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Soliton surface associated with the oriented associativity equation for n=3 case

Abstract. This paper describes the soliton surfaces approach to the Oriented Associativity equation for n=3 case. The equation of associativity arose from the 2D topological field theory. We constructed the surface associated with the Oriented Associativity equation for n=3 case equations using Sym-Tafel formula, which gives a connection between the classical geometry of manifolds immersed in R^m and the theory of solitons. The so-called Sym-Tafel formula simplifies the explicit reconstruction of the surface from the knowledge of its fundamental forms, unifies various integrable nonlinearities and enables one to apply powerful methods of the soliton theory to geometrical problems. The soliton surfaces approach is very useful in construction of the so-called integrable geometries. Indeed, any class of soliton surfaces is integrable. Geometrical objects associated with soliton surfaces (tangent vectors, normal vectors, foliations by curves etc.) usually can be identified with solutions to some nonlinear models (spins, chiral models, strings, vortices etc.). We consider the geometry of surfaces immersed in Euclidean spaces. The Oriented Associativity equation plays a fundamental role in the theory of integrable systems. Such soliton surfaces for the Oriented Associativity equation for n=3 case are considered, and first and second fundamental forms of soliton surfaces are found for this case. Also, we study an area of surfaces for the Oriented Associativity equation for n=3 case.
Key words: the Oriented Associativity equation, nonlinear equation, the Lax pair, first and second fundamental forms, soliton surfaces, area of surfaces.

Introduction

The equation of associativity relation for genus 0 Gromov-Witten (GW) invariants completely solves the classical problem of enumerating complex rational curves in the complex projective space Pn [1]. For genus-0 GW-theory, the associativity of quantum cohomology, which is equivalent to equation of associativity, led to Kontsevich's solution to the classical problem of counting degree d rational curves passing through 3d − 1 general points in P2 [2]. A system of PDE, called open WDVV, that constrains the bulkdeformed superpotential and associated open GW invariants of a Lagrangian submanifold $L \subset X$ with a bounding chain [3]. In this paper we shall consider so-called nonlinear partial differential equations of associativity in 2D topological field theories (see [4-7]) and give their description as integrable nondiagonalizable weakly nonlinear systems of hydrodynamic type. For systems of this type corresponding general differential geometric theory of integrability connected with Poisson structures of hydrodynamic type can be developed. For an arbitrary solution of the open equation of associativity, satisfying a certain homogeneity condition, constructed a descendent potential in genus 0 [8]. For any mechanics, given by the metric and the third order Codazzi tensor, it is possible to obtain the superfield Lagrangian [9] by solving a simple differential equation. Universal algebraic structure, closely related with that of the equation of associativity, govern quantum correlation functions of every quantum field theory [10]. Topological approach provides a general framework for lifting relations via morphisms between not necessarily orientable spaces [11]. For isotropic (so(n)-invariant) spaces provided admissible prepotentials for any solution to the curved equation of associativity [12]. For every flat-space equation of associativity solution subject to a simple constraint provided a curved-space solution on any isotropic space, in terms of the rotationally invariant conformal factor of the metric [13]. Flat structure was introduced by K. Saito and his collaborators at the end of 1970's. Independently the equation of associativity arose from the 2D topological field theory. B. Dubrovin

unified these two notions as Frobenius manifold structure [14]. The concepts of Frobenius manifold and Lenard complex must be strictly related. They provides two ways of looking at the same object from different perspectives and by using different geometrical structures [15]. In paper [16]compared two different geometrical interpretations of the equation of associativity of 2D topological field theory. The first is the classical interpretation proposed by Boris Dubrovin, based on the concept of Frobenius manifold. The second is a novel interpretation, based on the concept of Lenard complex on a Haantjes manifold. In paper [17]. determined correlators of topological quantum field theories and provided explicit solutions to the equation of associativity.

