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ON the *D*-saturation property

Abstract. In this paper we study the notion of a *D*-saturated model, which occupies an intermediate position between the notions of a homogeneous model and a saturated model. Both the homogeneous models and the saturated models play a very important role in model theory. For example, the saturated models are the universal domains of the corresponding theories in the sense that is used in algebraic geometry (one can also notice that the algebraically closed fields of infinite transcendence degree, which are the universal domains in algebraic geometry, are the saturated models of the theory of algebraically closed fields); this means that the saturated models realize all what is necessary for the effective study of the corresponding theory.The *D*-saturated models proved to be useful in various situations. Naturally the question arises on the conditions under which a model is *D*-saturated. In this paper we indicate some conditions of such sort for weakly o-minimal models and models of stable theories. Namely, for such models we prove that homogeneity and a certain approximation of *D*-saturation imply *D*-saturation.

Key words: *D*-saturated model, homogeneous model, weakly o-minimal model, stable theory.

Introduction

In this paper models of first-order theories are studied. The notion of a homogeneous model introduced by Vaught [1], Jonsson [2], Craig [3]plays a very important role in model theory. In particular, the saturated models which are in a sense the universal domains of the corresponding theories are also homogeneous. In [4], the notion of a *D*-saturated model which is intermediate between homogeneity and saturationwas defined and it was shown that the notion is useful. For example, for *D*-saturated models the problem of finding an elementary extension with the same diagram has a nice solution. In [5], results on the existence of *D*-saturated models in different cardinalities for stable theories were obtained. Some results on *D*-saturated models (where they are called the normal models) can be found in [6].Naturally the following question arises: under what conditions will a model be *D*-saturated? In the present paper some conditions of such sort for weakly o-minimal models (the notion of weak o-minimality was introduced in [7] and studied in detail in [8] and other papers) and models of stable theories (for a perfect presentation of stability theory see [9]) will be found. Namely, for the indicated models it will be proved that homogeneity and some approximation of *D*-saturation imply *D*-saturation.

Preliminaries

Let us fix a sufficiently saturated model of a first-order language L(the *universal domain*).All the models, sets, elements thatwe will work with, will be elementary sub-models, subsets, elements of the universal domain. We will use the same symbol for a model and its underlying set. The cardinality of a set A will be denoted by $|A|$, but for the language by $|L|$ we will denote the cardinality of the set of all its formulas. By$i, j, \alpha, \beta, \delta$ we will denote ordinals, and by κ, λ, μ infinite cardinals. The first infinite ordinal (cardinal) will be denoted by ω, and the first uncountable cardinal will be denoted by ω_1. The minimal cardinal, which is greater than λ, will be denoted by λ^+.

The definitions of all the model-theoretic notions that are used, but are not defined in this paper, can be found in [9].

Definition 1.1. (1) A model M is calledλ-*homogeneous* if for every set $A \subseteq M$ of cardinality $|A| < \lambda$ and every element $a \in M$ each elementary map from A to M extends to an elementary map from $A \cup \{a\}$ to M.

(2) A model M is called *homogeneous* if M is $|M|$-homogeneous.

Following Shelah [10], for a subset A of a model M by $D(A)$ we denote the set of all complete pure types that are realized by finite tuples of elements of M. We call the set $D = D(M)$ the *diagram* of the model M. A complete 1-type p over A is called a *D-type* over A if $D(A\cup\{a\})\subseteq D$ for some (equivalently, every) element a that realizes p.

The following result of Keisler and Morley [11] plays an important role in the study of homogeneous models.

Lemma 1.1. *Let M be a λ-homogeneous model, $A\subseteq B$, $|A|<\lambda$, $|B|\le\lambda$, and $D(B)\subseteq D(M)$. Then every elementary map from A to M extends to an elementary map from B to M.*

Lemma 1.1 implies the following

Lemma 1.2. *A model N is λ-homogeneous if and only if for every set $A\subseteq N$ of cardinality $|A|<\lambda$ each $D(N)$-type over A is realized in N.*

The next statement follows from definitions.

Lemma 1.3. *The union of an increasing chain of D-types is also a D-type.*

For a 1-type p over a subset of a model M let $p(M)$ be the set of all elements in M realizing p.

Definition 1.2. A model M is called *D-saturated* if $D(M)=D$ and for every set A of cardinality $|A|<|M|$ and every non-algebraic D-type p over A we have $|p(M)|=|M|$.

Proposition 1.4. *Every countable model of a countable language has, for an appropriate D, a countable D-saturated elementary extension.*

Proof. First, for an arbitrary countable model M of a countable language we find a countable model $M^* @ M$ such that $|p(M^*)|=\omega$ for every $p\in P_M$, where P_M is the set of all $D(M)$-types over finite subsets of M. To do it, we take an ω-saturated model $M' @ M$ and for every $p\in P_M$ choose a set $A_p\subseteq p(M')$ of cardinality ω. Since P_M is countable, by Lowenheim-Skolem Theorem, there exists a countable model $M^* \text{E}\, M'$ containing $M\cup ; \{A_p : p\in P_M\}$.

