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Nonexistence of travelling wave solution of the Korteweg-de Vries Benjamin Bona Mahony equation

Abstract. This paper is devoted to the Korteweg-de Vries Benjamin Bona Mahony equation in an infinite domain. The paper discusses weak solutions of the Korteweg-de Vries Benjamin Bona Mahony equation without any conditions at infinity. This particular problem arises from the phenomenon of long breaking wave with small amplitude in fluid. In fluid dynamics, a breaking wave is a wave whose amplitude reaches a critical level at which some process can suddenly start to occur that causes large amounts of wave energy to be transformed into turbulent kinetic energy.

For the Korteweg-de Vries Benjamin Bona Mahony equation, we obtain the conditions of blowing-up of travelling wave solutions in finite time. Moreover, there is an explicit upper bound estimate for the wavelength of the corresponding singular traveling wave, depending on the speed of waves. The proof of the results is based on the nonlinear capacity method. In closing, we provide the numerical examples.

Key words: Breaking waves, Korteweg-de Vries-Benjamin-Bona-Mahony equation; blow-up of solution, travelling wave solution.

1. Introduction

1.1. Breaking waves

In fluid dynamics, a breaking wave is a wave whose amplitude reaches a critical level at which some process can suddenly start to occur that causes large amounts of wave energy to be transformed into turbulent kinetic energy. At this point, simple physical models that describe wave dynamics often become invalid, particularly those that assume linear behavior.

Breaking of water surface waves may occur anywhere that the amplitude is sufficient, including in mid-ocean. However, it is particularly common on beaches because wave heights are amplified in the region of shallower water (because the group velocity is lower there). There are three basic types of breaking water waves [1]. They are spilling, plunging and surging:

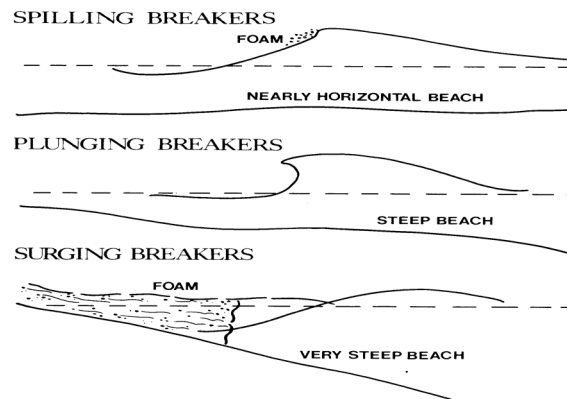


Figure 4 – Types of breaking water waves

1.2. Mathematical model

In this section we present the well-known mathematical model of the Korteweg de Vries-Benjamin-Bona-Mahony equation (see. [2]).

One of the well-known non-linear equations that embody both variance and non-linearity and is actively used in applications is the Korteweg-de Vries equation [3] which models the unidirectional propagation of weakly nonlinear and weakly dispersive waves:

$$\eta_t + c \left(1 + \frac{3h}{2} \eta \right) \eta_x + \frac{ch^2}{6} \eta_{xxx} = 0, \quad (1)$$

where η is the vertical excursion of the free surface above the still water level, h is the uniform undisturbed water depth and $c = \sqrt{gh}$ is the speed of linear gravity waves (g being the gravity acceleration).

The Benjamin-Bona-Mahony equation is an alternative to the Korteweg-de Vries equation [4] which is described as follows:

$$\eta_t + c \left(1 + \frac{3h}{2} \eta \right) \eta_x - \frac{ch^2}{6} \eta_{txx} = 0.$$

