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Construction of a solution for optimal control problem with phase and integral constraints

Abstract. A method for solving the Lagrange problem with phase restrictions for processes described by ordinary differential equations without involvement of the Lagrange principle is supposed. Necessary and sufficient conditions for existence of a solution of the variation problem are obtained, feasible control is found and optimal solution is constructed by narrowing the field of feasible controls. The basis of the proposed method for solving the variation problem is an immersion principle. The essence of the immersion principle is that the original variation problem with the boundary conditions with phase and integral constraints is replaced by equivalent optimal control problem with a free right end of the trajectory. This approach is made possible by finding the general solution of a class of Fredholm integral equations of the first order. The scientific novelty of the results is that: there is no need to introduce additional variables in the form of Lagrange multipliers; proof of the existence of a saddle point of the Lagrange functional; the existence and construction of a solution to the Lagrange problem are solved together.

Key words: immersion principle, feasible control, integral equations, optimal control, optimal solution, minimizing sequence.

Introduction

One of the methods for solving the variational calculus problem is the Lagrange principle. The Lagrange principle makes it possible to reduce the solution of the original problem to the search for the extremum of the Lagrange functional obtained by introducing auxiliary variables (Lagrange multipliers).

The Lagrange principle is the statement about the existence of Lagrange multipliers, satisfying a set of conditions when the original problem has a weak local minimum. The Lagrange principle gives the necessary condition for a weak local minimum and it does not exclude the existence of other methods for solving variational calculus problems unrelated to the Lagrange functional.

The Lagrange principle is devoted to the works [1-3]. A unified approach to different extremum problems based on the Lagrange principle is described in [4].

In the classical variational calculus, it is assumed that the solution of the differential equation belongs to the space $C^1(I,R^n)$, and the control $u(t),\ t \in I$ is from the space $C^1(I,R^m)$, in optimal control problems [5] the solution $x(t) \in KC^1(I,R^n)$, and the control $u(t) \in KC(I,R^m)$. In this work, the control $u(t)$, $t \in I$ is chosen from $L_2(I,R^m)$ and the solution $x(t)$, $t \in I$ is an absolutely continuous function on the interval $I = [t_0, t_1]$. For this case solvability and uniqueness of the initial problem for differential equation are given in [4, 6-8].

The purpose of this work is to create a method for solving the variational calculus problem for the processes described by ordinary differential equations with phase and integral constraints that differ from the known methods based on the Lagrange principle. It is a continuation of the research outlined [9-10].

Problem statement

We consider the following problem: minimize the functional

$$J(u(\cdot),x_0,x_1)= \\ =\int_{t_0}^{t_1} F_0(x(t),u(t),x_0,x_1,t)dt \to inf \quad (1.1)$$

at conditions

$$\dot{x} = A(t)x + B(t)f(x,u,t),\ t\in I=[t_0,t_1] \quad (1.2)$$

with boundary conditions

$$(x(t_0)) = x_0, x(t_1) = x_1) \in S_0 \times S_1 = S \subset R^{2n} \quad (1.3)$$

in the presence of phase constraints

$$x(t)\in G(t): G(t)= \\ =\{x\in R^n \,/\, \omega(t)\le Q(x,t)\le \phi(t),\ t\in I\},$$

and integral constraints

$$g_j(u(\cdot),x_0,x_1)\le 0, \\ j=\overline{1,m_1};\ g_j(u(\cdot),x_0,x_1)=0, \quad (1.4) \\ j=\overline{m_1+1,m_2},$$

$$g_j(u(\cdot),x_0,x_1)= \\ =\int_{t_0}^{t_1} f_{0j}(x(t),u(t),x_0,x_1,t)dt, \quad (1.5) \\ j=\overline{1,m_2}.$$

where the control

$$u(\cdot)\in L_2(I,R^m). \quad (1.6)$$

Here $A(t)$, $B(t)$ are matrices with piecewise-continuous elements of orders $n\times n$, $n\times r$, respectively, a vector function $f(x,u,t)=(f_1(x,u,t),\ldots,f_r(x,u,t))$ is continuous with respect to the variables $(x,u,t)\in R^n\times R^m\times I$, satisfies the Lipschitz condition by x, i.e.

$$|f(x,u,t)-f(y,u,t)|\le l(t)|x-y|, \\ \forall(x,u,t),(y,u,t)\in R^n\times R^m\times I \quad (1.7)$$

and the condition

$$|f(x,u,t)|\le c_0(|x|+|u|^2)+c_1(t), \\ \forall(x,u,t), \quad (1.8)$$

where $l(t)\ge 0$, $l(t)\in L_1(I,R^1)$, $c_0 = const > 0$, $c_1(t)\ge 0$, $c_1(t)\in L_1(I,R^1)$.

The vector function $F(x,t)=(F_1(x,t),\ldots,F_s(x,t))$ is continuous with respect to the variables $(x,t)\in R^n\times I$. Function $f_0(x,u,x_0,x_1,t)=(f_{01}(x,u,x_0,x_1,t),\ldots$ $\ldots,f_{0m_2}(x,u,x_0,x_1,t))$ satisfies the condition

$$|f_0(x,u,x_0,x_1,t)|\le c_2(|x|+|u|^2+|x_0|+|x_1|)+ \\ +c_3(t),$$

$$\forall(x,u,x_0,x_1,t),(y,u,x_0,x_1,t)\in R^n\times \\ \times R^m\times R^n\times R^n\times I,$$

$$c_2 = const\ge 0,\ c_3(t)\ge 0,\ c_3(t)\in L_1(I,R^1).$$

Scalar function $F_0(x,u,x_0,x_1,t)$ is defined and continuous with respect to the variables together with partial derivatives by variables (x,u,x_0,x_1), $\omega(t)$, $\varphi(t)$, $t\in I-$ are given $s-$ dimensional functions. S is given bounded convex closed set of R^{2n}, the time moments t_0,t_1 are fixed.

In particular, the set $S=\{(x_0,x_1)\in R^{2n}/H_j(x_0,x_1)\le 0,\quad j=\overline{1,p_1};$ $<a_j,(x_0,x_1)>=0,\quad j=\overline{p_1+1,p_2}\},$ where $H_j(x_0,x_1)$, $j=\overline{1,p_1}$ are convex functions, $a_j\in R^{2n}$, $j=\overline{p_1+1,p_2}$ are given vectors.

Note, that if the conditions (1.7), (1.8) are satisfied for any control $u(\cdot)\in L_2(I,R^m)$ and the initial condition $x(t_0)=x_0$ of the differential equation (1.2) has a unique solution. $x(t),\ t \in I$. This solution has derivative $\dot{x} \in L_2(I, R^n)$ and satisfies equation (1.2) for almost all $t \in I$.

It should be noted that integral constraints

$$g_j(u(\cdot), x_0, x_1) = \int_{t_0}^{t_1} f_{0j}(x(t), u(t), x_0, x_1, t)dt \le 0, \quad j = \overline{1, m_1}, \tag{1.9}$$

by introducing additional variables $d_j \ge 0$, $j = \overline{1, m_1}$, can be written in the form

$$g_j(u(\cdot), x_0, x_1) = -d_j, \ \ j = \overline{1, m_1}.$$

Let the vector be $\overline{c} = (-d_1, \ldots, -d_{m_1}, 0, 0, \ldots, 0) \in R^{m_2}$, where $d_j \ge 0$, $j = \overline{1, m_1}$. Let a set be $Q = \{\overline{c} \in R^{m_2} / d_j \ge 0,\ j = \overline{1, m_1}\}$, where $d_j \ge 0$, $j = \overline{1, m_1}$ are unknown numbers.

Definition 1.1. The triple $(u_*(t), x_0^*, x_1^*) \in U \times S_0 \times S_1$ is called by admissible control for the problem (1.1) – (1.6), if the boundary problem (1.2) – (1.6) has a solution. A set of all admissible controls is denote by Σ, $\Sigma \subset U \times S_0 \times S_1$.

