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An inverse problem for the pseudo-parabolic equation for Laplace operator

Abstract. A class of inverse problems for restoring the right-hand side of the pseudo-parabolic equation for 1D Laplace operator is considered. The inverse problem is to be well-posed in the sense of Hadamard whenever an overdetermination condition of the final temperature is given. Mathematical statements involve inverse problems for the pseudo-parabolic equation in which, solving the equation, we have to find the unknown right-hand side depending only on the space variable. We prove the existence and uniqueness of the classical solutions. The proof of the existence and uniqueness results of the solutions is carried out by using L-Fourier analysis. The mentioned results are presented as well as for the fractional time pseudo–parabolic equation. Inverse problems of identifying the coefficients of right hand side of the pseudo-parabolic equation from the local overdetermination condition have important applications in various areas of applied science and engineering, also such problems can be modeled using common homogeneous left-invariant hypoelliptic operators on common graded Lie groups.
Key words: Pseudo-parabolic equation, 1D Laplace operator, fractional Caputo derivative, inverse problem, well-posedness.

Introduction

In this paper we study inverse problem for the time-fractional pseudo-parabolic equation for one dimensional Laplace operator. We consider following equation

$$\mathcal{D}_t^\alpha[u(t,x)-u_{xx}(t,x)]-u_{xx}(t,x)=f(x), \quad (1)$$

for $(t,x)\in\Omega=\{(t,x)|0<t\le T<\infty, 0\le x\le l\}$, where $\mathcal{D}_t^\alpha$ is the Caputo derivative which is defined in the next section. The operator $-\frac{d^2}{dx^2}$ which is participating in the equation(1) is the well known 1D Laplace operator and we will denote it further by $\mathcal{L}$. We know the second order differential operator in $L^2(0,l)$ generated by the differential expression

$$\mathcal{L}u(x)=-u_{xx}(x), x\in(0,l) \quad (2)$$

and boundary conditions

$$u(0)=0, u(l)=0, \quad (3)$$

is self-adjoint in $L^2(0,l)$. The problem (2)-(3) has the following eigenvalues

$$\lambda_k=\left(\frac{k\pi}{l}\right)^2, k\in\mathbb{N},$$

and the corresponding system of eigenfunctions

$$e_k(x)=\sqrt{\frac{2}{l}}\sin\frac{k\pi}{l}(x), k\in\mathbb{N}.$$

It is known that the self-adjoint problem has real eigenvalues and their eigenfunctions form a complete orthonormal basis in $L^2(0,l)$.

The study of inverse problems for pseudo parabolic equations began in the 1980s. The first result obtained by Rundell [2] refers to the inverse identification problems for an unknown sourse function f in a following equation

$$\frac{\partial}{\partial t}[u(x,t)+\mathcal{L}u(x,t)]+\mathcal{L}u(x,t)=f. \qquad (4)$$

Where $\mathcal{L}$ is even order linear differential operator. Rundell proved global existence and uniqueness theorems for cases when f depends only x or only t. In a series of articles [6], [7], [8], [9], [10], [12], [13], [14], [15],[16],[17] some recent work has been done on inverse problems and spectral problems for the diffusion and anomalous diffusion equations.

Definitions of fractional operators

We begin this paper with a brief introduction of several concepts that are important for the further studies.

Definition 1. [5] The Riemann-Liouville fractional integral I^{α} of order $\alpha>0$ for an integrable function is defined by

$$I^{\alpha}[f](t)=\frac{1}{\Gamma(\alpha)}\int_c^t(t-s)^{\alpha-1}f(s)ds, t\in[c,d],$$

where Γ denotes the Euler gamma function.

