. Then, small-amplitude motion $u_i\left(x_{j,}t\right)$ is super-imposed over $\mathcal{B}_e$, resulting in the current configuration $\mathcal{B}_t$. Thus, the current position vector is given by

$$\tilde{x}_i\left(X_A,t\right)=x_i\left(X_A\right)+u_i\left(x_{j,}t\right). \qquad (2.1)$$

Consider the elastic half-space $x_2 \geq 0$, with the coordinate axis directed along the principal directions of primary deformation. Throughout the paper we are focusing on two-dimensional super-imposed motions for which $u_3 = 0$, and u_j (j = 1, 2) are independent of x_3.

Following [10], the governing equations of motion may be written as

$$\begin{gathered}
A_{1111}\frac{\partial^2 u_1}{\partial x_1^2}+A_{2121}\frac{\partial^2 u_1}{\partial x_2^2}+ \\
\left(A_{1122}+A_{1221}\right)\frac{\partial^2 u_2}{\partial x_1 \partial x_2}=\rho_e\frac{\partial^2 u_1}{\partial t^2}, \\
A_{1212}\frac{\partial^2 u_2}{\partial x_1^2}+A_{2222}\frac{\partial^2 u_2}{\partial x_2^2}+ \\
\left(A_{1122}+A_{1221}\right)\frac{\partial^2 u_1}{\partial x_1 \partial x_2}=\rho_e\frac{\partial^2 u_2}{\partial t^2},
\end{gathered} \qquad (2.2)$$

where

$$A_{ijkl}=J^{-1}\bar{F}_{iA}\bar{F}_{kC}\left.\frac{\partial^2 W}{\partial F_{jA}\partial F_{lC}}\right|_{F=\bar{F}}, \qquad (2.3)$$

are the components of the fourth-order elasticity tensor, with its non-zero elements defined by (see [25] for more detail)

$$\begin{aligned}
&A_{iijj}=J^{-1}\lambda_i\lambda_j\frac{\partial^2 W}{\partial\lambda_i\partial\lambda_j}, \\
&A_{ijij}=\begin{cases}\left(\lambda_i\dfrac{\partial W}{\partial\lambda_i}-\lambda_j\dfrac{\partial W}{\partial\lambda_j}\right)\dfrac{\lambda_i^2}{J\left(\lambda_i^2-\lambda_j^2\right)}, & i\neq j,\ \ \lambda_i\neq\lambda_j, \\ \dfrac{1}{2}\left(A_{iiii}-A_{iijj}+\dfrac{\lambda_i}{J}\dfrac{\partial W}{\partial\lambda_i}\right), & i\neq j,\ \ \lambda_i\neq\lambda_j,\end{cases} \\
&A_{ijji}=A_{jiij}=A_{ijij}-\sigma_i.
\end{aligned} \qquad (2.4)$$

In the above W is the strain-energy function,

$$W=W\left(I_1,I_2,J\right), \qquad (2.5)$$

depending on the invariants

$$I_1=\operatorname{tr}\mathrm{C},\quad I_2=\frac{1}{2}\left[\left(\operatorname{tr}\mathrm{C}\right)^2-\operatorname{tr}\left(\mathrm{C}^2\right)\right],\ \text{and}$$

$$J=\det F=\lambda_1\lambda_2\lambda_3=\frac{\rho_u}{\rho_e}, \qquad (2.6)$$

with F and $\overline{F}$ being the gradient deformation tensor associated with the mapping from $\mathcal{B}_u$ to $\mathcal{B}_t$, and $\mathcal{B}_u$ to $\mathcal{B}_e$, respectively, $C = FF^T$ denoting the right Cauchy-Green tensor, λ_1, λ_2 and λ_3 conventionally denoting the principal stretches, p_u and p_e stand for the material density in the configurations $\mathcal{B}_u$ and $\mathcal{B}_e$, respectively, whereas the principal Cauchy stresses σ_i are

$$\sigma_i = \frac{\lambda_i}{J}\frac{\partial W}{\partial \lambda_i}. \tag{2.7}$$

The linearised measure of the incremental traction is given by

$$\tau_j^{(n)} = A_{ijkl}\frac{\partial u_l}{\partial x_k} n_i, \tag{2.8}$$

for more detail see e.g. [24]. The boundary conditions at the surface $x_2 = 0$ are prescribed in the form of specified traction components, i.e.

$$A_{2121}\frac{\partial u_1}{\partial x_2} + \left(A_{2121} - \sigma_2\right)\frac{\partial u_2}{\partial x_1} = P_1, \quad A_{1122}\frac{\partial u_1}{\partial x_1} + A_{2222}\frac{\partial u_2}{\partial x_2} = P_2. \tag{2.9}$$

It should be noted that the material parameters are chosen within the range of stability of the material, i.e. strong ellipticity conditions are satisfied, see [10].