From this definition it follows that for each element of the set Σ the following properties are satisfied: 1) the solutions $x_*(t),\ t \in I$ of the differential equation (1.2), issuing from the point $x_0^* \in S_0$, satisfy the condition $x_*(t_1) = x_1^* \in S_1$, and also $(x_0^*, x_1^*) \in S_0 \times S_1 = S$; 2) the inclusion $x_*(t) \in G(t),\ t \in I$ holds; 3) for each element of the set Σ we have the equality $g(u(\cdot), x_0, x_1) = \overline{c}$, where

$$g(u_*(\cdot), x_0^*, x_1^*) = (g_1(u_*(\cdot), x_0^*, x_1^*), \ldots, g_{m_2}(u_*(\cdot), x_0^*, x_1^*)).$$

The following problems are set:

Problem 1.2. *Find the necessary and sufficient conditions for the existence of a solution of the boundary value problem (1.2) – (1.6).*

Note, that the optimal control problem (1.1) – (1.6) has a solution if and only if the boundary value problem (1.2) – (1.6) has a solution.

Problem 1.3. *Find an admissible control* $(u_*(t), x_0^*, x_1^*) \in \Sigma \subset U \times S_0 \times S_1$.

If problem 1.2. has a solution, then there exists an admissible control.

Problem 1.4. *Find the optimal control* $\overline{u}_*(t) \in U(t)$, *the point* $(\overline{x}_0^*, \overline{x}_1^*) \in S_0 \times S_1 = S$ *and the optimal trajectory* $\overline{x}_*(t; t_0, \overline{x}_0^*),\ t \in I$, where $\overline{x}_*(t) \in G(t),\ t \in I,\ \overline{x}_*(t_1) = \overline{x}_1^* \in S_1$, $g_j(\overline{u}_*(\cdot), \overline{x}_0^*, \overline{x}_1^*) \le 0,\ j = \overline{1, m_1},\ g_j(\overline{u}_*(\cdot), \overline{x}_0^*, \overline{x}_1^*) = 0$, $j = \overline{m_1 + 1, m_2},\ J(\overline{u}_*(\cdot), \overline{x}_0^*, \overline{x}_1^*) = \inf J(\overline{u}(\cdot), \overline{x}_0, \overline{x}_1)$, $\forall (\overline{u}(\cdot), \overline{x}_0, \overline{x}_1) \in L_2(I, R^m) \times S_0 \times S_1$.

One of the methods for solving the problem of variation calculus is the Lagrange principle. The Lagrange principle allows to reduce the solution of the original problem to the search for an extremum of the Lagrange functional obtained by introducing auxiliary variables (Lagrange multipliers).

In the classical variation calculus, it is assumed that the solution of the differential equation (1.2) belongs to the space $C^1(I, R^n)$ and the control *u(t), t ∈ I* of the space $C(I, R^m)$ in the optimal control problems [5], the solution $x \in KC^1(I, R^n)$ and control $u(t) \in KC^1(I, R^m)$. In this paper, the control *u(t), t ∈ I* is chosen from $L_2(I, R^m)$, and the solution *x(t), t ∈ I* is an absolutely continuous function on the interval $I = [t_0, t_1]$. For this case, the existence and uniqueness of the solutions of the initial problem for equation (1.2) are presented in the references [4, 6, 7, 8].

The purpose of this paper is to create a method for solving the problem of the variation calculus for processes described by ordinary differential equations with phase and integral constraints that differ from the known methods based on the Lagrange principle. It is a continuation of the scientific research presented in [9-16].

Existence of a solution

We consider the following optimal control problem: minimize the functional

$$I_1(u(\cdot), p(\cdot), v_1(\cdot), v_2(\cdot), x_0, x_1, d) = \int_{t_0}^{t_1} F_1(q(t), t) \to \inf \tag{2.1}$$

$$\dot{z} = A_1(t)z + B_1(t)v_1(t) + B_2 v_2(t),\ z(t_0) = 0,\quad t \in I \tag{2.2}$$

$$v_1(\cdot) \in L_2(I, R^r),\ v_2(\cdot) \in L_2(I, R^{m_2}), \tag{2.3}$$

$$p(t) \in V(t), u(\cdot) \in L_2(I, R^m),\quad (x_0, x_1) \in S_0 \times S_1 = S,\ d \in \Gamma. \tag{2.4}$$

We introduce the following notations: $H = L_2(I,R^m) \times L_2(I,R^s) \times L_2(I,R^r) \times \times L_2(I,R^{m_2}) \times R^n \times R^n \times R^{m_1}$, $X = L_2(I,R^m) \times V \times L_2(I,R^r) \times \times L_2(I,R^{m_2}) \times S_0 \times S_1 \times \Gamma \subset H$, vector function $\theta(t) = (u(t), p(t), v_1(t), v_2(t), x_0, x_1, d) \in X \subset H$, $q(t) = (z(t), z(t_1), \theta(t))$.

The optimization problem (2.3) – (2.6) can be represented in the form:

$$I_1(\theta(\cdot)) = \int_{t_0}^{t_1} F_1(q(t),t) \to \inf,\ \theta(\cdot) \in X \subset H.$$

Let the set be $X_* = \{\theta_*(\cdot) \in X \mid I_1(\theta_*(\cdot)) = \inf_{\theta \in X} I_1(\theta(\cdot))\}$.

Lemma 2.1. *Let the matrix be positive definite $T(t_0,t_1) > 0$. In order to the boundary value problem (1.2) – (1.6) have a solution, it is necessary and sufficient that $\lim_{n\to\infty} I_1(\theta_n) = I_{1*} = \inf_{\theta\in X} I_1(\theta) = 0$, where $\{\theta_n(\cdot)\} \subset X$ is a minimizing sequence in the problem (2.1) – (2.4).*

Proof of the lemma follows from Theorem 2.3. and Lemmas 2.4. and 2.5. [9].

Theorem 2.2. *Let the matrix be $T(t_0,t_1) > 0$, the function $F_1(q,t)$ be defined and continuous in the set of variables (q,t) together with the partial derivatives with respect to q and satisfies the Lipschitz conditions*

$$| F_{1q}(q + \Delta q, t) - F_{1q}(q,t) | \le l\, | \Delta q |,\ t \in I, \tag{2.5}$$

where

$$F_{1q}(q,t) = (F_{1z}(q,t), F_{1z(t_1)}(q,t), F_{1u}(q,t), F_{1p}(q,t), F_{1v_1}(q,t), F_{1v_2}(q,t), F_{1x_0}(q,t), F_{1x_1}(q,t), F_{1d}(q,t)),$$

$$q = (z, z(t_1), u, p, v_1, v_2, x_0, x_1, d) \in R^{n+m_2} \times R^{n+m_2} \times \times R^m \times R^s \times R^r \times R^{m_2} \times R^n \times R^n \times R^{m_1},$$

$$\Delta q = (\Delta z, \Delta z(t_1), \Delta u, \Delta p, \Delta v_1, \Delta v_2, \Delta x_0, \Delta x_1, \Delta d),\quad l = const > 0.$$

Then the functional (2.1) under the conditions (2.2) – (2.4) is continuously Frechet differentiable, the gradient $I_1'(\boldsymbol{\theta}) = (I_{1u}'(\boldsymbol{\theta}), I_{1p}'(\boldsymbol{\theta}), I_{1v_1}'(\boldsymbol{\theta}), I_{1v_2}'(\boldsymbol{\theta}), I_{1x_0}'(\boldsymbol{\theta}), I_{1x_1}'(\boldsymbol{\theta}), I_{1d}'(\boldsymbol{\theta})) \in H$ at any point $\theta \in X$ is calculated by the formula

$$I_{1u}'(\boldsymbol{\theta}) = F_{1u}(q(t),t),\ I_{1p}'(\boldsymbol{\theta}) = F_{1p}(q(t),t),\ I_{1v_1}'(\boldsymbol{\theta}) = F_{1v_1}(q(t),t) - B_1^*(t)\boldsymbol{\psi}(t),$$
$$I_{1v_2}'(\boldsymbol{\theta}) = F_{1v_2}(q(t),t) - B_2^*\boldsymbol{\psi}(t),\ I_{1x_0}'(\boldsymbol{\theta}) = \int_{t_0}^{t_1} F_{1x_0}(q(t),t)dt,$$
$$I_{1x_1}'(\boldsymbol{\theta}) = \int_{t_0}^{t_1} F_{1x_1}(q(t),t)dt,\ I_{1d}'(\boldsymbol{\theta}) = \int_{t_0}^{t_1} F_{1d}(q(t),t)dt, \tag{2.6}$$

where $z(t)$, $t \in I$ is the solution of the differential equation (2.2), and the function $\psi(t)$, $t \in I$ is the solution of the conjugate system

$$\dot{\boldsymbol{\psi}} = F_{1z}(q(t),t) - A_1^*(t)\boldsymbol{\psi},\ \boldsymbol{\psi}(t_1) = -\int_{t_0}^{t_1} F_{1z(t_1)}(q(t),t)dt. \tag{2.7}$$

