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NUMERICAL SOLUTION OF A CONTROL PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS WITH MULTIPOINT INTEGRAL CONDITION

Abstract.A linear boundary value problem with a parameter for ordinary differential equations with multipoint integral conditionis investigated.The method of parameterization is used for solving the considered problem. The linear boundary value problem with a parameter for ordinary differential equations with multipoint integral condition by introducing additional parameters at the partition points is reduced to equivalent boundary value problem with parameters. The equivalent boundary value problem with parameters consists of the Cauchy problem for the system of ordinary differential equations with parameters, multipoint integral condition and continuity conditions. The solution of the Cauchy problem for the system of ordinary differential equations with parameters is constructed using the fundamental matrix of differential equation. The system of linear algebraic equations with respect to the parameters are composed by substituting the values of the corresponding points in the built solutions to the multipoint integral condition and the continuity condition. Numerical method for finding solution of the problem is suggested, which based on the solving the constructed system and Runge-Kutta method of the 4-th order for solving Cauchy problem on the subintervals.

Key words: control problem with multipoint integral condition, numerical solution, algorithm.

Introduction

Control problems, which are also called boundary value problems with parameters and the problem of identification parameter for a system of ordinary differential and integro-differential equations with parameters, have been intensively investigated in recent years. Questions of existence, uniqueness and stability of solving problems with parameters are very important for development of numerical methods of identification of parameters of the mathematical models described by ordinary differential equations with multipoint integral condition [1-8].To solve these classes of control problems, there were used the optimization methods, topological methods, the maximum principle, etc. In spite of this, the questions of finding the effective signs of unique solvability and constructing the numerical algorithms for finding the optimal solutions of control problems for the systems of ordinary differential equations with parameters still remain open. One of the constructive methods for investigating and solving the boundary value problems with parameters for the system of ordinary differential equations is the parameterization method [9]. The parameterization method was developed for the investigating and solving the boundary value problems for the system of ordinary differential equations. On the basis of this method, coefficient criteria for the unique solvability of linear boundary value problems for the system of ordinary differential equations were obtained. Algorithms for finding the approximate solutions were also proposed and their convergence to the exact solution of the problem studied was established. Later, the parameterization method was developed for the two-point boundary value problems for the Fredholmintegro-differential equations [10-14]. Necessary and sufficient conditions for the solvability and unique solvability are established, the algorithms for finding the approximate solutions of the problems considered are the linear Fredholm integro-differential equation on the basis of new algorithms of parameterization method are constructed. This approach are applied to two-point boundary value problems for system of ordinary and ordinary loaded differential equations with parameter [16-17].

In the present paper, linear problem with a parameter for an ordinary differential equation with multipoint integral condition is investigated. Based on the parameterization method and numerical methods, the numerical method for solving the problem considered is developed, and the algorithms for their implementation are proposed. By introducing additional parameters as the values of the desired solution at some points of the interval $[0,T]$, where the problem is considered, the obtained problem is reduced to the equivalent problem consisting of a special Cauchy problem for the system of ordinary differential equations, multipoint integral conditions, and continuity conditions for the solution at the points of partition. Using the integral equation, that equivalent to the special Cauchy problem for the system of ordinary differential equation, we obtained a representation of the solution of the special Cauchy problem using the entered parameters at the assumption of invertibility of a some matrix. Based on this representation, a system of algebraic equations with respect to the parameters is constructed from the multipoint integral condition and the continuity conditions of the solution. We offer algorithm for solving the control problem for the ordinary differential equation with multipoint integral condition, and its numerical implementation.