Definition 2. [5] The Riemann-Liouville fractional derivative D^{α} of order $\alpha\in(0,1)$ of a continuous function is defined by

$$D^{\alpha}[f](t)=\frac{d}{dt}I^{\alpha}[f](t), t\in[c,d].$$

Definition 3. [5] The Caputo fractional derivative of order $0<\alpha<1$ of a differentiable function is defined by

$$\mathcal{D}_*^{\alpha}[f](t)=D^{\alpha}[f'(t)], t\in[c,d].$$

Definition 4.[5] (Caputo derivative). Let $f\in L^1[a,b], -\infty\le a<t<b\le+\infty$and $f*K_{m-\alpha}(t)\in W^{m,1}[a,b], m=[\alpha], \alpha>0$.The Caputo fractional derivative$\partial_{+a}^{\alpha}$of order$\alpha\in\mathbb{R}$ $(m-1<\alpha<m, m\in\mathbb{N})$ is defined as

$$\partial_{+a}^{\alpha}f(t)=$$
$$=D_{+a}^{\alpha}\left[f(t)-f(a)-f'(a)\frac{(t-a)}{1!}-\ldots-f^{(m-1)}(a)\frac{(t-a)^{m-1}}{(m-1)!}\right].$$

If $f\in C^m[a,b]$ then, the Caputo fractional derivative ∂_{+a}^{α} of order $\alpha\in\mathbb{R}$ ($m-1<\alpha<m, m\in\mathbb{N}$) is defined as

$$\partial_{+a}^{\alpha}[f](t)=I_{+a}^{m-\alpha}f^{(m)}(t)=$$
$$=\frac{1}{\Gamma(m-\alpha)}\int_a^t(t-s)^{m-1-\alpha}f^{(m)}(s)ds.$$

Formulation of the problem

Problem 1. We aim to find a couple of functions $(\mathrm{u}(\mathrm{t},\mathrm{x}),\mathrm{f}(\mathrm{x}))$ satisfying the equation(1), under the conditions

$$u(0,x)=\varphi(x), x\in[0,l] \qquad (5)$$

$$u(T,x)=\psi(x), x\in[0,l]. \qquad (6)$$

and the homogeneous Dirichlet boundary conditions

$$u(t,0)=u(t,l)=0, t\in[0,T]. \qquad (7)$$

By using $\mathcal{L}$–Fourier analysis we obtain existence and uniqueness results for this problem.

We say a solution of Problem 1 is a pair of functions $(u(t,x),f(x))$ such that they satisfy equation(1) and conditions(5)-(7) where $u(t,x)\in C^1([0,T];C^2([0,l]))$ and $f(x)\in C([0,l])$.

Main results

For Problem 3.1, the following theorem holds.

Theorem 1. Assume that$\varphi(x),\psi(x)\in C_0^3[0,\pi]$. Then the solution $u(t,x)\in C^1\big([0,T],\ C^2([0,l])\big)$, $f(x)\in C([0,l])$ of the Problem 3.1 exists, is unique, and can be written in the form

$$u(t,x) = \varphi(x) + \sum_{k=1}^{\infty} \frac{\left(\varphi_k^{(2)} - \psi_k^{(2)}\right)\left(1 - E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2} t^\alpha\right)\right)}{\left(\frac{k\pi}{l}\right)^2\left(1 - E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2} T^\alpha\right)\right)} \sin\frac{k\pi}{l}(x), \tag{8}$$

$$f(x) = -\varphi''(x) + \sum_{k=1}^{\infty} \frac{\varphi_k^{(2)} - \psi_k^{(2)}}{1 - E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2} T^\alpha\right)} \sin\frac{k\pi}{l}(x), \tag{9}$$

where $\varphi_k^{(2)} = (\varphi'', e_k)_{L^2(0,l)}, \psi_k^{(2)} = (\psi'', e_k)_{L^2(0,l)}$ and $E_{\alpha,\beta}(\lambda t)$ is the Mittag-Leffler type function (see [4]):

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}.$$

First of all, we start by proving an existence result.

Proof. Let us seek functions $u(x,t)$ and $f(x)$ in the forms:

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t) \sin\frac{k\pi}{l}(x), k \in \mathbb{N}, \tag{10}$$

and

$$f(x) = \sum_{k=1}^{\infty} f_k \sin\frac{k\pi}{l}(x), k \in \mathbb{N}, \tag{11}$$

where $u_k(t)$ and f_k are unknown. Substituting Equations (10) and (11) into Equation (1), we obtain the following equation for the functions $u_k(t)$ and the constants f_k:

$$D_t^\alpha u_k(t) + \frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2} u_k(t) = \frac{f_k}{1+\left(\frac{k\pi}{l}\right)^2}.$$