In addition, the gradient $I_1'(\theta)$, $\theta \in X$ satisfies the Lipschitz condition

$$\| I_1'(\boldsymbol{\theta}_1) - I_1'(\boldsymbol{\theta}_2) \| \le K \| \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \|,\ \forall \boldsymbol{\theta}_1,\ \boldsymbol{\theta}_2 \in X, \tag{2.8}$$

where $K = const > 0$.

Proof. Let $\theta(t), \theta(t) + \Delta\theta(t) \in X$, $z(t, v_1, v_2)$, $z(t, v_1 + \Delta v_1, v_2 + \Delta v_2)$, $t \in I$ be a solution of the system (2.2), (2.3). Let $z(t, v_1 + \Delta v_1, v_2 + \Delta v_2) = z(t, v_1, v_2) + \Delta z(t)$, $t \in I$. Then

$$|\Delta z(t)| \le C_1 \|\Delta v_1\| + C_2 \|\Delta v_2\|. \quad (2.9)$$

The increment of the functional (see (2.5))

$$\Delta I_1 = I_1(\boldsymbol{\theta} + \Delta\boldsymbol{\theta}) - I_1(\boldsymbol{\theta}) =$$

$$= \int_{t_0}^{t_1} [F_1(q(t) + \Delta q(t), t) - F_1(q(t), t)] dt =$$

$$= \int_{t_0}^{t_1} [\Delta u^*(t) F_{1u}(q(t),t) + \Delta p^*(t) F_{1p}(q(t),t) +$$

$$+ \Delta v_1^*(t) F_{1v_1}(q(t),t) +$$

$$+ \Delta v_2^*(t) F_{1v_2}(q(t),t) + \Delta x_0^* F_{1x_0}(q(t),t) +$$

$$+ \Delta x_1^* F_{1x_1}(q(t),t) + \Delta d^* F_{1d}(q(t),t) +$$

$$+ \Delta z^*(t) F_{1z}(q(t),t) + \Delta z^*(t_1) F_{1z(t_1)}(q(t),t)] dt +$$

$$+ \sum_{i=1}^{9} R_i, \quad (2.10)$$

where $|R_1| \le l_1 \int_{t_0}^{t_1} |\Delta u(t)| |\Delta q(t)| dt,$

$$|R_2| \le l_2 \int_{t_0}^{t_1} |\Delta p(t)| |\Delta q(t)| dt,$$

$$|R_3| \le l_3 \int_{t_0}^{t_1} |\Delta v_1(t)| |\Delta q(t)| dt,$$

$$|R_4| \le l_4 \int_{t_0}^{t_1} |\Delta v_2(t)| |\Delta q(t)| dt,$$

$$|R_5| \le l_5 \int_{t_0}^{t_1} |\Delta x_0| |\Delta q(t)| dt,$$

$$|R_6| \le l_6 \int_{t_0}^{t_1} |\Delta x_1| |\Delta q(t)| dt,$$

$$|R_7| \le l_7 \int_{t_0}^{t_1} |\Delta d| |\Delta q(t)| dt,$$

$$|R_8| \le l_8 \int_{t_0}^{t_1} |\Delta z(t)| |\Delta q(t)| dt,$$

$|R_9| \le l_9 \int_{t_0}^{t_1} |\Delta z(t_1)| |\Delta q(t)| dt$ by the Lipschitz condition (2.5). We note that (see (2.7), (2.9))

$$\int_{t_0}^{t_1} \Delta z^*(t_1) F_{1z(t_1)}(q(t),t) dt =$$

$$= -\int_{t_0}^{t_1} [\Delta v_1^*(t) B_1^*(t) + \Delta v_2^*(t) B_2^*] \boldsymbol{\psi}(t) dt - \quad (2.11)$$

$$- \int_{t_0}^{t_1} \Delta z^*(t) F_{1z}(q(t),t) dt.$$

From (2.10) and (2.11) we get

$$\Delta I_1 = \int_{t_0}^{t_1} \{\Delta u^*(t) F_{1u}(q(t),t) + \Delta p^*(t) F_{1p}(q(t),t) +$$

$$+ \Delta v_1^*(t)[F_{1v_1}(q(t),t) - B_1^*(t)\boldsymbol{\psi}(t)] +$$

$$+ \Delta v_2^*(t)[F_{1v_2}(q(t),t) - B_2^* \boldsymbol{\psi}(t)] +$$

$$+ \Delta x_0^* F_{1x_0}(q(t),t) + \Delta x_1^* F_{1x_1}(q(t),t) +$$

$$+ \Delta d^* F_{1d}(q(t),t)\} dt + \sum_{i=1}^{9} R_i = \langle I_1'(\theta), \Delta\theta \rangle_H + R,$$

where $R = \sum_{i=1}^{9} R_i$, $|R| \le C_3 \|\Delta\theta\|^2$, $\dfrac{|R|}{\|\Delta\theta\|} \to 0$, *at* $\|\Delta\theta\| \to 0$.

This implies the relation (2.6). Let

$$\boldsymbol{\theta}_1 = (u + \Delta u, p + \Delta p, v_1 + \Delta v_1, v_2 + + \Delta v_2, x_0 + \Delta x_0, x_1 + \Delta x_1, d + \Delta d),$$

$\theta_2 = (u, p, v_1, v_2, x_0, x_1, d) \in X$. Since

$$|I_1'(\theta_1) - I_1'(\theta_2)|^2 \le l_{10} |\Delta q(t)|^2 + l_{11} |\Delta\psi(t)|^2 + l_{12} |\Delta\theta|^2,$$

$$|\Delta q(t)| \le l_{13} \|\Delta\theta\|, |\Delta\psi(t)| \le l_{14} \|\Delta\theta\|,$$

that

$$\|I_1'(\boldsymbol{\theta}_1) - I_1'(\boldsymbol{\theta}_2)\|^2 =$$

$$= \int_{t_0}^{t_1} |I_1'(\boldsymbol{\theta}_1) - I_1'(\boldsymbol{\theta}_2)|^2 dt \le l_{15} \|\Delta\boldsymbol{\theta}\|^2,$$

where $l_i = const > 0$, $i = \overline{10,15}$. This implies the estimation (2.8), where $K = \sqrt{l_{15}}$. The theorem is proved.

Lemma 2.3. *Let the matrix be $T(t_0,t_1) > 0$, the function $F_1(q,t)$ be convex, with respect to the variable $q \in R^N$, $N = 4n+m+s+r+m_1$,* i.e.

$$F_1(\alpha q_1 + (1-\alpha)q_2) \le \alpha F_1(q_1,t) + (1-\alpha)F_1(q_2,t),$$
$$\forall q_1, q_2 \in R^N, \forall \alpha, \alpha \in [0,1]. \tag{2.12}$$

Then the functional (2.1) under the conditions (2.2) – (2.4) is convex.