Statement of problem and scheme of parametrization method

We consider a linear boundary value problem with a parameter for an ordinary differential equation with multipoint integral condition

$$\frac{dx}{dt} = A(t)x + A_0(t)\mu + f(t),$$

$$x \in R^n, \quad \mu \in R^m, \quad t \in (0,T), \tag{1}$$

$$\sum_{i=0}^{N+1} C_i x(t_i) + B_0\mu + \int_0^T M(t)x(t)dt = d, \quad d \in R^{n+m}, \tag{2}$$

where the $(n\times n)$-matrix $A(t)$, $(n\times m)$-matrix $A_0(t)$, $((n+m)\times n)$-matrix $M(t)$ and n-vector-function $f(t)$ are continuous on $[0,T]$, the $((n+m)\times n)$-matrices C_i, $i=\overline{0,N+1}$, the $((n+m)\times m)$-matrix B_0 are constants.

The solution to problem (1), (2) is a pair $(x^*(t),\mu^*)$, where continuous on $[0,T]$ and continuously differentiable on $(0,T)$ a function $x^*(t)$ satisfies the ordinary differential equation (1) and condition (2) with $\mu=\mu^*$.

To solve the problem with parameter (1), (2), the approach developed in [24-26] is used, based on the algorithms of the parameterization method and numerical methods for solving Cauchy problems.

Scheme of the method. Points $0=t_0<t_1<\dots<t_N<t_{N+1}=T$ are taken and the interval $[0,T)$ is divided into N subintervals:

$$[0,T) = \cup_{r=1}^{N+1}[t_{r-1},t_r).$$

Let $C([0,T],R^n)$ be the space of continuous on $[0,T]$ functions $x:[0,T]\to R^n$ with norm $\|x\|_1 = \max_{t\in[0,T]}\|x(t)\|$; $C([0,T],\Delta_N,R^{n(N+1)})$ - the space of systems of functions $x[t]=(x_1(t),x_2(t),\dots,x_{N+1}(t))$, where $x_r:[t_{r-1},t_r)\to R^n$ are continuous on $[t_{r-1},t_r)$ and have finite left-sided limits $\lim_{t\to t_r-0} x_r(t)$ for all r=$\overline{1,N+1}$, with norm $\|x[\cdot]\|_2 = \max_{r=\overline{1,N+1}} \sup_{t\in[t_{r-1},t_r)} \|x_r(t)\|$.

The restriction of the function $x(t)$ to the r-th interval $[t_{r-1},t_r)$ is denoted by $x_r(t)$, i.e. $x_r(t)=x(t)$ for $t\in[t_{r-1},t_r)$, r=$\overline{1,N+1}$. Then we reduce problem (1), (2) to the equivalent multipoint boundary value problem

$$\frac{dx_r}{dt} = A(t)x_r + A_0(t)\mu + \mathrm{f(t)},$$

$$t\in[t_{r-1},t_r), \quad r=\overline{1,N+1}\,, \tag{3}$$

$$\sum_{i=0}^{N} C_i x_{i+1}(t_i) + C_{N+1}\lim_{t\to T-0} x_{N+1}(t) + B_0\mu + \sum_{k=1}^{N+1}\int_{t_{k-1}}^{t_k} M(t)x_k(t)dt = d, \tag{4}$$

$$\lim_{t\to t_s-0} x_s(t) = x_{s+1}(t_s), \quad s=\overline{1,N}. \tag{5}$$

where (5) are conditions for matching the solution at the interior points of partition.

The solution of problem (3) - (5) is the pair $(x^*[t],\mu^*)$ with elements $x^*[t] = (x_1^*(t),x_2^*(t),\dots,x_{N+1}^*(t)) \in C([0,T],\Delta_N,R^{n(N+1)})$, $\mu^*\in R^m$, where functions

$x_r^*(t),\ r=\overline{1,N+1}$, are continuously differentiable on $[t_{r-1},t_r)$, which satisfies system of ordinary differential equations (3) and condition (4) with $\mu=\mu^*$ and continuity conditions (5).