The equation of associativity, in general, have the following form [4,18]:

$$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^p}\eta^{pq}\frac{\partial^3 F}{\partial t^q \partial t^k \partial t^r} = \\ = \frac{\partial^3 F}{\partial t^j \partial t^k \partial t^p}\eta^{pq}\frac{\partial^3 F}{\partial t^i \partial t^q \partial t^r},$$

$$\forall i, j, k, r \in \{1, ..., n\},$$

where F is a prepotential, η is a metric.

The Associativity equation, or WDVV equation, plays a fundamental role in the geometric theory of Integrable Systems. Its solutions define Frobenius manifolds, which correspond to integrable systems; Frobenius manifolds also play a fundamental role in the theory of quantum cohomology and Gromov - Witten invariants. These connections were shown by B. Dubrovin in his seminal paper [19].

In this paper we shall consider so-called nonlinear partial differential equations of associativity.

The nonlinear partial differential system of equations:

$$\frac{\partial^2 c^i}{\partial a^j \partial a^m}\frac{\partial^2 c^m}{\partial a^k \partial a^n} = \frac{\partial^2 c^i}{\partial a^k \partial a^m}\frac{\partial^2 c^m}{\partial a^j \partial a^n} \qquad (1)$$

on n unknown functions (c^i) of n independent variables (a^i) was introduced in [19] as a generalization of the Associativity equations. Its solution define F-manifolds, which are still in correspondence with integrable systems. The far-reaching implication of this generalization are an active subject of study: flat and bi-flat F-manifolds have interesting connections with Painlevé equations [20-22]; see also the papers [23-24] devoted to coisotropic deformations. We call the system (1) the Oriented Associativity equation.

Soliton surface associated with the Oriented Associativity equation for $n = 3$ case

The Oriented Associativity equation admits the scalar linear spectral problem

$$\frac{\partial^2 h}{\partial a^i \partial a^j} = \lambda \frac{\partial^2 c^m}{\partial a^i \partial a^j}\frac{\partial h}{\partial a^m}$$

(see, for instance, [25]) that ensure that the equation is integrable as it provides a Lax pair.

We observe that the Associativity equation [26] can be obtained from (1) by the potential reduction $c^i = \eta^{im}\partial F a^m$, where η^{ks} is a constant nondegenerate symmetric matrix.

The system of quadratic equations [26]

$$\begin{aligned} u_{xx} &= v_{xt}w_{xx} - v_{xx}w_{xt} + w_{xt}^2 - w_{xx}w_{tt}, \\ u_{xt} &= v_{tt}w_{xx} - v_{xt}w_{xt}, \\ u_{tt} &= v_{xt}^2 - v_{xx}v_{tt} + v_{tt}w_{xt} - v_{xt}w_{tt}, \end{aligned} \qquad (2)$$

is the Oriented Associativity equation in the simplest case $n = 3$. It is endowed by the Lax pair

$$\begin{pmatrix} \psi \\ \psi_1 \\ \psi_2 \end{pmatrix}_x = \lambda \begin{pmatrix} 0 & 1 & 0 \\ u_{xx} & v_{xx} & w_{xx} \\ u_{xt} & v_{xt} & w_{xt} \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\begin{pmatrix} \psi \\ \psi_1 \\ \psi_2 \end{pmatrix}_t = \lambda \begin{pmatrix} 0 & 0 & 1 \\ u_{xt} & v_{xt} & w_{xt} \\ u_{tt} & v_{tt} & w_{tt} \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \\ \psi_2 \end{pmatrix}$$

In the following sections we work with the system (2).

First fundamental form of a surface

The corresponding Lax pair for the Oriented Associativity equation for $n = 3$ case to the system (2) is given by

$$\Phi_x = U\Phi \quad (3)$$

$$\Phi_t = V\Phi \quad (4)$$

where $U = \lambda A$ and $V = \lambda B$. Here A and B matrices defined as follows:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ u_{xx} & v_{xx} & w_{xx} \\ u_{xt} & v_{xt} & w_{xt} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ u_{xt} & v_{xt} & w_{xt} \\ u_{tt} & v_{tt} & w_{tt} \end{pmatrix}$$

Geometrical objects associated with soliton surfaces usually can be identified with solutions to some nonlinear models [27-28]. The scalar square of the total differential dr of the radius-vector of the current point of a surface is called the first fundamental form I of the surface [29]:

$$I = \mathrm{d}r^2,$$

In expanded form, it is recorded as

$$I = r_x^2 \mathrm{d}x^2 + 2r_x r_t \mathrm{d}x\mathrm{d}t + r_t^2 \mathrm{d}t^2, \quad (6)$$

where x and t are the curvatures.