Proof. Let $\theta_1, \theta_2 \in X$, $\alpha \in [0,1]$. It can be shown, that

$$z(t, \alpha v_1 + (1-\alpha)\bar{v}_1, \alpha v_2 + (1-\alpha)\bar{v}_2) =$$
$$= \alpha z(t, v_1, v_2) + (1-\alpha) z(t, \bar{v}_1, \bar{v}_2),$$

$$\forall (v_1, v_2), (\bar{v}_1, \bar{v}_2) \in L_2(I, R^{r+m_2}).$$

Then

$$I_1(\alpha\theta_1 + (1-\alpha)\theta_2) = \int_{t_0}^{t_1} F_1(\alpha q_1(t) +$$
$$+(1-\alpha)q_2(t))dt \le \alpha I_1(\theta_1) + (1-\alpha) I_1(\theta_2),$$

$$\forall \theta_1, \theta_2 \in X, \theta_1 = (u_1, p_1, v_1, v_2, x_0, x_1, d),$$
$$\theta_2 = (\bar{u}_1, \bar{p}_1, \bar{v}_1, \bar{v}_2, \bar{x}_0, \bar{x}_1, \bar{d}).$$

The lemma is proved.

The initial optimal control problem (2.1) – (2.4) can be solved by numerical methods for solving extremal problems [9,10]. We introduce the following sets $U = \{u(\cdot) \in L_2(I, R^m) / \|u\| \le \beta\}$,

$$V_1(I, R^r) = \{v_1(\cdot) \in L_2(I, R^r) / \|v_1\| \le \beta\},$$

$$V_2(I, R^{m_2}) = \{v_2(\cdot) \in L_2(I, R^{m_2}) / \|v_2\| \le \beta\}),$$

$$\Gamma_1 = \{d \in R^{m_1} / d \ge 0, |d| \le \beta\},$$

$\beta > 0$ is a sufficiently large number. We construct sequences $\{\theta_n\} = \{u_n, p_n, v_1^n, v_2^n, x_0^n, x_1^n, d_n\} \subset X_1$, $n = 0,1,2,\ldots$ by the algorithm

$$u_{n+1} = P_U[u_n - \alpha_n I'_{1u}(\theta_n)],\ p_{n+1} = P_V[p_n - \alpha_n I'_{1p}(\theta_n)],$$

$$v_1^{n+1} = P_{V_1}[v_1^n - \alpha_n I'_{1v_1}(\theta_n)],\ v_2^{n+1} = P_{V_2}[v_2^n - \alpha_n I'_{1v_2}(\theta_n)],$$

$$x_0^{n+1} = P_{S_0}[x_0^n - \alpha_n I'_{1x_0}(\theta_n)],\ x_1^{n+1} = P_{S_1}[x_1^n - \alpha_n I'_{1x_1}(\theta_n)],$$

$$d_{n+1} = P_{\Gamma_1}[d_n - \alpha_n I'_{1d}(\theta_n)],\ n = 0,1,2,\ldots,$$
$$0 < \varepsilon_0 \le \alpha_n \le \frac{2}{K + 2\varepsilon},\ \varepsilon > 0, \tag{2.13}$$

where $P_\Omega[\cdot]$ is the projection of the point on the set Ω, $K = const > 0$ from (2.8).

Theorem 2.4. *Let the conditions of Theorem 2.2. be satisfied, in addition, the function $F_1(q,t)$ be convex with respect to the variable $q \in R^N$ and the sequence $\{\theta_n\} \subset X_1$ be determined by formula (2.13). Then:*

1) the lower bound of the functional (2.1) is reached under the conditions (2.2) – (2.4)

$$\inf_{\theta \in X_1} I_1(\theta) = I_1(\theta_*) = \min_{\theta \in X_1} I_1(\theta),\ \theta_* \in X_1;$$

2) the sequence $\{\theta_n\} \subset X_1$ is minimizing $\lim_{n\to\infty} I_1(\theta_n) = I_{1*} = \inf_{\theta \in X_1} I_1(\theta)$;

3) the sequence $\{\theta_n\} \subset X_1$ weakly converges to the point $\theta_* \in X_1$, $u_n \longrightarrow u_*$, $p_n \longrightarrow p_*$, $v_1^n \longrightarrow v_1^*$, $v_2^n \longrightarrow v_2^*$, $x_0^n \to x_0^*$, $x_1^n \to x_1^*$, $d_n \to d_*$ at $n \to \infty$, where $\theta_* = (u_*, p_*, v_1^*, v_2^*, x_0^*, x_1^*, d_*) \in X_1$;

4) in order to the problem (1.2) – (1.6) have a solution, it is necessary and sufficient that $\lim_{n\to\infty} I_1(\theta_n) = I_{1*} = 0$;

5) the following estimation of the rate of convergence holds

$$0 \le I_1(\theta_n) - I_{1*} \le \frac{C_0}{n},\ n = 1,2,\ldots,$$
$$C_0 = const > 0. \tag{2.14}$$

Proof. Since the function $F_1(q,t)$, $t \in I$ is convex, it follows from Lemma 3.3. that the functional $I_1(\theta)$, $\theta \in X_1$ is convex on a weekly bicompact set X_1. Consequently, $I_1(\theta) \in C^1(X_1)$ is weakly lower semicontinuous on a weakly bicompact set and reaches the lower bound on X_1. This implies the first statement of the theorem.

Using the properties of the projection of a point on a convex closed set X_1 and taking into account that $I_1(\theta) \in C^{1,1}(X_1)$ it can be shown that $I_1(\theta_n) - I_1(\theta_{n+1}) \geq \varepsilon \| \theta_n - \theta_{n+1} \|^2$, $n = 0,1,2,\ldots$, $\varepsilon > 0$. It follows that: 1) the numerical sequence $\{I_1(\theta_n)\}$ strictly decreases; 2) $\| \theta_n - \theta_{n+1} \| \to 0$ at $n \to \infty$.

Since the functional is convex and the set X_1 is bounded, the inequality holds

$$0 \leq I_1(\theta_n) - I_1(\theta_*) \leq C_1 \| \theta_n - \theta_{n+1} \|, \quad C_1 = const > 0,\, n = 0,1,2,\ldots. \tag{2.15}$$

Hence, taking into account that $\| \theta_n - \theta_{n+1} \| \to 0$ at $n \to \infty$,, we have: the sequence $\{\theta_n\}$ is minimizing. $\lim_{n\to\infty} I_1(\theta_n) = I_1(\theta_*) = \inf_{\theta \in X_1} I_1(\theta)$.

Since $\{\theta_n\} \subset X_1$, X_1 is weakly bicompact, that, $\theta_n \xrightarrow{weakly} \theta_*$ *at* $n \to \infty$.

As it follows from Lemma 3.1., if the value $I_1(\theta_*) = 0$, then the problem of optimal control (1.1) – (1.6) has a solution.

The estimation (2.14) follows directly from the inequalities (2.15),

$I_1(\theta_n) - I_1(\theta_{n+1}) \geq \varepsilon \| \theta_n - \theta_{n+1} \|^2$.

We briefly outlined above, the main steps in proof of the theorem. Detailed proof of an analogous theorem is given in [16]. The theorem is proved.

For the case when the function $F_1(q,t)$ is not convex with respect to the variable q, the following theorem is true.

Theorem 2.5. *It is supposed, that the conditions of Theorem 2.2. are satisfied, the sequence* $\{\theta_n\} \subset X_1$ *is determined by formula (2.13). Then: 1) the value of the functional* $I_1(\theta_n)$ *strictly decreases for* $n = 0,1,2,\ldots$*; 2)* $\| \theta_n - \theta_{n+1} \| \to 0$ *at* $n \to \infty$.

Proof of the theorem follows from Theorem 2.4.