We consider the following scaled dependent and independent variables:

$$\eta \leftarrow \frac{\eta}{a_0}, \quad x \leftarrow \frac{x}{l}, \quad t \leftarrow \frac{ct}{l},$$

where a_0 is the characteristic wave amplitude and l is the characteristic wavelength. In dimensionless variables KdV equation (1) reads:

$$\eta_t + \left(1 + \frac{3\epsilon}{2} \eta \right) \eta_x + \frac{\mu^2}{6} \eta_{xxx} = 0,$$

where parameter $\epsilon = \frac{a_0}{h}$ measures the nonlinearity and $\mu = \frac{h}{l}$ is the dispersion parameter. The relative importance of these two effects is measured by the so-called Stokes-Ursell number [5]:

$$S = \frac{\epsilon}{\mu^2} \equiv \frac{a_0 l^2}{h^3}.$$

The last equation can be further simplified if we perform an additional change of variables:

$$\eta \leftarrow \frac{3\mu^2}{a_0} \eta, \quad x \leftarrow \frac{\sqrt{6}}{\mu} (x - t), \quad t \leftarrow \frac{\sqrt{6}}{\mu} t,$$

which yields the following simple equation including explicitly the Stokes-Ursell number S :

$$\eta_t + S\eta\eta_x + \eta_{xxx} = 0.$$

The last scaled KdV equation can be further generalized by using the low-order asymptotic relations in order to alternate higher order terms as it was proposed by Bona and Smith [6]. This step is rather standard and we do not provide here the details of the derivation [7]:

$$\eta_t + S\eta\eta_x + \eta_{xxx} - \delta\eta_{txx} = 0, \quad (2)$$

where $\delta \in \mathbb{R}$. The equation (2) is so-called Korteweg-de Vries-Benjamin-Bona-Mahony equation.

We note that for a particular value of the Stokes-Ursell number $S = 1$ another simpler scaling is possible when all the lengths (x and η) are scaled by the mean water depth h .

1.3. Statement of the problem

We consider one of the mathematical problem of the breaking water waves, the Korteweg-de Vries-Benjamin-Bona-Mahony equation:

$$\begin{aligned} \eta_t + \eta\eta_x + \eta_{xxx} - \eta_{txx} - \eta_x &= \\ = 0, t > 0, x \in \mathbb{R}. \end{aligned} \quad (3)$$

The Korteweg-de Vries-Benjamin-Bona-Mahony equation has important application in different physical situations such as waves on shallow water, and processes in semiconductors with differential conductivity.

In [8], traveling-wave solutions $u(x, t) = f(x - ct)$ are sought for the equation (3) which describes wave the processes in semiconductors with strong spatial dispersion. In [8-12] the authors obtained sufficient conditions for the finite time blow-up of solutions of time and space initial problems for the Korteweg-de Vries and Benjamin-Bona-Mahony type equations.

In this paper, based on the method of nonlinear capacity [13-15], the existence of singular travelling wave solutions of the equation (3) is proved.

2. Singular travelling wave solutions

We consider the traveling wave type solutions of the Korteweg-de Vries-Benjamin Bona Mahony equation (3):

$$\eta(x, t) = \eta(\xi),$$

where $\xi = x - ct$ and c is the wave velocity. Then $\eta(\xi)$ satisfies

$$(1 + c)\eta''' + \eta\eta' - (1 + c)\eta' = 0. \quad (4)$$

Equation (4) admits the following integrals:

$$(1 + c)\eta'' + \frac{\eta^2}{2} - (1 + c)\eta + C = 0, \quad (5)$$

where C is an arbitrary constant.

2.1. Nonexistence of travelling wave solution

A weak solution of (5) is a function $\eta \in L^2(I), I \subset \mathbb{R}$ that satisfies the integral identity

$$\begin{aligned} & \int_I \eta^2(\xi)\varphi(\xi)d\xi = \\ & = -2(1 + c) \int_I \eta(\xi)(\varphi''(\xi) - \varphi(\xi))d\xi - \\ & \quad -2C \int_I \varphi(\xi)d\xi \end{aligned} \quad (6)$$

for $\varphi \in C_0^2(I)$.

