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Soliton solutions of a generalized Klein–Gordon equation with power-law nonlinearity via the first integral method

Abstract. This paper studies solitary wave solutions of a generalized nonlinear Klein-Gordon (KG) equation with power-law nonlinearity via the so-called first integral method. Using the method, some soliton solutions of the equation are obtained. The method is hereby shown to be an efficient and reliable mathematical tool for solving many nonlinear evolution equations arising in a number of problems in science and engineering.

Key words: Soliton, power-law nonlinearity, quantum field theory (QFT).

Introduction

The Klein-Gordon (KG) equation is an important equation in theoretical physics especially in quantum field theory (QFT) and relativistic quantum mechanics. It also appears in nonlinear optics and plasma physics. The Klein-Gordon equation often arises in physics in linear as well as nonlinear forms. In the past, the equation had been extensively studied by many physicists and applied mathematicians with the help of a variety of methods. This paper deals with solving a particular form of the generalized Klein-Gordon (GKG) equation with full nonlinearity via the first integral method [1 – 7].

The generalized Klein-Gordon equation [8 – 10] that is to be studied in this paper is written in the form

$$q_{tt} - \mu^2 q_{xx} + \alpha q - \beta q^n + \gamma q^{2n-1} = 0, \quad (1)$$

where the dependent variable $q(x,t)$ represents a wave profile, x and t are spatial and temporal variables, $\mu, \alpha, \beta, \gamma$ are real-valued constants and $n = 2, 3, 4, \ldots$

Reduction to Nonlinear Ordinary Differential Equation (NLODE)

To reduce Eq.(1) to a nonlinear ordinary differential equation (NLODE), we put

$$q(x,t) = u(\xi), \xi = x - vt \quad (2)$$

where v is a constant , generally the constant speed of wave propagation.

Now, from eq.(2), we have

$$q_{tt} = v^2 \frac{d^2u}{d\xi^2}, q_{xx} = \frac{d^2u}{d\xi^2}.$$

Substituting these derivatives in Eq.(1), we obtain

$$(v^2 - \mu^2)\frac{d^2u}{d\xi^2} + \alpha u - \beta u^n + \gamma u^{2n-1} = 0. \quad (3)$$

Thus, Eq.(1) is reduced to a NLODE.

Let us further simplify the reduced NLODE by putting

$$u(\xi) = [U(\xi)]^{\frac{1}{n-1}}. \quad (4)$$

Then, we have

$$\frac{du}{d\xi} = \frac{1}{n-1} U^{\frac{2-n}{n-1}} \frac{dU}{d\xi} \quad (5)$$

and

$$\frac{d^2u}{d\xi^2} = \frac{2-n}{(n-1)^2} U^{\frac{3-2n}{n-1}} \left(\frac{dU}{d\xi}\right)^2 + \frac{1}{n-1} U^{\frac{2-n}{n-1}} \frac{d^2U}{d\xi^2}. \quad (6)$$

Substituting Eqs.(4) to (6) in Eq. (3), we obtain

$$(n-1)(v^2-\mu^2)U\frac{d^2U}{d\xi^2}-$$

$$-(n-2)\,(v^2-\,\mu^2)\left(\frac{dU}{d\xi}\right)^2+$$

$$+\,(n-1)^2\,\alpha U^2\,-\,(n-1)^2\,\beta U^3\,+$$

$$+(n-1)^2\,\gamma U^4\,=0. \qquad (7)$$

Solving Eq. (7) and using Eqs. (2) and (4), we can obtain the solution $q(x,t)$ of Eq. (1).

In this paper, solutions of Eqs. (7) are to be obtained via a method known as the first integral method.

Algorithm of the First Integral method

Before applying the first integral method in finding the solutions of Eq. (7), we introduce an algorithm of the method as in the following.