To construct the surface, we now use the Sym-Tafel formula [30]. It has the form

$$r = \Phi^{-1}\Phi_\lambda,$$

where $r = \sum r_j \sigma_j$ is the matrix form of the position vector of the surface, Φ is a solution of the equations (3)-(4). We have

$$r_x = \Phi^{-1} U_\lambda \Phi, \quad r_t = \Phi^{-1} V_\lambda \Phi.$$

In terms of the Lax representation, equation (6) will be rewritten as follows:

$$I = \frac{1}{2}\left(\mathrm{tr}(U_\lambda^2)dx^2 + 2\mathrm{tr}(U_\lambda V_\lambda)dxdt + \mathrm{tr}(V_\lambda^2)dt^2\right). \quad (7)$$

We now turn to finding the first fundamental form of soliton surface for the Oriented Associativity equation for n = 3 case to the system (2)

$$\mathrm{tr}(U_\lambda^2) = v_{xx}^2 + w_{xt}^2 + 2\left(u_{xx} + v_{xt}w_{xx}\right), \quad (8)$$

$$\mathrm{tr}(U_\lambda V_\lambda) = 2u_{xt} + v_{xt}v_{xx} + w_{xx}v_{tt} + v_{xt}w_{xt} + w_{xt}w_{tt}, \quad (9)$$

$$\mathrm{tr}(V_\lambda^2) = v_{xt}^2 + w_{tt}^2 + 2\left(u_{tt} + v_{tt}w_{xt}\right). \quad (10)$$

Substituting equations (8)-(10) into equation (7) we have the first fundamental form of soliton surface for the Oriented Associativity equation to the system (2)

$$I = \frac{1}{2}\Big[\left\{v_{xx}^2 + w_{xt}^2 + 2\left(u_{xx} + v_{xt}w_{xx}\right)\right\}\mathrm{d}x^2 + \left(2u_{xt} + v_{xt}v_{xx} + w_{xx}v_{tt} + v_{xt}w_{xt} + w_{xt}w_{tt}\right)\mathrm{d}x\mathrm{d}t + \left\{v_{xt}^2 + w_{tt}^2 + 2\left(u_{tt} + v_{tt}w_{xt}\right)\right\}\mathrm{d}t^2\Big]$$

Second fundamental form of a surface

The scalar product of the total differential of the second order $\mathrm{d}^2 r$ of the radius-vector r of the current point of a surface by the orbit of the normal n at this point is called the second quadratic form of the surface:

$$II = -\mathrm{d}n \cdot \mathrm{d}r,$$

where

$$n = \frac{r_x \wedge r_t}{|r_x \wedge r_t|}.$$

In an expanded form, it is recorded as

$$II = b_{11}dx^2 + 2b_{12}dxdt + b_{22}dt^2, \quad (11)$$

where the coefficients b_{11}, b_{12} and b_{22} are given as

$$b_{11} = r_{xx} \cdot n, \quad (12)$$

$$b_{12} = r_{xt} \cdot n, \quad (13)$$

$$b_{22} = r_{tt} \cdot n, \quad (14)$$

or

$$b_{11} = \frac{1}{2} tr(r_{xx} n),$$

$$b_{12} = \frac{1}{2} tr(r_{xt} n),$$

$$b_{22} = \frac{1}{2} tr(r_{tt} n),$$

here

$$r_{xx} = \Phi^{-1}(U_{\lambda x} + [U_\lambda, U])\Phi,$$

$$r_{xt} = \Phi^{-1}(U_{\lambda t} + [U_\lambda, V])\Phi,$$

$$r_{tt} = \Phi^{-1}(V_{\lambda t} + [V_\lambda, V])\Phi$$

The normal vector n is given by

$$n = \pm \frac{\Phi^{-1}[U_\lambda, V_\lambda]\Phi}{\sqrt{\frac{1}{2} tr([U_\lambda, V_\lambda]^2)}}.$$