We introduce additional parameters $\lambda_r = x_r(t_{r-1}),\ r=\overline{1,N+1},\ \lambda_{N+2}=\mu$. Making the substitution $x_r(t)=u_r(t)+\lambda_r$ on every r-th interval $[t_{r-1},t_r),\ r=\overline{1,N+1}$, we obtain multipoint boundary value problem with parameters

$$\frac{du_r}{dt}=A(t)[u_r+\lambda_r]+A_0(t)\lambda_{N+2}+\mathrm{f(t)},$$

$$t\in[t_{r-1},t_r),\ r=\overline{1,N+1}, \tag{6}$$

$$u_r(t_{r-1})=0,\ r=\overline{1,N+1}, \tag{7}$$

$$\sum_{i=0}^{N} C_i\lambda_{i+1}+C_{N+1}\lambda_{N+1}+$$

$$+C_{N+1}\lim_{t\to T-0}u_{N+1}(t)+B_0\lambda_{N+2}+$$

$$+\sum_{k=1}^{N+1}\int_{t_{k-1}}^{t_k}M(t)[u_k(t)+\lambda_k]dt=d, \tag{8}$$

$$\lambda_s+\lim_{t\to t_s-0}u_s(t)=\lambda_{s+1},\quad s=\overline{1,N}. \tag{9}$$

A pair $(u^*[t],\lambda^*)$ with elements $u^*[t]=(u_1^*(t),u_2^*(t),\dots,u_{N+1}^*(t))\in C([0,T],\Delta_N,R^{n(N+1)})$, $\lambda^*=(\lambda_1^*,\lambda_2^*,\dots,\lambda_{N+1}^*,\lambda_{N+2}^*)\in R^{n(N+1)+m}$ is said to be a solution to problem (6)-(9) if the functions $u_r^*(t),\ r=\overline{1,N+1}$, are continuously differentiable on $[t_{r-1},t_r)$ and satisfy (6) and additional conditions (8), (9) with $\lambda_j=\lambda_j^*,\ j=\overline{1,N+2}$, and initial conditions (7).

If the pair $(x^*(t),\mu^*)$ is a solution of problem (1), (2), then the pair $(u^*[t],\lambda^*)$ with elements $u^*[t]=(u_1^*(t),u_2^*(t),\dots,u_{N+1}^*(t))\in C([0,T],\Delta_N,R^{n(N+1)})$, $\lambda^*=(\lambda_1^*,\lambda_2^*,\dots,\lambda_{N+1}^*,\lambda_{N+2}^*)\in R^{n(N+1)+m}$, where $\lambda_r^*=x_r^*(t_{r-1}),\ u_r^*(t)=x_r^*(t)+x_r^*(t_{r-1}),\ t\in[t_{r-1},t_r),\ \mathrm{r}=\overline{1,N+1},\ \lambda_{N+2}^*=\mu^*\in R^m$, is the solution of problem (3)-(6). Conversely, if a pair $(\tilde u[t],\ \tilde\lambda)$ with elements $\tilde u[t]=(\tilde u_1(t),\tilde u_2(t),\dots,\tilde u_{N+1}(t))\in C([0,T],\Delta_N,R^{n(N+1)})$, $\tilde\lambda=(\tilde\lambda_1,\tilde\lambda_2,\dots,\tilde\lambda_{N+2})\in R^{n(N+1)+m}$, is a solution of (3)-(6) , then the pair $(\tilde x(t),\tilde\mu)$ defined by the equalities $\tilde x(t)=u_r(t)+\tilde\lambda_r,\ t\in[t_{r-1},t_r),\ \mathrm{r}=\overline{1,N+1}$, $\tilde x(T)=\lim_{t\to T-0}\tilde u_N(t)+\tilde\lambda_N$ and $\tilde\mu=\tilde\lambda_{N+2}$, will be the solution of the original boundary value problem with parameter (1), (2).

Let $X_r(t)$ be a fundamental matrix to the differential equation $\frac{dx}{dt}=A(t)x$ on $[t_{r-1},t_r),\ \mathrm{r}=\overline{1,N+1}$.