Solving these equation,we obtain

$$u_k(t) = \frac{f_k}{\left(\frac{k\pi}{l}\right)^2} + C_k E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2} t^\alpha\right),$$

where the constants f_k and C_k are unknown. To find these constants, we use conditions(5), (6). Let φ_k and ψ_k be the coefficients of the expansions of $\varphi(x)$ and $\psi(x)$:

$$\varphi_k = \sqrt{\frac{2}{l}} \int_0^l \varphi(x) \sin\frac{k\pi}{l}(x) dx, k \in \mathbb{N},$$

$$\psi_k = \sqrt{\frac{2}{l}} \int_0^l \psi(x) \sin\frac{k\pi}{l}(x) dx, k \in \mathbb{N}.$$

We first find C_k:

$$u_k(0) = \frac{f_k}{\left(\frac{k\pi}{l}\right)^2} + C_k = \varphi_k,$$

$$u_k(T) = \frac{f_k}{\left(\frac{k\pi}{l}\right)^2} + C_k E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2} T^\alpha\right) = \psi_k.$$

Then

$$C_k = \frac{\varphi_{1k} - \psi_{1k}}{1 - E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2} T^\alpha\right)}.$$

The constant f_k is represented as

$$f_k = \varphi_k \left(\frac{k\pi}{l}\right)^2 - C_k \left(\frac{k\pi}{l}\right)^2.$$

Substituting $u_k(t)$ and f_k into expansion (10), we find

$$u(t,x)=\varphi(x)+\sum_{k=1}^{\infty} C_k\left(E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2}t^{\alpha}\right)-1\right)\sin\frac{k\pi}{l}(x). \quad (12)$$

By the supposition of the theorem we know

$$\varphi^{(n)}(0)=0, \varphi^{(n)}(\pi)=0, n=0,1,2,$$
$$\psi^{(n)}(0)=0, \psi^{(n)}(\pi)=0, n=0,1,2.$$

then using they we have

$$C_k=\frac{\varphi_k-\psi_k}{1-E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2}T^{\alpha}\right)}=-\frac{\varphi_k^{(2)}-\psi_k^{(2)}}{\left(\frac{k\pi}{l}\right)^2\left(1-E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2}T^{\alpha}\right)\right)}.$$

Putting this into equations (10) and (11) we obtain

$$u(t,x)=\varphi(x)+\sum_{k=1}^{\infty}\frac{\left(\varphi_k^{(2)}-\psi_k^{(2)}\right)\left(1-E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2}t^{\alpha}\right)\right)}{\left(\frac{k\pi}{l}\right)^2\left(1-E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2}T^{\alpha}\right)\right)}\sin\frac{k\pi}{l}(x). \quad (13)$$

Similarly,

$$f(x)=-\varphi''(x)+\sum_{k=1}^{\infty}\frac{\varphi_k^{(2)}-\psi_k^{(2)}}{1-E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2}T^{\alpha}\right)}\sin\frac{k\pi}{l}(x). \quad (14)$$

The following Mittag-Leffler function's estimate is known by [11]:

$$|E_{\alpha,\beta}(z)|\le\frac{M}{1+|z|}, arg(z)=\pi, |z|\to\infty. \quad (15)$$

Now, we show that $u(t,x)\in C^1([0,T];C^2([0,l]))$, $f(x)\in C([0,l])$, that is

$$\| u \|_{C^1([0,T];C^2([0,l]))}=\max_{t\in[0,T]}\| u(t,\cdot) \|_{C^2([0,l])}+\max_{t\in[0,T]}\| \mathcal{D}_t^{\alpha}u(t,\cdot) \|_{C^2([0,l])}<\infty,$$

and

$$\| f \|_{C([0,l])}<\infty.$$

By using (15), we get following estimates

$$\begin{aligned}|u(t,x)| &\lesssim |\varphi(x)|+\sum_{k=1}^{\infty}\frac{\left|\varphi_k^{(2)}\right|+\left|\psi_k^{(2)}\right|}{\left(\frac{k\pi}{l}\right)^2\left(1-E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2}T^{\alpha}\right)\right)}\\ &\lesssim |\varphi(x)|+\sum_{k=1}^{\infty}\frac{\left|\varphi_k^{(2)}\right|+\left|\psi_k^{(2)}\right|}{\left(\frac{k\pi}{l}\right)^2},\end{aligned} \quad (16)$$