Now let N be a countable model of a countable language. By induction on $i=\omega$, we construct countable models N_i such that $N_0=N$ and $N_{i+1}=N_i^*$. Let $N_\omega = ;_{i<\omega} N_i$. Then $N \text{E}\, N_\omega$ and N_ω is $D(N_\omega)$-saturated and countable.

Proposition 1.4 is proved.

Remark. (1) A similar construction for an uncountable model gives a D-saturated elementary extension of cardinality μ such that $\mu=\mu^{<\mu}$. The existence of such uncountable cardinals is not provable in ZFC.

(2) Some results on the existence of D-saturated models of different cardinalities for a stable theory T under the assumption that T has a D-saturated model M of a certain cardinality (say, $|M|>|T|$) can be found in [5].

***D*-saturation and weak o-minimality**

Let M be a model of a language L that contains, among others, a binary relation symbol $<$, which is interpreted as a linear order on the underlying set of the model.

A subset A of the model M is called *convex* if for any $a,b\in A$ and $c\in M$ the condition $a<c<b$ implies $c\in A$.

For example, intervals are convex sets. Singletons are also convex sets.

In the following "definable" will mean "definable with parameters".

Definition 2.1. A model is called *weakly o-minimal* if every its definable subset is the union of finitely many convex sets.

Definition 2.2. (1) We say that a model M is (κ, λ)-*normal* if for every set $A\subseteq M$ of cardinality $|A|<\kappa$ and every non-algebraic 1-type p over A, which is realized in M, we have $|p(M)|\ge\lambda$.

(2) We say that a model M is λ-*normal* if M is $(|M|,\lambda)$-normal.

(3) We say that a model M is *normal* if M is $|M|$-normal.

Lemma 2.1. *A model M is D(M)-saturated if and only if M is normal and homogeneous.*

Proof. Follows from Lemma 1.2.

Theorem 2.2. *Let M be a weakly o-minimal model. Suppose that M is λ-homogeneous and $\left(\kappa,\left(2^{|L|}\right)^+\right)$-normal, where $\kappa\ge\omega$ and $\lambda>2^{|L|}$. Then M is (κ, λ)-normal.*

In order to prove Theorem 2.2, we need the following notions and results from [12].

Definition 2.3. We say that a sub-order of a given linear order is an α-*sequence* if it is isomorphic or anti-isomorphic to the ordinal α.

Lemma 2.3. *Every linear order of cardinality at least $\left(2^\kappa\right)^+$ contains a κ^+-sequence.*

Proof. See [12, Lemma 3.1].

Now let N be a weakly o-minimal model that contains a κ^+-sequence I. We can assume that I increases (the case of decrease is considered similarly). Let $L(A)$ be the set of all formulas of the language L with parameters from A. By weak o-minimality, the set, which is definable in N by a formula $\varphi(x) \in L(N)$, is the union of finitely many convex sets. Let J_φ be the rightmost of these convex sets. Then either $\left|I \cap J_\varphi\right| < \kappa^+$ and hence $\left|I \setminus J_{\neg\varphi}\right| < \kappa^+$ or $\left|I \setminus J_\varphi\right| < \kappa^+$. It follows that for any $A \subseteq N$ the set

$$AV(I, A) = \left\{\varphi(x) \in L(A) : \left|I \setminus J_\varphi\right| < \kappa^+\right\}$$

is a complete 1-type over A.

Lemma 2.4. *If $|L| < \kappa^+$, then for every $B \subseteq N$ of cardinality $|B| < \kappa^+$ there exists $I_B \subseteq I$ of cardinality $|I_B| < \kappa^+$ such that every element from $I|I_B$ realizes $AV(I,B)$.*

Proof. See [12, Lemma 3.2].

Corollary 2.5. *If I is an $|L|^+$-sequence in a model N and B is a subset of N, then $AV(I,B)$ is a $D(N)$-type.*

Proof of Theorem 2.2. Let $A \subseteq M$, $|A| < \kappa$, and let p be a non-algebraic 1-type over A that is realized in M. We must prove that $|p(M)| \geq \lambda$.

