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Asymptotic behavior of the solution of a singularly perturbed three-point boundary value problem with boundary jumps

Abstract. In this paper, the three-point boundary value problem is considered for the third-order linear differential equation with a small parameter with at the two highest derivatives when the roots of additional characteristic equation have negative signs. The aim of this paper is to bring asymptotic estimation of the solution of a singularly perturbed three-point boundary value problem with boundary jumps and the asymptotic convergence of the solution of a singularly perturbed initial value problem to the solution of an unperturbed initial value problem. In this paper the fundamental system of solutions, initial functions of a singularly perturbed homogeneous di erential equation are constructed and their asymptotic estimates are obtained. An asymptotic behavior of the solution of the three-point boundary value problem at the points of initial jumps is established. A degenerate boundary-value problem is constructed. It is proved that the solution of the original singularly perturbed boundary value problem tends to the solution of the degenerate boundary value problem.

Key words: singular perturbation, small parameter, asymptotic, initial jumps,asymptotic estimate, boundary value problem, boundary functions, fundamental solutions.

Introduction

Equations containing a small parameter with the highest derivatives are called singularly perturbed equations. Such equations are of great applied importance. They act as mathematical models in the study of various processes in physics, chemistry, biology and technology.

The study of initial problems for singularly perturbed equations with unbounded initial data as the small parameter tends to zero, which are called Cauchy problems with an initial jump, first began with the work of M. I. Vishik, L. A. Lyusternik [1] and K. A. Kasymov [2]. A feature of such problems is that the solution of a singularly perturbed problem tends to the solution of the degenerate equation with modified initial conditions when the small parameter tends to zero. In this case, we say that there is a phenomenon of an initial jump in the solution. K. A. Kasymov and his students in [3-11] continued research on initial and two-point boundary value problems with initial jumps. Three-point boundary value problems for ordinary differential and integro-differential equations with a small parameter at only the highest derivative, which have the phenomenon of initial jumps, were considered in [12,13].

The present work is devoted to the study of the three-point boundary value problem for linear ordinary differential equations of the third order with a small parameter for two highest derivatives, which has the phenomenon of boundary jumps. The scientific novelty of this problem is that the fast variable of the solution increases unlimitedly not only at the so-called starting point, but also at the other end of the considered segment when the small parameter tends to zero.

Statement of the problem and preliminaries

We consider third-order linear differential equation with a small parameter at the two highest derivatives

$$\begin{aligned} L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_0(t) y'' + \\ + A_1(t) y' + A_2(t) y = F(t) \end{aligned} \tag{1}$$

with the boundary conditions

$$\begin{aligned} &h_1 y(t,\varepsilon) \equiv y(0,\varepsilon) = \alpha, \\ &h_2 y(t,\varepsilon) \equiv y(t_0,\varepsilon) = \beta, \\ &h_3 y(t,\varepsilon) \equiv y(1,\varepsilon) = \gamma, \end{aligned} \tag{2}$$

where $\varepsilon > 0 -$ small parametr, $0 < t_0 < 1$, and $\alpha, \beta, \gamma -$ known constants.

Let us assume that:

I. $A_i(t) \in C^2[0,1], i = \overline{0,2}, F(t) \in C[0,1]$

II. The roots of the equation

$$\mu^2 + A_0(t)\mu + A_1(t) = 0$$

satisfy the conditions

$$\mu_1(t) < -\gamma_1 < 0, \quad \mu_2(t) > \gamma_2 > 0.$$

We consider the followinghomogeneous equation associatedwith (1):

$$\begin{aligned} L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_0(t) y'' + \\ + A_1(t) y' + A_2(t) y = 0 \end{aligned} \tag{3}$$

Lemma1. If the conditions I, II are satisfied, then the fundamental set of solutionsof the equation (3) has the following asymptotic representation as $\varepsilon \to 0$ [6]:

$$y_1^{(i)}(t,\varepsilon) = \frac{1}{\varepsilon^i} e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx} (\mu_1^i(t) y_{10}(t) + O(\varepsilon)),$$

$$i = \overline{0,2},$$

$$y_2^{(i)}(t,\varepsilon) = \frac{1}{\varepsilon^i} e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx} (\mu_2^i(t) y_{20}(t) + O(\varepsilon)),$$

$$i = \overline{0,2}, \tag{4}$$

$$y_3^{(i)}(t,\varepsilon) = y_{30}^{(i)}(t) + O(\varepsilon), \ i = \overline{0,2},$$

where $y_3(t) = \exp\left(-\int_0^t \frac{A_2(x)}{A_1(x)} dx\right)$, and the functions $y_{i0}(t), i = 1,2$ are solutions of the problems

$$p_i(t) y_{i0}'(t) + q_i(t) y_{i0}(t) = 0, \ y_{i0}(0) = 1, \quad i = 1,2,$$

where

$$\begin{aligned} p_i(t) &= (A_0(t) + 2\mu_i(t))\mu_i(t); \\ q_i(t) &= A_2(t) + A_0(t)\mu_i'(t) + 3\mu_i(t)\mu_i'(t). \end{aligned}$$