1.1. Explicit model for surface wave field

Following the procedure in [18], the following ansatz for displacement components may be adopted

$$u_i = u_i\left(\xi, x_2, \tau\right), \quad i = 1, 2, \tag{2.10}$$

where $\xi = x_1 - ct$, and $\tau = \varepsilon t$ is slow time, with the physical meaning of the small parameter being the deviation of the phase speed from the surface wave speed. Then, the displacement components u_i are expanded as asymptotic series

$$u_i = \frac{1}{\varepsilon}\left(u_{i0} + \varepsilon u_{i1} + ...\right), \quad i = 1, 2. \tag{2.11}$$

The leading order problem is then given by

$$\begin{aligned} &A_{2121}\frac{\partial^2 u_{10}}{\partial x_2^2} + \left(A_{1111} - \hat{c}^2\right)\frac{\partial^2 u_{10}}{\partial \xi^2} + \left(A_{1122} + A_{1221}\right)\frac{\partial^2 u_{20}}{\partial x_2 \partial \xi} = 0, \\ &A_{2222}\frac{\partial^2 u_{20}}{\partial x_2^2} + \left(A_{1212} - \hat{c}^2\right)\frac{\partial^2 u_{20}}{\partial \xi^2} + \left(A_{1122} + A_{1221}\right)\frac{\partial^2 u_{10}}{\partial x_2 \partial \xi} = 0, \end{aligned} \tag{2.12}$$

where $\hat{c}^2 = \rho_e c^2$, subject to the boundary conditions

$$\begin{aligned} &A_{2121}\frac{\partial u_{10}}{\partial x_2} + \left(A_{2121} - \sigma_2\right)\frac{\partial u_{20}}{\partial \xi} = 0, \\ &A_{1122}\frac{\partial u_{10}}{\partial \xi} + A_{2222}\frac{\partial u_{20}}{\partial x_2} = 0. \end{aligned} \tag{2.13}$$

Equations (2.12) may be transformed to a single fourth order PDE in respect of one displacement component, say, u_1, giving in operator form

$$\left[\partial_{22} + k_1^2 \partial_{\xi\xi}\right]\left[\partial_{22} + k_2^2 \partial_{\xi\xi}\right] u_{10} = 0, \tag{2.14}$$

where

$$k_1^2 + k_2^2 = \frac{\alpha_{22}\left(\alpha_{11} - \hat{c}^2\right) + \gamma_2\left(\gamma_1 - \hat{c}^2\right) - \chi^2}{\alpha_{22}\gamma_2},$$

$$k_1^2 k_2^2 = \frac{\left(\alpha_{11} - \hat{c}^2\right)\left(\gamma_1 - \hat{c}^2\right)}{\alpha_{22}\gamma_2},$$

$$\alpha_{ij} = A_{iijj}, \quad \gamma_m = A_{mjmj},$$

$$\chi = \alpha_{12} + \gamma_2 - \sigma_2, \quad i,j,m = 1,2; \quad j \neq m.$$

The solution of (2.14) is given by

$$u_{10}\left(\xi, x_2, \tau\right) = \\ = \varphi_{10}\left(\xi, k_1 x_2, \tau\right) + \varphi_{20}\left(\xi, k_2 x_2, \tau\right), \quad (2.15)$$

where φ_{10} and φ_{20} are arbitrary functions, harmonic in the first two arguments. Using (2.12) along with the Cauchy-Riemann identities, we deduce

$$u_{20}\left(\xi, x_2, \tau\right) = f\left(k_1\right)\varphi_{10}^{*}\left(\xi, k_1 x_2, \tau\right) + \\ + f\left(k_2\right)\varphi_{20}^{*}\left(\xi, k_2 x_2, \tau\right), \quad (2.16)$$

where the asterisk denotes the harmonic conjugate, and

$$f(k_i) = \frac{\gamma_2 k_i^2 - \left(\alpha_{11} - \hat{c}^2\right)}{k_i \chi}, \quad i = 1,2. \quad (2.17)$$