$$|f(x)| \lesssim |\varphi''(x)| + \sum_{k=1}^{\infty} \frac{\left|\varphi_k^{(2)}\right| + \left|\psi_k^{(2)}\right|}{1 - E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2} T^\alpha\right)} \lesssim |\varphi''(x)| + \sum_{k=1}^{\infty} \left|\varphi_k^{(2)}\right| + \left|\psi_k^{(2)}\right|. \quad (17)$$

Where, $L \lesssim Q$ \$L denotes $L \leq CQ$ for some positive constant C independent of L and Q.

By supposition of the theorem we know $\varphi^{(2)}$ and $\psi^{(2)}$ are continuous on $[0, l]$.

Then by the Bessel inequality for the trigonometric series (see [1]) and by the Weierstrass M-test (see [3]), series (16) and (17) converge absolutely and uniformly in the region $\overline{\Omega}$. Now we show.

$$|u_{xx}(t,x)| \lesssim |\varphi''(x)| + \sum_{k=1}^{\infty} \frac{\left|\varphi_k^{(2)}\right| + \left|\psi_k^{(2)}\right|}{1 - E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2} T^\alpha\right)} \lesssim |\varphi''(x)| + \sum_{k=1}^{\infty} \left|\varphi_k^{(2)}\right| + \left|\psi_k^{(2)}\right| < \infty,$$

$$|\mathcal{D}_t^\alpha u(t,x)| \lesssim \sum_{k=1}^{\infty} \frac{\left|\varphi_k^{(2)}\right| + \left|\psi_k^{(2)}\right|}{\left(1 + \left(\frac{k\pi}{l}\right)^2\right)\left(1 - E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2} T^\alpha\right)\right)} \lesssim \sum_{k=1}^{\infty} \frac{\left|\varphi_k^{(2)}\right| + \left|\psi_k^{(2)}\right|}{1 + \left(\frac{k\pi}{l}\right)^2} < \infty,$$

$$|\mathcal{D}_t^\alpha u_{xx}(t,x)| \lesssim \sum_{k=1}^{\infty} \frac{\left(\frac{k\pi}{l}\right)^2 \left(\left|\varphi_k^{(2)}\right| + \left|\psi_k^{(2)}\right|\right)}{\left(1 + \left(\frac{k\pi}{l}\right)^2\right)\left(1 - E_{\alpha,1}\left(-\frac{\left(\frac{k\pi}{l}\right)^2}{1+\left(\frac{k\pi}{l}\right)^2} T^\alpha\right)\right)} \lesssim \sum_{k=1}^{\infty} \left|\varphi_k^{(2)}\right| + \left|\psi_k^{(2)}\right| + \sum_{k=1}^{\infty} \frac{\left|\varphi_k^{(2)}\right| + \left|\psi_k^{(2)}\right|}{1 + \left(\frac{k\pi}{l}\right)^2} < \infty.$$

Finally, we obtain

$$\| u \|_{C^1([0,T], C^2[0,\pi])} \leq C < \infty, C = const,$$

and

$$\| f \|_{C([0,l])} < \infty.$$

Existence of the solution is proved.

Now, we start proving uniqness of the solution.Let us suppose that $\{u_1(t,x), f_1(x)\}$ and $\{u_2(t,x), f_2(x)\}$ are solution of the Problem 1. Then $u(t,x) = u_1(t,x) - u_2(t,x)$ and $f(x) = f_1(x) - f_2(x)$ are solution of following problem:

$$\mathcal{D}_t^\alpha [u(t,x) - u_{xx}(t,x)] - u_{xx}(t,x) = f(x), \quad (18)$$

$$u(0, x) = 0, \tag{19}$$

$$u(T, 0) = 0. \tag{20}$$

By using (13) and (14) for (18)-(20) we easily see $u(x, t) \equiv 0, f(x) \equiv 0$. Uniqness of the solution of the Problem 1.

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