Since the model M is $\left(\kappa, \left(2^{|L|}\right)^+\right)$-normal, we have $|p(M)| \geq \left(2^{|L|}\right)^+$. Then by Lemma 2.3, $p(M)$ contains an $|L|^+$-sequence $I = \left\{a_i : i < \left|L^+\right|\right\}$. By induction on $j \geq |L|^+$, we define elements $a_j \in M$ such that α_j realizes the type $p_j = AV\left(I, \{a_i : i < j\}\right)$. We can do it for all $j < \lambda$ because M is a λ-homogeneous model and, by Corollary 2.5, p_j is a $D(M)$-type. Since $p \subseteq p_j$ for all $j < \lambda$, we have $|p(M)| \geq \lambda$. Moreover, $\{a_i : i < \lambda\}$ is a λ-sequence because the formula $a_i < x$ belongs to the type p_j and hence $a_i < a_j$ for all $i < j < \lambda$.

Theorem 2.2 is proved.

Corollary 2.6. *Let M be a weakly o-minimal model. Suppose that M is homogeneous and $\left(2^{|L|}\right)^+$-normal. Then M is $D(M)$-saturated.*

Proof. Follows from Theorem 2.2 and Lemma 2.1.

Let us notice that from the proof of Theorem 2.2 the following statement follows.

Proposition 2.7. *Let M be a weakly o-minimal model. Suppose that M is λ-homogeneous, where $\lambda > |L|$. Then every $|L|^+$-sequence in M can be extended to a λ-sequence in M.*

***D*-saturation and stability**

Let $\lambda(T)$ be the minimal cardinal in which the theory T is stable.

Theorem 3.1. *Let M be a model of a stable theory T. Then*

(1) for every $\lambda > \lambda(T)$ and $\kappa \leq \lambda$ if M is λ-homogeneous and $(\kappa, \lambda(T)^+)$-normal, then M is (κ, λ)-normal;

(2) if M is homogeneous and $\lambda(T)^+$-normal, then M is $D(M)$-saturated;

(3) *for every $\lambda > |T|$ if M is λ-homogeneous and $(|T|^+, |T|^+)$-normal, then M is$(|T|^+ \lambda)$-normal.*

In order to prove Theorem 3.1, we need the following facts.

Lemma 3.2. *Every maximal infinite indiscernible set in a λ-homogeneous model has the cardinality at least λ.*

Proof. See [5, Lemma 3.2].

Definition 3.1. We say that a sequence $\{b_\alpha : \alpha < \mu\}$ of elements of a model is a *Morley pre-sequence over A,* where A is a subset of the model, if $p_\alpha \subseteq p_\gamma$ for all $\alpha < \gamma < \mu$, where p_α is the type realized by b_α over $A \cup \{b_\beta : \beta < \alpha\}$.

In the following lemma we use the cardinal $\kappa(T)$, the definition and properties of which can be found in [9]. We only notice that $\kappa(T) \leq |T|^+$, and the equality $\kappa(T)=\omega$, is equivalent to superstability of the theory T. Let $\kappa_r(T)$ be the minimal regular cardinal that is greater than or equal to $\kappa(T)$.

Lemma 3.3. *Let $\{b_\alpha : \alpha < \mu\}$ be a Morley pre-sequence over A in a model of a stable theory T, where $\mu \geq \kappa_r(T) + \omega_1$ is a regular cardinal. Then there exists an ordinal $\alpha_0 < \mu$ such that the set $\{b_\alpha : \alpha_0 < \alpha < \mu\}$ is indiscernible over A.*

Proof. We use the notation from Definition 3.1. From the definition of $\kappa(T)$ (see [9, p. 100]) and regularity of μ, it follows that there exists an ordinal $\alpha_1 < \mu$ such that for every $\alpha \geq \alpha_1$ the type p_α does not fork over $\{b_\beta : \beta < \alpha_1\}$. By [9, Chapter III, Lemma 2.12], the type $p_{\alpha_1+\omega}$ is stationary. Then by [9, Chapter III, Lemma 1.10(1)], the set $\{b_\alpha : \alpha_1 + \omega \leq \alpha < \mu\}$ is indiscernible over A.

Lemma 3.3 is proved.

Lemma 3.4. *If M is a model of a λ-stable theory, $I \subseteq M$ and $|I| > \lambda \geq |A|$, then there exists $J \subseteq I$ such that $|J| > \lambda$ and J is indiscernible over A.*

Proof. See [9, Chapter I, Theorem 2.8].

Lemma 3.5. *Let M be a model of a stable theory T. Suppose that M is $|T|^+$-homogeneous and $(|T|^+, |T|^+)$-normal, and p is a non-algebraic 1-type over $A \subseteq M$, $|A| \leq |T|$, that is realized in M. Then p(M) contains an indiscernible set of cardinality $|T|^+$.*

Proof. By induction on $\alpha < |T|^+$, we define elements $a_\alpha \in M$ and non-algebraic *D(M)*-types p_α over $A_\alpha = A \cup \{a_\beta : \beta < \alpha\}$ as follows.