Then the unique solution to the Cauchy problem for the system of ordinary differential equations (6), (7) at the fixed values $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{N+1},\lambda_{N+2})$ has the following form

$$u_r(t)=X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)A(\tau)d\tau\,\lambda_\mathrm{r}+X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)A_0(\tau)d\tau\,\lambda_{\mathrm{N+2}}+$$
$$+X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)f(\tau)d\tau,\quad t\in[t_{r-1},t_r),\ \mathrm{r}=\overline{1,N+1}. \tag{10}$$

Substituting the corresponding right-hand sides of (10) into the conditions (8), (9), we obtain a system of linear algebraic equations with respect to the parameters $\lambda_r,\ r=\overline{1,N+2}$:

$$\sum_{i=0}^{N}C_i\lambda_{i+1}+C_{N+1}\lambda_{N+1}+B_0\lambda_{N+2}+C_{N+1}X_{N+1}(T)\int_{t_N}^{T}X_{N+1}^{-1}(\tau)A(\tau)d\tau\,\lambda_{\mathrm{N+1}}+$$
$$+C_{N+1}X_{N+1}(T)\int_{t_N}^{T}X_{N+1}^{-1}(\tau)A_0(\tau)d\tau\,\lambda_{\mathrm{N+2}}+$$

$$+\sum_{k=1}^{N+1}\int_{t_{k-1}}^{t_k} M(t)\left[X_k(t)\int_{t_{k-1}}^{t} X_k^{-1}(\tau)A(\tau)d\tau\,\lambda_k+\lambda_k\right]dt+$$
$$+\sum_{k=1}^{N+1}\int_{t_{k-1}}^{t_k} M(t)\left[X_k(t)\int_{t_{k-1}}^{t} X_k^{-1}(\tau)A_0(\tau)d\tau\,\lambda_{N+2}\right]dt=$$
$$=d-C_{N+1}X_{N+1}(T)\int_{t_N}^{T} X_{N+1}^{-1}(\tau)f(\tau)d\tau-$$
$$-\sum_{k=1}^{N+1}\int_{t_{k-1}}^{t_k} M(t)X_k(t)\int_{t_{k-1}}^{t} X_k^{-1}(\tau)f(\tau)d\tau\,dt, \qquad (11)$$

$$\lambda_s+X_s(t_s)\int_{t_{s-1}}^{t_s} X_s^{-1}(\tau)A(\tau)d\tau\,\lambda_s+X_s(t_s)\int_{t_{s-1}}^{t_s} X_s^{-1}(\tau)A_0(\tau)d\tau\,\lambda_{N+2}-\lambda_{s+1}=$$
$$=-X_s(t_s)\int_{t_{s-1}}^{t_s} X_s^{-1}(\tau)f(\tau)d\tau,\quad s=\overline{1,N}. \qquad (12)$$

We denote the matrix corresponding to the left side of the system of equations (11), (12) by $Q_*(\Delta_N)$ and write the system in the form

$$Q_*(\Delta_N)\lambda=-F_*(\Delta_N),\ \ \lambda\in R^{n(N+1)+m}, \qquad (13)$$

where

$$F_*(\Delta_N)=$$
$$\begin{pmatrix} -d+C_{N+1}X_{N+1}(T)\int_{t_N}^{T} X_{N+1}^{-1}(\tau)f(\tau)d\tau+\sum_{k=1}^{N+1}\int_{t_{k-1}}^{t_k} M(t)X_k(t)\int_{t_{k-1}}^{t} X_k^{-1}(\tau)f(\tau)d\tau\,dt \\ X_1(t_1)\int_{t_0}^{t_1} X_1^{-1}(\tau)f(\tau)d\tau \\ \cdots\ \ \cdots\ \ \cdots \\ X_N(t_N)\int_{t_{N-1}}^{t_N} X_N^{-1}(\tau)f(\tau)d\tau \end{pmatrix}.$$

It is not difficult to establish that the solvability of the boundary value problem (1), (2) is equivalent to the solvability of the system (13). The solution of the system (13) is a vector $\lambda^*=(\lambda_1^*,\lambda_2^*,\dots,\lambda_{N+1}^*,\lambda_{N+2}^*)\in R^{n(N+1)+m}$ consists of the values of the solutions of the original problem (1), (2) in the initial points of subintervals, i.e. $\lambda_r^*=x^*(t_{r-1})$, $r=\overline{1,N+1}$, $\lambda_{N+2}^*=\mu^*$.