From the results it follows that 1) if $\theta_* = (u_*, p_*, v_1^*, v_2^*, x_0^*, x_1^*, d_*) \in X_1$ is the solution of optimal control problem (2.1) – (2.4), for which $I_1(\theta_*) = 0$, then $(u_* = u_*(t), x_0^*, x_1^*) \in \Sigma \subset U \times S_0 \times S_1$ is admissible control; 2) the function $x_*(t; t_0, x_0^*)$, $t \in I$ is the solution of differential equation (1.2), satisfies the conditions: $x(t_1; t_0, x_0^*) = x_1^*$, $x_*(t; t_0, x_0^*) \in G(t)$, $t \in I$, the functionals $g_j(u_*(\cdot), x_0^*, x_1^*) \leq 0$, $j = \overline{1, m_1}$, $g_j(u_*(\cdot), x_0^*, x_1^*) = 0$, $j = \overline{m_1 + 1, m_2}$; 3) the necessary and sufficient condition for the existence of a solution of the boundary value problem (1.2) – (1.6) is $I_1(\theta_*) = 0$ where $\theta_* \in X_1$ is the solution of problem (2.1) – (2.4); 4) for the admissible control, the value of the functional (1.1) equals to

$$J(u_*(\cdot), x_0^*, x_1^*) = = \int_{t_0}^{t_1} F_0(x_*(t), u_*(t), x_0^*, x_1^*, t)\,dt = \gamma_*, \tag{2.16}$$

where $x_*(t) = x_*(t; t_0, x_0^*)$, $t \in I$. In the general case, the value

$J(u_*(\cdot), x_0^*, x_1^*) \neq J(\bar{u}_*, \bar{x}_0^*, \bar{x}_1^*) =$
$\inf J(u(\cdot), x_0, x_1),$
$(u(\cdot), x_0, x_1) \in L_2(I, R^m) \times S_0 \times S_1.$

Construction of an optimal solution

We consider the optimal control problem (1.1) – (1.6). We define a scalar function $\sigma(t)$, $t \in I$ as:

$$\sigma(t) = \int_{t_0}^{t} F_0(x(\tau), u(\tau), x_0, x_1, \tau)\,d\tau, \; t \in I.$$

Then $\dot{\sigma}(t) = F_0(x(t), u(t), x_0, x_1, t)$, $\sigma(t_0) = 0$, $\sigma(t_1) = \gamma = I(u(\cdot), x_0, x_1) \in \Omega$, $\Omega = \{\gamma \in R^1 /$ $\gamma \geq \gamma_0$, $\gamma_0 > -\infty$ $\}$, *where* $\gamma = \inf I(u(\cdot), x_0, x_1) \geq \gamma_0$, the value γ_0 is bounded from below, in particular $\gamma_0 = 0$, if $F_0 \geq 0$.

Now the problem of optimal control (1.1) – (1.6) can be written in the form (see (2.1))

$$\sigma(t_1) = \gamma = I(u(\cdot), x_0, x_1) \to \inf \qquad (3.1)$$

at conditions

$$\dot{\boldsymbol{\sigma}}(t) = F_0(x(t), u(t), x_0, x_1, t), \quad \boldsymbol{\sigma}(t_0) = 0, \boldsymbol{\sigma}(t_1) = \boldsymbol{\gamma}, \qquad (3.2)$$

$$\dot{x} = A(t)x + B(t)f(x,u,t), (x(t_0) = x_0, \quad x(t_1) = x_1) \in S_0 \times S_1, \qquad (3.3)$$

$$\dot{\boldsymbol{\eta}} = f_0(x(t), u(t), x_0, x_1, t), \quad \boldsymbol{\eta}(t_0) = 0, \boldsymbol{\eta}(t_1) = \bar{c} \in Q, \qquad (3.4)$$

$$x(t) \in G(t), u(\cdot) \in L_2(I, R^m), t \in I. \qquad (3.5)$$

We introduce the notations

$$\boldsymbol{\mu}(t) = \begin{pmatrix} \boldsymbol{\sigma}(t) \\ x(t) \\ \boldsymbol{\eta}(t) \end{pmatrix},$$

$$A_2(t) = \begin{pmatrix} O_{1,1} & O_{1,n} & O_{1,m_2} \\ O_{n,1} & A(t) & O_{n,m_2} \\ O_{m_2,1} & O_{m_2,n} & O_{m_2,m_2} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 1 \\ O_{n,1} \\ O_{m_2,1} \end{pmatrix},$$

$$C_0(t) = \begin{pmatrix} O_{1,r} \\ B(t) \\ O_{m_2,r} \end{pmatrix}, D_0(t) = \begin{pmatrix} O_{1,m_2} \\ O_{n,m_2} \\ I_{m_2} \end{pmatrix},$$

$$P_0 = \begin{pmatrix} 1, & O_{1,n}, & O_{1,m_2} \end{pmatrix}, P_1 = \begin{pmatrix} O_{n,1}, & I_n, & O_{n,m_2} \end{pmatrix},$$

where $P_0\mu(t_1) = \sigma(t_1)$, $P_1\mu = x$.

Then the optimal control problem (3.1) – (3.5) has the form:

$$P_0\mu(t_1) = \gamma = I(u(\cdot), x_0, x_1) \to \inf, \qquad (3.6)$$

at conditions

$$\dot{\boldsymbol{\mu}} = A_2(t)\boldsymbol{\mu} + B_0 F_0(P_1\boldsymbol{\mu}, u, x_0, x_1, t) + \\ + C_0(t) f(P_1\boldsymbol{\mu}, u, t) + D_0 f_0(P_1\boldsymbol{\mu}, u, x_0, x_1, t), \qquad (3.7)$$

$$\boldsymbol{\mu}(t_0) = \boldsymbol{\mu}_0 = \begin{pmatrix} \boldsymbol{\sigma}(t_0) \\ x(t_0) \\ \boldsymbol{\eta}(t_0) \end{pmatrix} = \\ = \begin{pmatrix} O_{1,1} \\ x_0 \\ O_{m_2,1} \end{pmatrix} \in O_{1,1} \times S_0 \times O_{m_2,1} = T_0, \qquad (3.8)$$

$$\boldsymbol{\mu}(t_1) = \boldsymbol{\mu}_1 = \begin{pmatrix} \boldsymbol{\sigma}(t_1) \\ x(t_1) \\ \boldsymbol{\eta}(t_1) \end{pmatrix} = \\ = \begin{pmatrix} \gamma \\ x_1 \\ \bar{c} \end{pmatrix} \in \boldsymbol{\Omega} \times S_1 \times Q = T_1, \qquad (3.9)$$

$$P_1\mu(t) \in G(t), u(\cdot) \in L_2(I, R^m), d \in \Gamma, \qquad (3.10)$$

where $x(t) = P_1\mu(t)$, $\sigma(t) = P_0\mu(t)$, $t \in I$, γ is determined by formula (3.6).

The immersion principle. We consider the boundary value problem (3.7) – (3.10). The corresponding linear controlled system has the form

$$\dot{\boldsymbol{\zeta}} = A_2(t)\boldsymbol{\zeta} + B_0\bar{w}_1(t) + C_0(t)\bar{w}_2(t) + \\ + D_0\bar{w}_3(t), t \in I, \qquad (3.11)$$

$$\bar{w}_1(\cdot) \in L_2(I, R^1), \ \bar{w}_2(\cdot) \in L_2(I, R^r), \\ \bar{w}_3(\cdot) \in L_2(I, R^{m_2}), \qquad (3.12)$$

$$\zeta(t_0) = \mu_0 \in T_0, \quad \zeta(t_1) = \mu_1 \in T_1. \qquad (3.13)$$

We introduce the following notations:

$$\bar{B}_0(t) = (B_0, C_0(t), D_0), \bar{w}(t) = \\ = (\bar{w}_1(t), \bar{w}_2(t), \bar{w}_3(t)), \boldsymbol{\Psi}(t, \boldsymbol{\tau}) = K(t)K^{-1}(\boldsymbol{\tau}), \\ \bar{a} = \boldsymbol{\Psi}(t_0, t_1)\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0, R(t_0, t_1) = \\ = \int_{t_0}^{t_1} \boldsymbol{\Psi}(t_0, t)\bar{B}_0(t)\bar{B}_0^*(t)\boldsymbol{\Psi}^*(t_0, t)dt,$$