On substituting (2.15) and (2.16) into (1.62), the solvability of the system implies

$$\sqrt{\frac{\gamma_2\left(\alpha_{11} - \hat{c}^2\right)\left(\gamma_1 - \hat{c}^2\right)}{\alpha_{22}}}\left[\alpha_{22}\left(\alpha_{11} - \hat{c}^2\right) - \alpha_{12}^2\right] + \left(\alpha_{11} - \hat{c}^2\right)\left[\gamma_2\left(\gamma_1 - \hat{c}^2\right) - \left(\gamma_2 - \sigma_2\right)^2\right] = 0, \quad (2.18)$$

which coincides with the surface wave speed equation obtained in [10], hence the speed c in the definition of the moving co-ordinate ξ coincides with the surface wave speed c_R, being the unique root of (2.18). An addition, a relation between the functions φ_{10} and φ_{20} is established

$$\varphi_{20}\left(\xi, k_2 x_2, \tau\right) = -\frac{g(k_1)}{g(k_2)}\varphi_{10}\left(\xi, k_2 x_2, \tau\right), \quad (2.19)$$

with

$$g(k_i) = \gamma_2 \alpha_{12} k_i + \frac{\left(\gamma_2 - \sigma_2\right)\left(\alpha_{11} - \hat{c}^2\right)}{k_i}, \quad i = 1,2. \quad (2.20)$$

Thus, the leading order displacements u_{10} and u_{20} are expressed in terms of a single harmonic function as

$$u_{10}\left(\xi, x_2, \tau\right) = \varphi_{10}\left(\xi, k_1 x_2, \tau\right) - \frac{g(k_1)}{g(k_2)}\varphi_{10}\left(\xi, k_2 x_2, \tau\right), \quad (2.21)$$

$$u_{20}\left(\xi, x_2, \tau\right) = f\left(k_1\right)\varphi_{10}^{*}\left(\xi, k_1 x_2, \tau\right) - \frac{f\left(k_2\right) g(k_1)}{g(k_2)}\varphi_{10}^{*}\left(\xi, k_2 x_2, \tau\right). \quad (2.22)$$

Next order problem may now be formulated, involving the equations of motion

$$A_{2121}\frac{\partial^2 u_{11}}{\partial x_2^2} + \left(A_{1111} - \hat{c}^2\right)\frac{\partial^2 u_{11}}{\partial \xi^2} + \left(A_{1122} + A_{1221}\right)\frac{\partial^2 u_{21}}{\partial x_2 \partial \xi} = -2\rho_e c_R \frac{\partial^2 u_{10}}{\partial \xi \partial \tau}, \\ A_{2222}\frac{\partial^2 u_{21}}{\partial x_2^2} + \left(A_{1212} - \hat{c}^2\right)\frac{\partial^2 u_{21}}{\partial \xi^2} + \left(A_{1122} + A_{1221}\right)\frac{\partial^2 u_{11}}{\partial x_2 \partial \xi} = -2\rho_e c_R \frac{\partial^2 u_{20}}{\partial \xi \partial \tau}, \quad (2.23)$$

subject to the boundary conditions

$$A_{2121}\frac{\partial u_{11}}{\partial x_2}+\left(A_{2121}-\sigma_2\right)\frac{\partial u_{21}}{\partial \xi}=P_1,$$
$$A_{1122}\frac{\partial u_{11}}{\partial \xi}+A_{2222}\frac{\partial u_{21}}{\partial x_2}=P_2. \quad (2.24)$$

The solutions of (2.23) may be found in a similar way to that employed in [18]. Furthermore, on substituting the latter into (2.24), from the solvability on the boundary $x_2=0$, a hyperbolic equation may be deduced for the auxiliary function $\varphi_1=\varepsilon^{-1}\varphi_{10}$, namely

$$\varphi_{1,11}-\frac{1}{c_R^2}\varphi_{1,tt}=A_{R1}P_{1,1}^{*}+A_{R2}P_{2,1} \ \text{(at } x_2=0\text{)}. \quad (2.25)$$