We construct auxiliary functions:

$$\begin{aligned} K_0(t,s,\varepsilon) &= \frac{P_0(t,s,\varepsilon)}{W(s,\varepsilon)}; \\ K_1(t,s,\varepsilon) &= \frac{P_1(t,s,\varepsilon)}{W(s,\varepsilon)}; \end{aligned} \tag{5}$$

where $W(s,\varepsilon)$ is the Wronskianof the fundamental set of solutions of the equation (3), and $P_0(t,s,\varepsilon)$, $P_1(t,s,\varepsilon)$ are determinantsobtained from the Wronskian $W(s,\varepsilon)$ by replacing its third rows with the corresponding rows $y_1(t,\varepsilon), 0, y_3(t,\varepsilon)$ and $0, y_2(t,\varepsilon), 0$.

In view of(4),for the functions $K_0(t,s,\varepsilon), K_1(t,s,\varepsilon)$ the following asymptotic representations holdas $\varepsilon \to 0$:

$$K_0^{(j)}(t,s,\varepsilon) = \varepsilon^2 \left(\frac{y_{30}^{(j)}(t)}{y_{30}(s)\mu_1(s)\mu_2(s)} - \frac{\mu_1^j(t) y_{10}(t)}{\varepsilon^j y_{10}(s)\mu_1(s)(\mu_2(s) - \mu_1(s))} e^{\frac{1}{\varepsilon}\int_s^t \mu_1(x)dx} + O(\varepsilon) \right), \tag{6}$$

$$s \le t,$$

$$K_1^{(j)}(t,s,\varepsilon) = \varepsilon^2 \left(\frac{\mu_2^j(t) y_{20}(t)}{\varepsilon^j y_{20}(s)\mu_2(s)(\mu_2(s) - \mu_1(s))} e^{-\frac{1}{\varepsilon}\int_t^s \mu_2(x)dx} + O(\varepsilon) \right), t \le s, \ j = \overline{0,2}.$$

From (6) for the functions $K_0(t,s,\varepsilon), K_1(t,s,\varepsilon)$ we obtainasymptotic estimates as $\varepsilon \to 0$:

$$\left|K_0^{(j)}(t,s,\varepsilon)\right| \le C\varepsilon^2 + \frac{C}{\varepsilon^{j-2}} e^{-\gamma_1 \frac{t-s}{\varepsilon}},$$
$$\left|K_1^{(j)}(t,s,\varepsilon)\right| \le \frac{C}{\varepsilon^{j-2}} e^{\gamma_2 \frac{s-t}{\varepsilon}}, \quad j=\overline{0,2}, \tag{7}$$

where $C>0, \gamma_i>0, i=1,2$ areconstants independent of ε.

Main results

Let the functions $\Phi_i(t,\varepsilon), i=1,2,3$ be a solutions of the problem

$$L_\varepsilon \Phi_i(t,\varepsilon)=0,\ h_k \Phi_i(t,\varepsilon)=\delta_{ki},\ i,k=1,2,3.$$

We call these a boundary functions and determine by the formula

$$\Phi_i(t,\varepsilon)=\frac{I_i(t,\varepsilon)}{I(\varepsilon)},\ i=1,2,3, \tag{8}$$

where

$$I(\varepsilon)=\begin{vmatrix} h_1 y_1(t,\varepsilon) & h_1 y_2(t,\varepsilon) & h_1 y_3(t,\varepsilon) \\ h_2 y_1(t,\varepsilon) & h_2 y_2(t,\varepsilon) & h_2 y_3(t,\varepsilon) \\ h_3 y_1(t,\varepsilon) & h_3 y_2(t,\varepsilon) & h_3 y_3(t,\varepsilon) \end{vmatrix},$$

and $I_i(t,\varepsilon)$ are determinantsobtained from the $I(\varepsilon)$ by replacing its i-th rows with the row $y_1(t,\varepsilon), y_2(t,\varepsilon), y_3(t,\varepsilon)$.