$$R(t_0,t)=\int_{t_0}^{t}\Psi(t_0,\tau)\overline{B}_0(\tau)\overline{B}_0^*(\tau)\Psi^*(t_0,\tau)d\tau,$$

$$R(t_0,t_1)=R(t_0,t)+R(t,t_1),$$

$$\overline{\Lambda}_1(t,\mu_0,\mu_1)=\overline{B}_0^*(t)\Psi^*(t_0,t)R^{-1}(t_0,t_1)a=$$

$$=\begin{pmatrix} B_0^*\Psi^*(t_0,t)R^{-1}(t_0,t_1)\overline{a} \\ C_0^*\Psi^*(t_0,t)R^{-1}(t_0,t_1)\overline{a} \\ D_0^*\Psi^*(t_0,t)R^{-1}(t_0,t_1)\overline{a}\end{pmatrix}=\begin{pmatrix}\overline{\Lambda}_{11}(t,\mu_0,\mu_1)\\ \overline{\Lambda}_{12}(t,\mu_0,\mu_1)\\ \overline{\Lambda}_{13}(t,\mu_0,\mu_1)\end{pmatrix},$$

$$K_1(t)=-\overline{B}_0^*\Psi^*(t)(t_0,t)R^{-1}(t_0,t_1)\Psi(t_0,t_1)=$$

$$=\begin{pmatrix} -B_0^*\Psi^*(t_0,t)R^{-1}(t_0,t_1)\Psi(t_0,t_1) \\ -C_0^*\Psi^*(t_0,t)R^{-1}(t_0,t_1)\Psi(t_0,t_1) \\ -D_0^*\Psi^*(t_0,t)R^{-1}(t_0,t_1)\Psi(t_0,t_1)\end{pmatrix}=\begin{pmatrix}K_{11}(t)\\ K_{12}(t)\\ K_{13}(t)\end{pmatrix},$$

$$\overline{\Lambda}_2(t,\mu_0,\mu_1)=\Psi(t,t_0)R(t,t_1)R^{-1}(t_0,t_1)\mu_0+$$
$$+\Psi(t,t_0)R(t_0,t)R^{-1}(t_0,t_1)\Psi(t_0,t_1)\mu_1,$$

$$K_2(t)=-\Psi(t,t_0)R(t_0,t)R^{-1}(t_0,t_1)\Psi(t_0,t_1),t\in I.$$

Theorem 3.1. *Let the matrix be* $R(t_0,t_1)>0.$ *Then the control* $\overline{w}(t)=(\overline{w}_1(t),\overline{w}_2(t),\overline{w}_3(t))\in L_2(I,R^{1+r+m_2})$ *transforms the trajectory of the system (3.11) – (3.13) from any initial point* $\mu_0\in R^{1+n+m_2}$ *to any given finite state* $\mu_1\in R^{1+n+m_2}$ *if and only if*

$$\overline{w}_1(t)\in\overline{W}_1=\{\overline{w}_1(\cdot)\in L_2(I,R^1)/\overline{w}_1(t)=$$
$$=\overline{v}_1(t)+\overline{\Lambda}_{11}(t,\mu_0,\mu_1)+K_{11}(t)\overline{z}(t_1,\overline{v}),$$

$$\forall\overline{v}_1(\cdot)\in L_2(I,R^1),t\in I\}, \qquad (3.14)$$

$$\overline{w}_2(t)\in\overline{W}_2=\{\overline{w}_{2(\cdot)}\in L_2(I,R^r)/\overline{w}_2(t)=$$
$$=\overline{v}_2(t)+\overline{\Lambda}_{12}(t,\mu_0,\mu_1)+K_{12}(t)\overline{z}(t_1,\overline{v}),$$

$$\forall\overline{v}_2(\cdot)\in L_2(I,R^r),t\in I\}, \qquad (3.15)$$

$$\overline{w}_3(t)\in\overline{W}_3=\{\overline{w}_3(\cdot)\in L_2(I,R^{m_2})/\overline{w}_3(t)=$$
$$=\overline{v}_3(t)+\overline{\Lambda}_{13}(t,\mu_0,\mu_1)+K_{13}(t)\overline{z}(t_1,\overline{v}),$$

$$\forall\overline{v}_3(\cdot)\in L_2(I,R^{m_2}),t\in I\}, \qquad (3.16)$$

where $\overline{v}(t)=(\overline{v}_1(t),\overline{v}_2(t),\overline{v}_3(t)),\overline{z}(t)=\overline{z}(t,\overline{v}),t\in I$ is the solution of the differential equation

$$\dot{\overline{z}}=A_2(t)\overline{z}+B_0\overline{v}_1(t)+C_0(t)\overline{v}_2(t)+$$
$$+D_0\overline{v}_3(t),\overline{z}(t_0)=0, \qquad (3.17)$$

$$\overline{v}_1(\cdot)\in L_2(I,R^1),\overline{v}_2(\cdot)\in L_2(I,R^r),$$
$$\overline{v}_3(\cdot)\in L_2(I,R^{m_2}). \qquad (3.18)$$

Solution of the system (3.11) – (3.13) has the form

$$\zeta(t)=\overline{z}(t,\overline{v})+\overline{\Lambda}_2(t,\mu_0,\mu_1)+$$
$$+K_2(t)\overline{z}(t_1,\overline{v}),t\in I. \qquad (3.19)$$

The proof of the analogous theorem is presented in the work [10].

Lemma 3.2. *Let the matrix be* $R(t_0,t_1)>0$. *Then the boundary value problem (3.7) – (3.10) is equivalent to the following problem*

$$\overline{w}_1(t)\in\overline{W}_1,\overline{w}_1(t)=F_0(P_1\zeta,u,x_0,x_1,t),$$
$$t\in I, \qquad (3.20)$$

$$\overline{w}_2(t)\in\overline{W}_2,\overline{w}_2(t)=f(P_1\zeta,u,t),t\in I, \qquad (3.21)$$

$$\overline{w}_3(t)\in\overline{W}_3,\overline{w}_3(t)=f_0(P_1\zeta,u,x_0,x_1,t),$$
$$t\in I, \qquad (3.22)$$

$$p(t)\in V(t)=\{p(\cdot)\in L_2(I,R^s)/p(t)=$$
$$=F(P_1\zeta,t),\omega(t)\le p(t)\le\phi(t),t\in I\}, \qquad (3.23)$$

$$\dot{\overline{z}}=A_2(t)\overline{z}+B_0\overline{v}_1(t)+C_0(t)\overline{v}_2(t)+$$
$$+D_0\overline{v}_3(t),\overline{z}(t_0)=0,t\in I, \qquad (3.24)$$

$$\overline{v}_1(\cdot)\in L_2(I,R^1),\overline{v}_2(\cdot)\in L_2(I,R^r),$$
$$\overline{v}_3(\cdot)\in L_2(I,R^{m_2}), \qquad (3.25)$$

$$(x_0,x_1)\in S_0\times S_1,u(\cdot)\in L_2(I,R^m),$$
$$\gamma\in\Omega,d\in\Gamma, \qquad (3.26)$$

where $\zeta(t),\ t\in I$ *is determined by formula (3.19),* $\overline{z}(t,\overline{v})$ *is the solution of system (3.17), (3.18).*

We consider the following optimal control problem: minimize the functional

$$J_2(\bar{v},u,p,x_0,x_1,d,\boldsymbol{\gamma})=\int_{t_0}^{t_1}F_2(\bar{q}(t),t)dt=$$

$$=\int_{t_0}^{t_1}[|\bar{w}_1(t)-F_0(P_1\boldsymbol{\zeta}(t),u(t),x_0,x_1,t)|^2+$$

$$+|\bar{w}_2(t)-f(P_1\boldsymbol{\zeta}(t),u(t),t)|^2+$$

$$+|\bar{w}_3(t)-f_0(P_1\boldsymbol{\zeta}(t),u(t),x_0,x_1,t)|^2+$$

$$+|p(t)-F(P_1\zeta(t),t)|^2]dt\to\inf \qquad (3.27)$$

under the conditions (3.24) – (3.26), where $\bar{w}_1(t)\in\bar{W}_1$, $\bar{w}_2(t)\in\bar{W}_2$, $\bar{w}_3(t)\in\bar{W}_3$, $\bar{v}=(\bar{v}_1,\bar{v}_2,\bar{v}_3)$,

$\bar{q}(t)=(\bar{v}_1,\bar{v}_2,\bar{v}_3,u,p,x_0,x_1,d,\gamma,\bar{z}(t),\bar{z}(t_1))$.