Thus, the equation (12)-(14) is written as follows

$$b_{11} = \frac{1}{2} \frac{tr\left(\left(U_{\lambda x} + [U_\lambda, U]\right)[U_\lambda, V_\lambda]\right)}{\sqrt{\frac{1}{2} tr([U_\lambda, V_\lambda]^2)}}, \quad (15)$$

$$b_{12} = \frac{1}{2} \frac{tr\left(\left(U_{\lambda t} + [U_\lambda, V]\right)[U_\lambda, V_\lambda]\right)}{\sqrt{\frac{1}{2} tr([U_\lambda, V_\lambda]^2)}}, \quad (16)$$

$$b_{22} = \frac{1}{2} \frac{tr\left(\left(V_{\lambda t} + [V_\lambda, V]\right)[U_\lambda, V_\lambda]\right)}{\sqrt{\frac{1}{2} tr([U_\lambda, V_\lambda]^2)}}, \quad (17)$$

Substituting equations (15)-(17) into equation (11) we have the second fundamental form of a soliton surface for the Oriented Associativity equation to the system (2)

$$II = \frac{1}{2} \frac{\alpha(w_{xxt} - v_{xxx}) + w_{xxx}\beta + v_{xxt}\gamma}{\sqrt{\alpha^2 + \gamma\beta}} dx^2 +$$

$$+ \frac{\alpha(w_{xtt} - v_{xxt} + 2\lambda\alpha) + \beta w_{xxt} + 2\lambda\beta\gamma + \gamma v_{xtt}}{\sqrt{\alpha^2 + \gamma\beta}} dxdt$$

$$+ \frac{1}{2} \frac{\alpha(w_{ttt} - v_{xtt}) + w_{xtt}\beta + v_{ttt}\gamma}{\sqrt{\alpha^2 + \gamma\beta}} dt^2$$

where

$$\alpha = (u_{xt} + v_{xt}w_{xt} - v_{tt}w_{xx}),$$

$$\beta = (v_{xt}^2 - u_{tt} - v_{tt}v_{xx} + v_{tt}w_{xt} - v_{xt}w_{tt}),$$

$$\gamma = (u_{xx} - w_{xt}^2 - v_{xt}w_{xx} + v_{xx}w_{xt} + w_{tt}w_{xx})$$

Area of surfaces for Oriented Associativity equation for $n = 3$ case

In this section we consider the area of surfaces for the Oriented Associativity equation for $n = 3$ to the system (2). Area of surfaces is evaluated by

$$S = \iint \sqrt{\frac{1}{2} tr\left(\left\{U_{\lambda x} + [U_\lambda, U]\right\}^2\right)} dxdt \quad (18)$$

where the matrix A is defined as in equation (5). So, that $[U_\lambda, U] = 0$, we have

$$(U_{\lambda x})^2 = \begin{pmatrix} 0 & 0 & 0 \\ u_{xxx}v_{xxx} + w_{xxx}u_{xxt} & v_{xxx}^2 + w_{xxx}v_{xxt} & v_{xxx}w_{xxx} + w_{xxx}w_{xxt} \\ u_{xxx}v_{xxt} + w_{xxt}u_{xxt} & v_{xxx}v_{xxt} + v_{xxt}w_{xxt} & v_{xxt}w_{xxx} + w_{xxt}^2 \end{pmatrix}$$

Area of surfaces (18) for the Oriented Associativity equation to the system (2) is given by

$$S = \iint \sqrt{\frac{v_{xxx}^2 + w_{xxt}^2}{2} + v_{xxt}w_{xxx}} dxdt$$

Conclusions

In this work we considered the Oriented Associativity equation for $n = 3$ case. Soliton surfaces for the Oriented Associativity equation for $n = 3$ case was obtained. Area of surfaces for the

Oriented Associativity equation for $n = 3$ case was investigated.

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