In view of (4), for the boundary functions $\Phi_i(t,\varepsilon), i=1,2,3$ we get asymptotic representations as $\varepsilon \to 0$:

$$\Phi_1^{(j)}(t,\varepsilon)=\frac{1}{\varepsilon^j} e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx} \left(\mu_1^j(t) y_{10}(t)+O(\varepsilon)\right),$$

$$\Phi_2^{(j)}(t,\varepsilon)=\frac{y_{30}^{(j)}(t)}{y_{30}(t_0)}-\frac{\mu_1^j(t) y_{10}(t)}{\varepsilon^j y_{30}(t_0)} e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx} -$$
$$-\frac{\mu_2^j(t) y_{20}(t) y_{30}(1)}{\varepsilon^j y_{20}(1) y_{30}(t_0)} e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx} + \tag{9}$$
$$+O\left(\varepsilon+\frac{1}{\varepsilon^{j-1}} e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx}+\frac{1}{\varepsilon^{j-1}} e^{-\frac{1}{\varepsilon}\int_t^1 \mu_1(x)dx}\right),$$

$$\Phi_3^{(j)}(t,\varepsilon)=\frac{1}{\varepsilon^j} e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx}\left(\frac{\mu_2^j(t) y_{20}(t)}{y_{20}(1)}+O(\varepsilon)\right).$$

From (9)for the boundary functions $\Phi_i(t,\varepsilon), i=1,2,3$, we get the following asymptotic estimates as $\varepsilon \to 0$:

$$\left|\Phi_1^{(j)}(t,\varepsilon)\right| \le \frac{C}{\varepsilon^j} e^{-\gamma_1 \frac{t}{\varepsilon}}, \quad \left|\Phi_3^{(j)}(t,\varepsilon)\right| \le \frac{C}{\varepsilon^j} e^{-\gamma_2 \frac{1-t}{\varepsilon}},$$

$$\left|\Phi_2^{(j)}(t,\varepsilon)\right| \le C+\frac{C}{\varepsilon^j} e^{-\gamma_1 \frac{t}{\varepsilon}}+\frac{C}{\varepsilon^j} e^{-\gamma_2 \frac{1-t}{\varepsilon}}, \tag{10}$$
$$j=0,1,2,$$

where $C>0, \gamma_i>0, i=1,2$ areconstants independent of ε.

We seek a solution of the boundary value problem (1), (2) in the form

$$y(t,\varepsilon)=C_1\Phi_1(t,\varepsilon)+C_2\Phi_2(t,\varepsilon)+C_3\Phi_3(t,\varepsilon)+$$
$$+\frac{1}{\varepsilon^2}\int_0^t K_0(t,s,\varepsilon)F(s)ds-\frac{1}{\varepsilon^2}\int_t^1 K_1(t,s,\varepsilon)F(s)ds$$

where $\Phi_i(t,\varepsilon), i=1,2,3-$boundary functions, and $K_0(t,s,\varepsilon),\ K_1(t,s,\varepsilon)-$auxiliary functions given by the formula (5). Now, by using condition (2) we will find the constants $C_i, i=1,2,3$. Then the following theorem is valid.

Theorem 1. Under conditions I, II, the solution of the problem (1), (2) can be represented in the form

$$y^{(j)}(t,\varepsilon)=\left(\alpha+\frac{1}{\varepsilon^2}\int_0^1 K_1(0,s,\varepsilon)F(s)ds\right)\Phi_1^{(j)}(t,\varepsilon)+$$
$$+\left(\beta+\frac{1}{\varepsilon^2}\int_0^{t_0} K_0(t_0,s,\varepsilon)F(s)ds-\frac{1}{\varepsilon^2}\int_{t_0}^1 K_1(t_0,s,\varepsilon)F(s)ds\right)\Phi_2^{(j)}(t,\varepsilon)+ \tag{11}$$
$$+\left(\gamma-\frac{1}{\varepsilon^2}\int_0^1 K_0(1,s,\varepsilon)F(s)ds\right)\Phi_3^{(j)}(t,\varepsilon)+\frac{1}{\varepsilon^2}\int_0^t K_0^{(j)}(t,s,\varepsilon)F(s)ds-\frac{1}{\varepsilon^2}\int_t^1 K_1^{(j)}(t,s,\varepsilon)F(s)ds.$$

Theorem 2. Under conditions I, II, for the solution of the problem (1), (2) the following asymptotic estimates hold as $\varepsilon\to 0$:

$$\left|y^{(j)}(t,\varepsilon)\right|\le C(|\beta|+\max_{0\le t\le 1}|F(t)|)+$$
$$+\frac{C}{\varepsilon^j}(|\alpha|+|\beta|+\max_{0\le t\le 1}|F(t)|)e^{-\gamma_1\frac{t}{\varepsilon}}+$$
$$+\frac{C}{\varepsilon^j}(|\beta|+|\gamma|+\max_{0\le t\le 1}|F(t)|)e^{-\gamma_1\frac{1-t}{\varepsilon}}+ \tag{12}$$
$$+\frac{C}{\varepsilon^{j-1}}\left|\frac{\mu_2^{j-2}(t)-\mu_1^{j-2}(t)}{\mu_2(t)-\mu_1(t)}\right|\max_{0\le t\le 1}|F(t),\ j=\overline{0,2},$$

where $C>0,\gamma_i>0,i=1,2$ areconstants independentof ε.