Here the asterisk may be interpreted in the sense of the Hilbert transform, and the material constants A_{R1} and A_{R2} are given by

$$A_{Ri}=\frac{S_i}{\rho_e c_R^2 \vartheta R_1},\ i=1,2, \quad (2.26)$$

with

$$R_1=\eta\sqrt{\alpha_{22}\gamma_2}+\left(\alpha_{11}+\gamma_1-2\rho_e c_R^2\right)\left[\gamma_2+\frac{1}{2\eta}\sqrt{\frac{\gamma_2}{\alpha_{22}}}\left(\alpha_{22}(\alpha_{11}-\rho_e c_R^2)-\alpha_{12}^2\right)\right]-\left(\gamma_2-\sigma_2\right)^2,$$
$$S_1=\eta\sqrt{\alpha_{11}\alpha_{22}+\gamma_1\gamma_2-\beta^2+\left(\alpha_{22}+\gamma_2\right)\rho_e c_R^2+2\eta\sqrt{\alpha_{22}\gamma_2}},\qquad \vartheta=1-\frac{g(k_1)}{g(k_2)},$$
$$S_2=\left(\gamma_2-\sigma_2\right)\left(\alpha_{11}-\rho_e c_R^2\right)-\eta\alpha_{12}\sqrt{\frac{\gamma_2}{\alpha_{22}}},\qquad \eta=\sqrt{\left(\gamma_1-\rho_e c_R^2\right)\left(\alpha_{11}-\rho_e c_R^2\right)}.$$

Note, the hyperbolic equation (2.24) serves as a boundary condition to the elliptic equation

$$\varphi_{1,22}+k_1^2\varphi_{1,11}=0, \quad (2.27)$$

following from (2.14) and (2.15).

Results and Discussion

Thus, the hyperbolic-elliptic model for surface wave in a pre-stressed compressible elastic half-space under the plane-strain assumption has been derived, comprised of the hyperbolic equation (2.25) and elliptic equation (2.27). It should be noted that once the auxiliary function φ_1 is determined, the displacements follow from the expressions of (2.21) and (2.22). Moreover, displacements satisfy the following hyperbolic equations on the surface

$$x_2=0\quad u_{1,11}-\frac{1}{c_R^2}u_{1,tt}=\vartheta\left(A_{R1}P_{1,1}^{*}+A_{R2}P_{2,1}\right) \quad (3.1)$$

and

$$u_{2,11}-\frac{1}{c_R^2}u_{2,tt}=\left(f(k_1)-\frac{f(k_2)g(k_1)}{g(k_2)}\right)\left(-A_{R1}P_{1,1}+A_{R2}P_{2,1}^{*}\right). \quad (3.2)$$

It may be shown that in case of no pre-stress the obtained results simplify to the known results for classical Rayleigh waves in linearly elastic, isotropic media (cf., for example, equation (3.1) in absence of tangential load (P_1 = 0) with equation (98) in [15]).

Another observation which may be made is related to similarity of the derivation procedure between the orthorhombic case in [18] and the current problem, which is highlighting once again the formal parallels between anisotropy and pre-stress, having although an important difference related to symmetry/non-symmetry of the stress tensor.

Another remark can be made regarding the similarity of the slow-time perturbation procedures in the current problem with that for the dispersive bending edge wave on a semi-infinite Kirchhoff plate, presented in [20]. Indeed, both of the cases are dealing with the scaled bi-harmonic equation, in respect of the displacements. The auxiliary function φ_1 may be interpreted within the sense of a partial potential decomposition, since its analogue in the isotropic case would be a derivative of the longitudinal Lamé potential.

Conclusion

A hyperbolic-elliptic model for Rayleigh-type wave induced by prescribed load on the surface of a pre-stressed, compressible, elastic half-space has been formulated in terms of a single plane harmonic function. The decay over the interior is governed by a scaled Laplace equation, whereas wave propagation is modelled by a hyperbolic equation. The results complement existing ones for isotropic media [15], as well as particular type of orthorhombic media [18] and pre-stressed incompressible media [19]. Scalar hyperbolic equations for surface displacements could potentially be beneficial for further development of methods of non-destructive evaluation.

The advantage of the obtained formulation is clearly associated with a reduction of the vector problem of elasticity to a scalar problem for a Laplace equation, thus opening the path to a number of analytical solutions for prescribed forms of surface loading. At the same time, it is emphasised that the model is only accounting for surface wave contribution and would therefore be efficient in situations when the surface wave field is dominant, and the contribution of the bulk waves is negligible. Examples of such behaviour include the far-field analysis or near-resonant regimes of the moving load. Moreover, the proposed approach may be further developed for seismic meta-surfaces, see e.g. [22], [26], as well as for layered half-space [15], [23]. Finally, we mention a less obvious generalisation to inhomogeneous media, see e.g. [27].

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