Further we consider the Cauchy problems for ordinary differential equations on subintervals

$$\frac{dz}{dt}=A(t)z+P(t),$$

$$z(t_{r-1})=0,\quad t\in[t_{r-1},t_r],\ \ r=\overline{1,N+1}, \qquad (14)$$

where $P(t)$ is either $(n\times n)$ matrix, or n vector, both continuous on $[t_{r-1},t_r]$, $r=\overline{1,N+1}$. Consequently, solution to problem (14) is a square matrix or a vector of dimension n. Denote by $a(P,t)$ the solution to the Cauchy problem (14). Obviously,

$$a(P,t)=X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)P(\tau)d\tau,$$
$$t\in[t_{r-1},t_r],$$

where $X_r(t)$ is a fundamental matrix of differential equation (14) on the r-th interval.

Numerical implementation of parametrization method

We offer the following numerical implementation of algorithm based on the Runge–Kutta method of 4th order and Simpson's method.

1. Suppose we have a partition Δ_N: $0=t_0<t_1<\dots<t_N<t_{N+1}=T$. Divide each r-th interval

$[t_{r-1}, t_r]$, $r = \overline{1, N+1}$, into N_r parts with step $h_r = (t_r - t_{r-1})/N_r$. Assume on each interval $[t_{r-1}, t_r]$ the variable $\hat{t}$ takes its discrete values: $\hat{t} = t_{r-1}$, $\hat{t} = t_{r-1} + h_r, \dots$, $\hat{t} = t_{r-1} + (N_r - 1)h_r$, $\hat{t} = t_r$, and denote by $\{t_{r-1}, t_r\}$ the set of such points.

2. Solving the Cauchy problems for ordinary differential equations

$$\frac{dz}{dt} = A(t)z + A(t),$$
$$z(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r],$$

$$\frac{dz}{dt} = A(t)z + A_0(t),$$
$$z(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r],$$

$$\frac{dz}{dt} = A(t)z + f(t),$$
$$z(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, N+1},$$

by using again the Runge–Kutta method of 4th order, we find the values of $(n \times n)$ matrices $a_r(A, \hat{t})$, $a_r(A_0, \hat{t})$ and n vector $a_r(f, \hat{t})$ on $\{t_{r-1}, t_r\}$, $r = \overline{1, N+1}$.

3. Applying Simpson's method on the set $\{t_{r-1}, t_r\}$, we evaluate the definite integrals

$$m_r^{h_r} = \int_{t_{r-1}}^{t_r} M(\tau) d\tau,$$

$$m_r^{h_r}(A) = \int_{t_{r-1}}^{t_r} M(\tau) a_r^{h_r}(A, \tau) d\tau,$$

$$m_r^{h_r}(A_0) = \int_{t_{r-1}}^{t_r} M(\tau) a_r^{h_r}(A_0, \tau) d\tau,$$

$$m_r^{h_r}(f) = \int_{t_{r-1}}^{t_r} M(\tau) a_r^{h_r}(f, \tau) d\tau, \ r = \overline{1, N+1}.$$

4. Construct the system of linear algebraic equations with respect to parameters

$$Q_*^{\tilde{h}}(\Delta_N)\lambda = -F_*^{\tilde{h}}(\Delta_N), \quad \lambda \in R^{n(N+1)+m}, \quad (15)$$

Solving the system (15), we find $\lambda^{\tilde{h}}$. As noted above, the elements of $\lambda^{\tilde{h}} = (\lambda_1^{\tilde{h}}, \lambda_2^{\tilde{h}}, \dots, \lambda_{N+2}^{\tilde{h}})$ are the values of approximate solution to problem (1), (2) in the starting points of subintervals: $x^{\tilde{h}_r}(t_{r-1}) = \lambda_r^{\tilde{h}}$, $r = \overline{1, N+1}$, $\mu^* = \lambda_{N+2}^*$.