We multiply equation (5) by a nonnegative test function $\varphi \in C_0^2(I)$ with compact support. Then after integration we obtain (6). Hence, by the Young inequality with parameter $a > 0$, we find that

$$\begin{aligned} \int_I \eta^2(\xi)\varphi(\xi)d\xi & \leq \frac{(1+c)}{a} \int_I \eta^2(\xi)\varphi(\xi)d\xi + \\ & + a(1 + c) \int_I \frac{(\varphi''(\xi) - \varphi(\xi))^2}{\varphi(\xi)} d\xi - \\ & -2C \int_I \varphi(\xi)d\xi. \end{aligned} \quad (7)$$

We now take the test function:

$$\varphi(\xi) = \varphi_0(\tau), \tau = \frac{\xi}{L},$$

where $L \geq 2$ is a free parameter and the function $0 \leq \varphi_0 \in C^2(I)$ such that

$$\varphi_0(\tau) = \begin{cases} 1 & \text{if } |\tau| \leq 1, \\ 0 & \text{if } |\tau| \geq 2. \end{cases}$$

Let the function φ_0 satisfies the following properties

$$\alpha = \int_{-2}^2 \frac{|L^2\varphi_0(\tau) - \varphi_0''(\tau)|^2}{\varphi_0(\tau)} d\tau < \infty,$$

and

$$\beta = \int_{-2}^2 \varphi_0(\tau)d\tau < \infty.$$

Then, if $a = c + 1$ the inequality (7) implies

$$(1 + c)(2 + c) \frac{\alpha}{L^2} \geq 2C\beta.$$

From this it directly follows that if there exist C such that the inequality (7) holds, then there is no such bounded travelling wave solution of equation (5).

Then the following results are true

Theorem 1. The equation (4) with support $L \geq 2$, satisfying the inequality

$$C > \frac{(c+1)(c+2)\alpha}{2\beta L^2} \quad (8)$$

does not admit a solution.

Thus, a sufficient condition for the existence of an unbounded traveling wave with a wavelength L_* is the fulfillment of the inequality

$$\frac{2L^2}{(c + 1)(c + 2)} C > \frac{\alpha}{\beta}$$

with $L > L_*$.

2.2. Numerical examples

In this subsection we consider some numerical examples for equation (5) with different viscosities. We consider some initial data (at $x = ct$) for a traveling wave. In this case, we note that the nonexistence of a solution to equation (5) depends on the conditions (8).

First, consider an example where the wave velocity is small enough. That is, consider a fluid with a velocity between zero and one. Then, as seen from the Figure 1, the traveling wave breaks relatively quickly.

Let us now study a fluid with a velocity between 50 and 100. In this case, the time of breaking the wave slightly increases. It is easy to see from the Figure 2.