Note, that the optimization problem (3.27), (3.24) – (3.26) is obtained on the basis of relations (3.20) – (3.23).

Theorem 3.3. *Let the matrix be, the derivative* $\dfrac{\partial F_2(\bar{q},t)}{\partial q}$ *satisfies the Lipschitz condition.*

Then:

1. The functional (3.27) under conditions (3.24) – (3.26) is continuously differentiable by Frechet, gradient of the functional

$$J_2'(\bar{\boldsymbol{\theta}})=(J_{2\bar{v}_1}'(\bar{\boldsymbol{\theta}}),J_{2\bar{v}_2}'(\bar{\boldsymbol{\theta}}),J_{2\bar{v}_3}'(\bar{\boldsymbol{\theta}}),J_{2u}'(\bar{\boldsymbol{\theta}}),J_{2p}'(\bar{\boldsymbol{\theta}}),$$

$$J_{2x_0}'(\bar{\boldsymbol{\theta}}),J_{2x_1}'(\bar{\boldsymbol{\theta}}),J_{2d}'(\bar{\boldsymbol{\theta}}),J_{2\gamma}'(\bar{\boldsymbol{\theta}})),$$

$$\bar{\theta}=(\bar{v}_1,\bar{v}_2,\bar{v}_3,u,p,x_0,x_1,d,\gamma)\in\bar{X},$$

$$\bar{X}=L_2(I,R^1)\times L_2(I,R^r)\times L_2(I,R^{m_2})\times$$

$$\times L_2(I,R^m)\times V\times S_0\times S_1\times\Gamma\times\boldsymbol{\Omega}$$

$$H_1=L_2(I,R^1)\times L_2(I,R^r)\times L_2(I,R^{m_2})\times$$

$$\times L_2(I,R^m)\times L_2(I,R^s)\times R^n\times R^n\times R^{m_1}\times R^1,$$

$$\bar{X}\subset H_1,\quad J_2'(\bar{\boldsymbol{\theta}})\in H_1$$

for any point $\bar{\theta}\in\bar{X}$ is calculated by the formulas

$$J_{2\bar{v}_1}'(\bar{\boldsymbol{\theta}})=\frac{\partial F_2(\bar{q}(t),t)}{\partial\bar{v}_1}-B_0^*\bar{\boldsymbol{\psi}}(t),$$

$$J_{2\bar{v}_2}'(\bar{\boldsymbol{\theta}})=\frac{\partial F_2(\bar{q}(t),t)}{\partial\bar{v}_2}-C_0^*\bar{\boldsymbol{\psi}}(t),$$

$$J_{2\bar{v}_3}'(\bar{\boldsymbol{\theta}})=\frac{\partial F_2(\bar{q}(t),t)}{\partial\bar{v}_3}-D_0^*\bar{\boldsymbol{\psi}}(t),$$

$$J_{2u}'(\bar{\boldsymbol{\theta}})=\frac{\partial F_2(\bar{q}(t),t)}{\partial u},\quad J_{2p}'(\bar{\boldsymbol{\theta}})=\frac{\partial F_2(\bar{q}(t),t)}{\partial p}$$

$$J_{2x_0}'(\bar{\boldsymbol{\theta}})=\int_{t_0}^{t_1}\frac{\partial F_2(\bar{q}(t),t)}{\partial x_0}dt,$$

$$J_{2x_1}'(\bar{\boldsymbol{\theta}})=\int_{t_0}^{t_1}\frac{\partial F_2(\bar{q}(t),t)}{\partial x_1}dt,$$

$$J_{2d}'(\bar{\boldsymbol{\theta}})=\int_{t_0}^{t_1}\frac{\partial F_2(\bar{q}(t),t)}{\partial d}dt,$$

$$J_{2\gamma}'(\bar{\boldsymbol{\theta}})=\int_{t_0}^{t_1}\frac{\partial F_2(\bar{q}(t),t)}{\partial\gamma}dt,$$

where $\bar{\psi}(t)$, $t\in I$ is the solution of the adjoint system

$$\dot{\bar{\psi}}=\frac{\partial F_2(\bar{q}(t),t)}{\partial\bar{z}}-A_2^*(t)\bar{\psi},$$

$$\bar{\psi}(t_1)=-\int_{t_0}^{t_1}\frac{\partial F_2(\bar{q}(t),t)}{\partial\bar{z}(t_1)}dt;$$

2. gradient $J_2'(\bar{\theta}),\bar{\theta}\in\bar{X}$ satisfies the Lipchitz condition

$$\|J_2'(\bar{\boldsymbol{\theta}}_1)-J_2'(\bar{\boldsymbol{\theta}}_2)\|\le l\|\bar{\boldsymbol{\theta}}_1-\bar{\boldsymbol{\theta}}_2\|, \qquad (3.28)$$

$$\forall\bar{\boldsymbol{\theta}}_1,\bar{\boldsymbol{\theta}}_2\in\bar{X}.$$

The proof of the analogous theorem can be found in the work [16]. We construct the following sequences $\{\bar\theta_n\} = \{\bar v_1^n, \bar v_2^n, \bar v_3^n, u_n, p_n, x_0^n, x_1^n, d_n, \gamma_n\} \subset \bar X_2$ by the algorithm

$$\bar v_1^{n+1} = P_{\bar V_1}[\bar v_1^n - \alpha_n J'_{2\bar v_1}(\bar\theta_n)],$$

$$\bar v_2^{n+1} = P_{\bar V_2}[\bar v_2^n - \alpha_n J'_{2\bar v_2}(\bar\theta_n)],$$

$$\bar v_3^{n+1} = P_{\bar V_3}[\bar v_3^n - \alpha_n J'_{2\bar v_3}(\bar\theta_n)],$$

$$u_{n+1} = P_U[u_n - \alpha_n J'_{2u}(\bar\theta_n)],$$

$$p_{n+1} = P_V[p_n - \alpha_n J'_{2p}(\bar\theta_n)],$$

$$x_0^{n+1} = P_{S_0}[x_0^n - \alpha_n J'_{2x_0}(\bar\theta_n)],$$

$$x_1^{n+1} = P_{S_1}[x_1^n - \alpha_n J'_{2x_1}(\bar\theta_n)],$$

$$d_{n+1} = P_{\bar\Gamma}[d_n - \alpha_n J'_{2d}(\bar\theta_n)],$$

$$\gamma_{n+1} = P_{\bar\Omega}[\gamma_n - \alpha_n J'_{2\gamma}(\bar\theta_n)], \qquad n = 0,1,2,...,$$

$$0 \le \alpha_n \le \frac{2}{l+2\varepsilon}, \quad \varepsilon > 0, \quad l = const > 0, \qquad (3.29)$$

where $\bar V_1 = \{\bar v_1(\cdot) \in L_2(I, R^1)/ \|\bar v_1\| \le \bar\beta\}$,
$\bar V_2 = \{\bar v_2(\cdot) \in L_2(I, R^r)/ \|\bar v_2\| \le \bar\beta\}$,
$\bar V_3 = \{\bar v_3(\cdot) \in L_2(I, R^{m_2})/ \|\bar v_3\| \le \bar\beta\}$,
$U = \{u(\cdot) \in L_2(I, R^m)/ \|u\| \le \bar\beta\}$,
$\bar\Gamma = \{d \in R^{m_1}/d \ge 0, |d| \le \bar\beta\}$,
$\bar\Omega = \{\gamma \in R^1 / a \le \gamma \le \gamma_*\}$,
$\bar X_2 = \bar V_1 \times \bar V_2 \times \bar V_3 \times U \times V \times S_0 \times S_1 \times \bar\Gamma \times \bar\Omega \subset H_1$,
$U = \{u(\cdot) \in L_2(I, R^m)/ \ \|u\| \le \bar\beta\}$, $\bar\beta > 0$ is a sufficiently large number.