5. To define the values of approximate solution at the remaining points of set $\{t_{r-1}, t_r\}$, we solve the Cauchy problems

$$\frac{dx}{dt} = A(t)x + A_0(t)\lambda_{N+2}^{\tilde{h}} + f(t),$$

$$x(t_{r-1}) = \lambda_r^{\tilde{h}}, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, N+1}.$$

And the solutions to Cauchy problems are found by the Runge–Kutta method of 4th order. Thus, the algorithm allows us to find the numerical solution to the problem (1), (2).

To illustrate the proposed approach for the numerical solving linear boundary value problem with a parameter for an ordinary differential equation with multipoint integral condition (1), (2) on the basis of parameterization method, let us consider the following example.

Example. We consider a linear boundary value problem with a parameter for an ordinary differential equation with multipoint integral condition

$$\frac{dx}{dt} = A(t)x + A_0(t)\mu + f(t),$$

$$x \in R^2, \quad \mu \in R^3, \quad t \in (0,1), \quad (16)$$

$$C_0 x(t_0) + C_1 x(t_1) + C_2 x(t_2) +$$

$$+ B_0\mu + \int_0^T M(t)x(t)dt = d, \quad d \in R^5. \quad (17)$$

Here $t_0 = 0, t_1 = \frac{1}{2}, t_2 = 1,$

$$A(t) = \begin{pmatrix} t^2 & 2t \\ 1 & t+9 \end{pmatrix}, \quad A_0(t) = \begin{pmatrix} 2 & t & t+3 \\ t^2 & 0 & 3t \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 2 & 0 \\ 4 & -4 \\ 1 & 6 \\ 0 & 2 \\ 9 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -3 & 1 \\ 5 & 2 \\ 3 & 0 \\ 8 & 6 \\ 1 & 9 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -6 & 1 \\ 5 & 3 \\ 8 & 1 \\ 2 & 6 \\ 0 & 9 \end{pmatrix}, B_0 = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & -4 \\ 0 & 7 & -6 \\ -2 & 1 & 8 \\ 6 & 1 & 0 \end{pmatrix},$$

$$d = \begin{pmatrix} -42 \\ \frac{349}{3} \\ \frac{553}{12} \\ \frac{505}{2} \\ \frac{369}{2} \end{pmatrix}, \quad M(t) = \begin{pmatrix} 1 & 0 \\ t & -2 \\ t+3 & t^2 \\ 0 & 9 \\ -3 & 1 \end{pmatrix},$$

$$f(t) = \begin{pmatrix} -8t^4 - t^3 - 25t^2 - 2t - 30 \\ -4t^4 - 36t^3 + t^2 - 107t + 20 \end{pmatrix}.$$

We use the numerical implementation of algorithm. Accuracy of solution depends on the accuracy of solving the Cauchy problem on subintervals and evaluating definite integrals. We provide the results of the numerical implementation of algorithm by partitioning the subintervals [0, 0.5], [0.5, 1] with step h = 0.025.