We let $p_0 = p$ and arbitrarily choose $a_0 \in p(M)$.

Suppose that the type p_α and the element $a_\alpha \in p_\alpha(M)$ have been defined. Since the model M is $(|T|^+, |T|^+)$-normal, we have $|p_\alpha(M)| < |T|$. Since $|acl(A_{\alpha+1})| \leq |T| + |A_\alpha| = |T|$, we can choose $a_{\alpha+1} \in p_\alpha(M) \setminus acl(A_{\alpha+1})$. Let $p_{\alpha+1} = tp(a_{\alpha+1} / A_{\alpha+1})$. Clearly, $p_\alpha \subseteq p_{\alpha+1}$.

Suppose that a_α and p_α have been defined for all $\alpha < \delta$, where $\delta < |T|^+$ is a limit ordinal, and $p_\alpha \subseteq p_\beta$ for all $\alpha < \beta < \delta$. Let $p_\delta = ;_{\alpha<\delta} p_\alpha$. Since for every $\alpha < \delta$ the type p_α is non-algebraic, the type p_δ is also non-algebraic. Since all the p_α are *D(M)*-types, by Lemma 1.3 the type p_δ is a *D(M)*-type over A_δ, where $|A_\delta| = |A| + |\delta| < |T|^+$. Since the model M is $|T|^+$-homogeneous, by Lemma 1.2 the type p_δ is realized by some $a_\delta \in M$.

By construction, $\{a_\alpha : \alpha < |T|^+\}$ is a Morley pre-sequence over A and hence, by Lemma 3.3, contains a set of cardinality $|T|^+$, which is indiscernible over A.

Lemma 3.5 is proved.

Proof of Theorem 3.1. (1) Let us consider a non-algebraic type p over $A \subseteq M$, $|A| < \kappa$, that is realized in *M*. We must prove that $|p(M)| \geq \lambda$.

Since the model *M* is $(\kappa, \lambda(T)^+)$-normal, we have $|p(M)| > \lambda(T)$. By Lemma 3.4, *p(M)* contains an indiscernible set *I* over *A* such that $|I| > \lambda(T)$. Let us extend *I* to a maximal indiscernible over *A* set $J \subseteq M$. Since $I \subseteq p(M)$ and *J* is indiscernible over *A*, we have $J \subseteq p(M)$. Since the model *M* is λ-homogeneous and $|A| < \kappa \leq \lambda$, the model $(M, a)_{a \in A}$ is also λ-homogeneous. Then by Lemma 3.2, we have $|J| \geq \lambda$ and hence $p(M) \geq \lambda$.

(2) Follows from (1) and Lemma 2.1.

(3) We repeat the proof of (1) replacing κ by $|T|^+$ and replacing $\lambda(T)$ by $|T|$, and using Lemma 3.5 instead of Lemma 3.4.

Theorem 3.1 is proved.

Conclusion

In this paper we study the notion of a *D*-saturated model. We prove that for weakly o-minimal models and models of stable theories homogeneity and some approximation of *D*-saturation imply *D*-saturation.

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References

1. Vaught R. "Homogeneous universal models of complete theories." Notices AMS. 5(1958): 775.
2. Jonsson B. "Homogeneous universal relational systems." Math. Scand. 8(1960): 137-142.
3. Craig W. "$\aleph_0$-homogeneous relatively universal systems." Notices AMS. 8(1961): 265.

4. Kudaibergenov K.Zh. "On extensions of homogeneous models." Algebra and Logic. 28(1)(1989): 61-74.

5. Kudaibergenov K.Zh. "Homogeneous models of stable theories." Siberian Advances in Mathematics. 3(3)(1993): 1-33.

6. Kudaibergenov K.Zh. "The small index property and the cofinality of the automorphism group." Siberian Advances in Mathematics. 27(1)(2017): 1-15.

7. Dickmann M.A. "Elimination of quantifiers for ordered valuation rings." In: Proc. of the 3rd Easter Conf. on Model Theory (Gross Koris, 1985). Berlin: Humboldt Univ. (1985): 64-88.

8. Macpherson D., Marker D., Steinhorn C. "Weakly o-minimal structures and real closed fields." Trans. Amer. Math. Soc. 352(12)(2000): 5435-5483.

9. Shelah S. Classification theory and the number of non-isomorphic models. North-Holland, Amsterdam, 1978.

10. Shelah S. "Finite diagrams stable in power." Ann. Math. Logic. 2(1970): 69-118.

11. Keisler H.J., Morley M.D. "On the number of homogeneous models of a given power." Israel J. Math. 5(2)(1967): 73-78.

12. Kudaibergenov K.Zh. "Small extensions of models of o-minimal theories and absolute homogeneity." Siberian Advances in Mathematics. 18(1)(2008): 118-123.