Theorem 3.4. *Let the conditions of Theorem 3.3. be satisfied* $\bar X_1$ *is a bounded convex closed set, the sequence* $\{\bar\theta_n\} \subset \bar X_2$ *is determined by the formula (3.29). Then:*

1. the numerical sequence $\{J_2(\bar\theta_n)\}$ *is strictly decreasing* $\|\bar\theta_n - \bar\theta_{n+1}\| \to 0$*, at* $n \to \infty$*.*

If, in addition,, $F_2(\bar q, t)$ *is a convex function with respect to a variable* $\bar q$*, then:*

2. the lower bound of the functional (3.27) is obtained under the conditions (3.24) – (3.26)

$$J_2(\bar\theta_*) = \inf_{\bar\theta \in \bar X_2} J_2(\bar\theta) = \min_{\theta \in X_2} J_2(\bar\theta) = J_{2*};$$

3. the sequence $\{\bar\theta_n\} \subset \bar X_2$ *is minimizing* $\lim_{n\to\infty} J_2(\bar\theta_n) = J_{2*} = \inf_{\bar\theta \in \bar X_2} J_2(\bar\theta)$;

4. the sequence { $\{\bar\theta_n\} \subset \bar X_1$ *weakly converges to the point* $\bar\theta_* \in \bar X_{1*}$, $\bar X_{2*} = \{\bar\theta_*/J_2(\bar\theta_*) = J_{2*} = \inf_{\bar\theta \in \bar X_1} J_2(\bar\theta) = \min_{\theta \in \bar X_1} J_2(\bar\theta)\}$ *, where* $\bar v_1^n \longrightarrow \bar v_1^*$, $\bar v_2^n \longrightarrow \bar v_2^*$, $\bar v_3^n \longrightarrow \bar v_3^*$, $u_n \longrightarrow \bar u_*$, $p_n \longrightarrow \bar p_*$, $x_0^n \to \bar x_0^*$, $x_1^n \to \bar x_1^*$, $d_n \to \bar d_*$, $\gamma_n \to \gamma_*$ *at* $n \to \infty$, $\bar\theta_* = (\bar v_1^*, \bar v_2^*, \bar v_3^*, \bar u_*, p_*, \bar x_0^*, \bar x_1^*, \bar d_*, \bar\gamma_*)$;

5. if $J_2(\bar\theta_*) = 0$*, then the optimal control for problem (1.1) – (1.6) is* $\bar u_* \in U$, $\bar x_0^* \in S_0$, $\bar x_1^* \in S_1$*, and the optimal trajectory*

$$\bar x_*(t) = P_1\zeta_*(t) = P_1[\bar z(t, \bar v_*) +$$

$$+ \bar\Lambda_2(t, \mu_0^*, \mu_1^*) + K_2(t)\bar z(t_1, \bar v_*)], t \in I,$$

where

$$\bar v_* = (\bar v_1^*, \bar v_2^*, \bar v_3^*), \mu_0^* = (O_{1,1}, \bar x_0^*, O_{m_2,1}),$$

$$\mu_1^* = (\gamma_*, \bar x_1^*, \bar c_*), \bar c_* \in Q = \{\bar c_* \in R^{m_2} / \bar c_{j*} =$$

$$= c_j - \bar d_j^*, \bar d_j^* \ge 0, j = \overline{1, m_1}; \bar c_{j*} = c_j, j = \overline{m_1 + 1, m_2}\},$$

the inclusion $\bar x_*(t) \in G(t)$ *and limitations (1.4) – (1.6)* $J(\bar u_*, \bar x_0^*, \bar x_1^*) = \bar\gamma_*$*; hold.*

6. The following estimation of the rate of convergence holds

$$0 \le J_2(\bar\theta_n) - J_{2*} \le \frac{\bar c_0}{n}, n = 1,2,..., \bar c_0 = const > 0.$$

Proof of the analogous theorem is given above.

A more obvious method for solving problem (1.1) – (1.6) is the method of narrowing the domain of admissible controls.

Theorem 3.5. *Let the conditions of Theorem 3.3. be satisfied,* $\overline{X}_3=\overline{V}_1\times\overline{V}_2\times\overline{V}_3\times U\times V\times S_0\times S_1\times\overline{\Gamma}$ *be a bounded convex closed set, the sequence* $\{\overline{\theta}_n\}\subset\overline{X}_2$ *be defined by (3.28) with the exception of the sequence* $\{\gamma_n\}\subset\Omega$. *Then:*

1. *the numerical sequence* $\{J_2(\overline{\theta}_n)\}$, $\{\overline{\theta}_n\}\subset X_3$ *is strictly decreasing;*
2. $\|\overline{\theta}_n-\overline{\theta}_{n+1}\|\to 0$ *at* $n\to\infty$, $\{\overline{\theta}_n\}\subset\overline{X}_3$;

If, in addition, the function $F_2(\overline{q},t)$ *is convex with respect to a variable* $\overline{q}$ *for fixed* γ, *then:*

3. *the sequence* $\{\overline{\theta}_n\}\subset\overline{X}_3$, *for a fixed* $\gamma=\overline{\gamma}$ *is minimizing;*
4. $\overline{\theta}_n\longrightarrow\overline{\theta}_*\in\overline{X}_3$ *at* $n\to\infty$, $\gamma=\overline{\gamma}$;
5. $J_2(\overline{\theta}_*)=\inf\limits_{\overline{\theta}_n\in\overline{X}_3}J_2(\overline{\theta}_n)=\min\limits_{\overline{\theta}_n\in\overline{X}_3}J_2(\overline{\theta}_n)$;
6. *the following estimation holds*

$$0\le J_2(\overline{\theta}_n)-J_2(\overline{\theta}_*)\le\frac{c_1}{n},$$

$c_1=const>0, n=1,2,...,\{\overline{\theta}_n\}\subset\overline{X}_3$.

The proof of the analogous theorem is presented in the work [10] for a fixed $\gamma\in\Omega$, $\gamma=\overline{\gamma}$.

Let the solution of the problem be $\overline{\theta}_*\in\overline{X}_2$ (3.27), (3.24) – (3.26) with $\gamma=\gamma_*\in\Omega$. There are the possible cases:

1. the value $J_2(\overline{\theta}_*)>0$;
2. the value $J_2(\overline{\theta}_*)=0$.

Note, that $J_2(\overline{\theta})\ge 0$, $\overline{\theta}\in\overline{X}_3$.

If $J_2(\overline{\theta}_*)>0$, then a new value of γ is selected as $\gamma=2\gamma_*$, and if $J_2(\overline{\theta}_*)=0$, then a new value $\gamma=\frac{\gamma_*}{2}$. According to this scheme, by dividing the uncertainty segment in half, the smallest value of the functional (1.1), under the conditions (1.2) – (1.6) can be found.

Conclusion

The Lagrange problem of the variation calculus is investigated in the presence of phase and integral constraints for processes described by ordinary differential equations. The particular cases of which are the simplest problem, the Bolz problem, the isoperimetric problem, the conditional extremum problem.

In contrast to the well-known method for solving the problem of the variation calculus on the basis of the Lagrange principle, an entirely new approach an "immersion principle" is proposed. The immersion principle is based on the investigation of the Fredholm integral equation of the first kind. For the Fredholm integral equation of the first kind, the existence theorem for the solution as well as the theorem on its general solution are proved.

The main scientific results are:

- reduction of the boundary value problem connected to the conditions in the Lagrange problem to the initial optimal control problem with a specific functional;
- necessary and sufficient conditions for the existence of the admissible control;
- method of constructing an admissible control on the limit point of the minimizing sequence;
- necessary and sufficient conditions for the existence of a solution of the Lagrange problem;
- method for constructing the solution of the Lagrange problem.

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