Solving the system of equations (15), we obtain the numerical values of the parameters

$$\lambda_1^{\tilde{h}} = \begin{pmatrix} 7.00000083 \\ -1.99999956 \end{pmatrix},$$

$$\lambda_2^{\tilde{h}} = \begin{pmatrix} 7.50000135 \\ 3.00000449 \end{pmatrix},$$

$$\lambda_3^{\tilde{h}} = \begin{pmatrix} 2.00000297 \\ -3.00000131 \\ 8.99999832 \end{pmatrix}.$$

We find the numerical solutions at the other points of the subintervals using Runge-Kutta method of the 4-th order to the following Cauchy problems

$$\frac{d\tilde{x}_r}{dt} = A(t)\tilde{x}_r + A_0(t)\lambda_3^{\tilde{h}} + f(t),$$

$$\tilde{x}_r(t_{r-1}) = \lambda_r^{\tilde{h}}, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1,2}.$$

Exact solution of the problem (16), (17) is pair $(x^*(t), \mu^*)$, where $x^*(t) = \begin{pmatrix} t+7 \\ 4t^3 + 9t - 2 \end{pmatrix}$, $\mu^* = \begin{pmatrix} 2 \\ -3 \\ 9 \end{pmatrix}$.

The results of calculations of numerical solutions at the partition points are presented in the following table:

t	$\tilde{x}_1(t)$	$\tilde{x}_2(t)$	t	$\tilde{x}_1(t)$	$\tilde{x}_2(t)$
0	7.00000083	-1.99999956	0.5	7.50000135	3.00000449
0.025	7.02500086	-1.77493687	0.525	7.52500139	3.30381719
0.05	7.05000088	-1.54949918	0.55	7.55000142	3.61550487
0.075	7.0750009	-1.32331148	0.575	7.57500146	3.93544255
0.1	7.10000092	-1.09599878	0.6	7.60000149	4.26400522
0.125	7.12500094	-0.86718609	0.625	7.62500152	4.60156787
0.15	7.15000097	-0.63649839	0.65	7.65000155	4.94850549
0.175	7.17500099	-0.40356069	0.675	7.67500158	5.30519309
0.2	7.20000101	-0.16799798	0.7	7.7000016	5.67200565
0.225	7.22500104	0.07056472	0.725	7.72500161	6.04931816
0.25	7.25000106	0.31250243	0.75	7.75000162	6.4375056
0.275	7.27500109	0.55819013	0.775	7.77500161	6.83694297
0.3	7.30000111	0.80800284	0.8	7.80000158	7.24800522
0.325	7.32500114	1.06231555	0.825	7.82500153	7.67106734
0.35	7.35000117	1.32150326	0.85	7.85000144	8.10650427
0.375	7.3750012	1.58594096	0.875	7.8750013	8.55469097
0.4	7.40000123	1.85600367	0.9	7.90000111	9.01600237
0.425	7.42500126	2.13206638	0.925	7.92500083	9.49081337
0.45	7.45000129	2.41450409	0.95	7.95000044	9.97949887
0.475	7.47500132	2.70369179	0.975	7.9749999	10.48243371
0.5	7.50000135	3.00000449	1	7.99999916	10.99999269

$\tilde{\mu}_1 = \lambda_{31}^{\tilde{h}}$	$\tilde{\mu}_2 = \lambda_{32}^{\tilde{h}}$	$\tilde{\mu}_3 = \lambda_{33}^{\tilde{h}}$
2.00000297	-3.00000131	8.99999832

For the difference of the corresponding values of the exact and constructed solutions of the problem the following estimate is true:

$$\max_{j=\overline{0,40}} \left\| x^*(t_j) - \tilde{x}(t_j) \right\| < 0.000007 \text{ and}$$
$$max\|\mu^* - \tilde{\mu}\| < 0.000003.$$

Conclusion

In this work, we propose a numerical implementation of parametrization method for finding solutions to linear boundary value problem with a parameter for an ordinary differential equation with multipoint integral condition. Using the parametrization method, we reduce the considered problem to the equivalent boundary value problem with parameters. The unknown functions are determined from the Cauchy problems for the system of ordinary differential equations, and the introduced parameters are determined from the system of algebraic equations. A numerical algorithm for finding solution to the considered problem is constructed. The Cauchy problem is solved by Runge– Kutta method of 4th-order accuracy. The examples illustrating the numerical algorithms of parametrization method are provided.

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