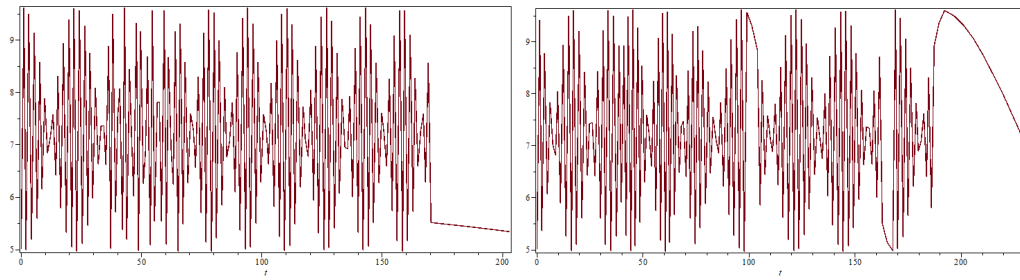


Figure 2 – Singular travelling waves

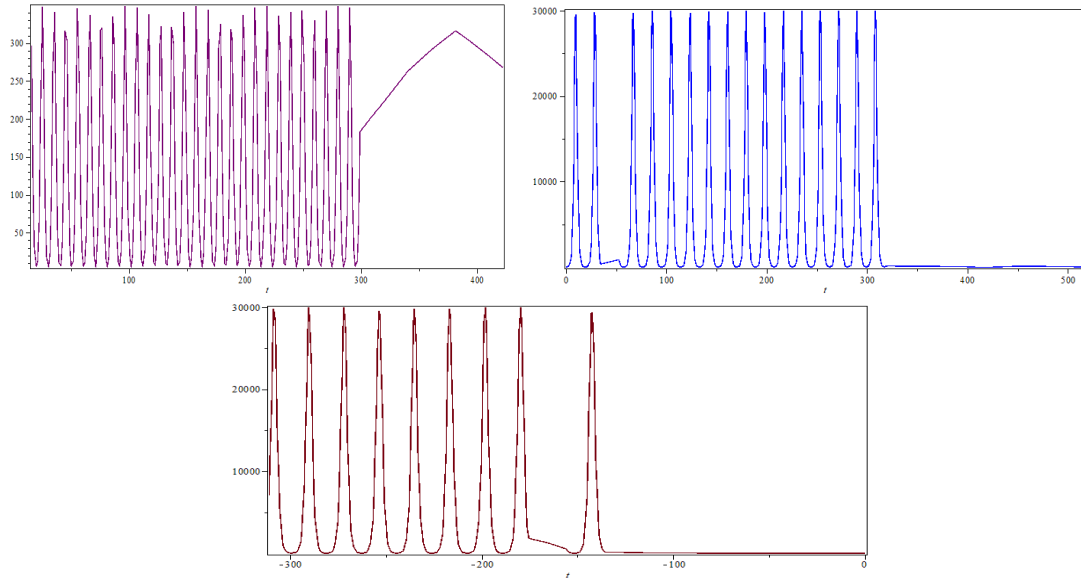


Figure 3 – Breaking travelling waves

Now let the fluid velocity be large enough. That is, consider a fluid with a velocity of about one

thousand. In this case, as seen from the Figure 3 the time of breaks of traveling waves will be quite large.

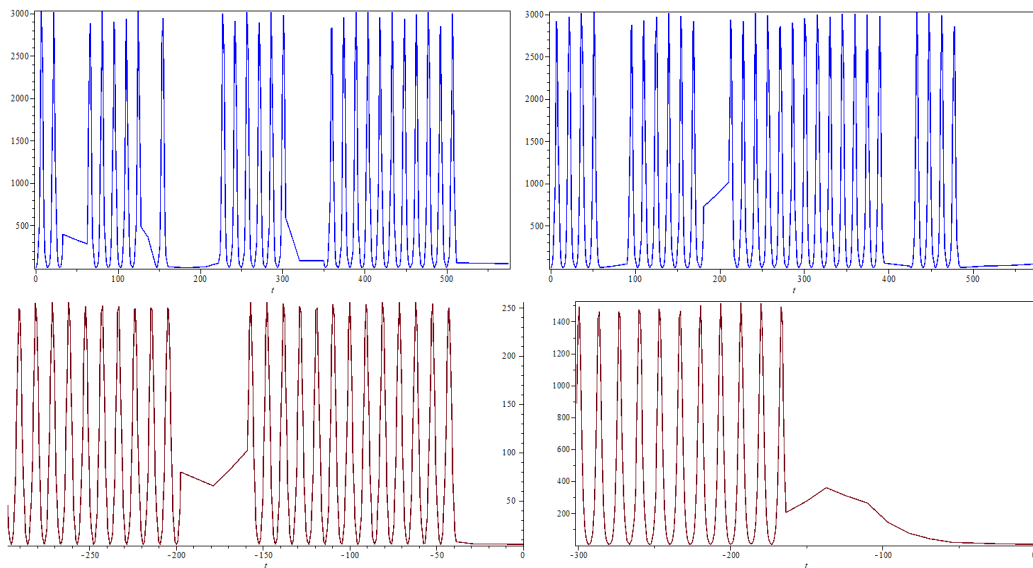


Figure 4 – Breaking travelling waves

Analyzing the above examples, we come to the conclusion that with an increase of the wave velocity, the time of wave break-up increases.

Conclusion

The present paper is devoted to the Korteweg-de Vries-Benjamin-Bona-Mahony equation in an infinite interval. This particular problem arises from the phenomenon of long breaking waves with small amplitude in fluid. For the Korteweg-de Vries-Benjamin-Bona-Mahony equation, we proved the nonexistence of the singular travelling wave solutions. Moreover, we provide some examples.

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