The proof of the Theorem 1 and Theorem 2 follow from (11), and in view of the estimates (7), (10).

By the Theorem 2, one can obtain

$$y'(0,\varepsilon)=O\left(\frac{1}{\varepsilon}\right),\ y'(1,\varepsilon)=O\left(\frac{1}{\varepsilon}\right),\ \varepsilon\to 0.$$

It means that, the solution of the boundary value problem (1), (2) has the phenomena of initial jumps of zero order at the points $t=0$ and $t=1$.

We consider the following degenerate problem

$$L_0\overline{y}\equiv A_1(t)\overline{y}'+A_2(t)\overline{y}=F(t),\quad \overline{y}(t_0)=\beta. \tag{13}$$

Let the initial jump condition be satisfied

III. $\Delta_0\equiv\alpha-\overline{y}(0)\neq 0,\quad \Delta_1\equiv\gamma-\overline{y}(1)\neq 0.$

Then following theorem holds true.

Theorem 3. Let the conditions I-III are satisfied, then for the difference between the solutions $y(t,\varepsilon)$ and $\overline{y}(t)$ of the singularly perturbed boundary value problem (1), (2) and the degenerate problem (13) following asymptotic estimate holds as $\varepsilon\to 0$:

$$\left|y^{(j)}(t,\varepsilon)-\overline{y}^{(j)}(t)\right|\le$$
$$\le\frac{C}{\varepsilon^j}\left(|\alpha-\overline{y}(0)|e^{-\gamma_1\frac{t}{\varepsilon}}+|\gamma-\overline{y}(1)|e^{-\gamma_2\frac{1-t}{\varepsilon}}\right)+ \tag{14}$$
$$+C\varepsilon,\ j=0,1,$$

where $C>0,\gamma_i>0,i=1,2$ areconstants independentof ε.

Proof. We denote by $u(t\varepsilon)=y(t,\varepsilon)-y(t)$. Then in view of (13), we get the singularly perturbed problem for the function $u(t,\varepsilon)$:

$$\varepsilon^2u'''+\varepsilon A_0(t)u''+A_1(t)u'+$$
$$+A_2(t)u=-\varepsilon^2\overline{y}'''-\varepsilon A_0(t)y'',$$

$$u(0,\varepsilon)=\alpha-\overline{y}(0),\quad u(t_0,\varepsilon)=0,\quad u(1,\varepsilon)=\gamma-\overline{y}(1). \tag{15}$$

The problems (15) and (1), (2) are the same type. Therefore, by using the estimates (12) for the singularly perturbed boundary value problem (15), we obtain estimates (14).

The estimates (14) imply the following limit transitions

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = \overline{y}(t),\ 0 < t < 1,$$
$$\lim_{\varepsilon\to 0} y'(t,\varepsilon) = \overline{y}'(t),\ 0 < t < 1,$$

where $\overline{y}(t)$ is the solution of the degenerate problem (13).

The values of the initial jumps are determined from the following equalities:

$$\Delta_0 \equiv y(0,\varepsilon) - \overline{y}(0) =$$
$$+\alpha - \beta e^{\int_0^{t_0}\frac{A_2(x)}{A_1(x)}dx} + \int_0^{t_0} \frac{F(s)}{A_1(s)} e^{\int_0^{s}\frac{A_2(s)}{A_1(s)}dx} ds,$$

$$\Delta_1 \equiv y(1,\varepsilon) - \overline{y}(1) =$$
$$= \gamma - \beta e^{-\int_{t_0}^{1}\frac{A_2(x)}{A_1(x)}dx} - \int_{t_0}^{1} \frac{F(s)}{A_1(s)} e^{-\int_s^{1}\frac{A_2(x)}{A_1(x)}dx} ds.$$

Conclusion

In this paper, we consider a three-point boundary value problem for a third-order linear differential equation with a small parameter at two highest derivatives when the roots of the “additional characteristic equation” have negative signs. An analytical formula and asymptotic estimates of the solution are obtained. A degenerate boundary value problem is defined.It is shown that the solution of the original singularly perturbed boundary value problem tends to the solution of the degenerate boundary value problem. It is established that the solution of this boundary value problem has the phenomenon of boundary jumps. This means that the points of the initial jump are not only the left, but also the right point of the segment. Moreover, at both boundary points, the orders of the initial jumps coincide.

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