Let us consider a general NLPDE in the form

$$F(u,u_x,u_t\,,u_{xx}\,,u_{xt},u_{tt}\,,u_{xxx}\,,\dots)\,=\,0, \quad (8)$$

where $u=u(x,t)$ is its solution, x and t represent the spatial and the temporal variables and F represents a polynomial in u and its partial derivatives. Here, the subscripts denote differentiations with respect to them.

Let us introduce the transformations,

$$u=u(x,t)=U(\xi),\xi=x\,-vt, \qquad (9)$$

where v is a constant to be determined latter.

Now, we have,

$$u_x=\frac{\partial u}{\partial x}=\frac{dU}{d\xi}\,,u_t=\frac{\partial u}{\partial t}=-v\,\frac{dU}{d\xi}\,,$$

$$u_{xx}=\frac{\partial^2u}{\partial x^2}=\frac{d^2U}{d\xi^2},u_{xt}=\frac{\partial^2u}{\partial x\,\partial t}=\,-v\,\frac{d^2U}{d\xi^2}\,,$$

$$u_{tt}=\frac{\partial^2u}{\partial t^2}=\,v^2\,\frac{d^2U}{d\xi^2}\,,etc. \qquad (10)$$

Using Eqs. (9) and (10), we can reduce Eq. (8) to a nonlinear ordinary differential equation (NLODE) of the form

$$G(U,U',U'',U''',\dots)=0 \qquad (11)$$

where the primes denote derivatives with respect to the same variable (ξ) such that $U'=\frac{dU}{d\xi}\,,U''=\frac{d^2U}{d\xi^2}\,,etc.$ and $G(U,U',U'',\dots)$ denotes another polynomial in U and its derivatives with respect to ξ.

This is exactly the way by which Eq. (1) was reduced to Eq. (7).

Now, Let us suppose that the solution of the Non Linear Ordinary Differential Equation (NLODE) (11) can be expressed as

$$U(\xi)=f(\xi). \qquad (12)$$

We further introduce the following new variables

$$X(\xi)=f(\xi),Y(\xi)=\,f'(\xi)=$$

$$=X'(\xi)=\frac{df}{d\xi}=\frac{dX}{d\xi} \qquad (13)$$

leading to the plane autonomous system

$$Y(\xi)=X'(\xi\,),Y'(\xi\,)=\,H(\,X(\xi),Y(\xi\,)\,)\,, \quad (14)$$

where H is a polynomial in X and Y.

If we can find two first integrals to the system of equations in (14) under the same conditions, then the analytic solutions of equations (14) can be obtained directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there exists neither a systematic theory that can tell us how to find its first integrals nor a logical way for telling us what these first integrals are. We will apply the Division Theorem to obtain a first integral to the system of equations (14) which reduces eqn. (11) to a first order integrable ODE. An exact solution of eqn. (8) is then obtained by solving this ODE. For convenience, let us recall the division theorem for two variables in the complex domain C [*w, z*].

Division Theorem: For two polynomials *P*(*w, z*) and *Q*(*w, z*) in a complex domain C [*w, z*], if *P*(*w, z*) is irreducible in C[*w, z*] and if *Q*(*w, z*) vanishes at all zero points of *P*(*w, z*), then there exists another polynomial *G*(*w, z*) in C[*w, z*] such that *Q*(*w, z*) = *P*(*w, z*) *G*(*w, z*). The division theorem follows immediately from Hilbert – Nullstellensatz theorem of commutative algebra.

Application of the First Integral method in solving the Generalized Klein-Gordon equation

In this section, the first integral method is applied in finding soliton solutions of Eq. (7) and hence of Eq. (1).

In Eq. (7), let us put

$$X(\xi) = U(\xi), Y(\xi) = X'(\xi) = \frac{dX}{d\xi}, \quad (15a)$$

$$Y'(\xi) = \frac{dY}{d\xi} = \frac{(n-2)}{(n-1)}\frac{(U')^2}{U} + \frac{(n-1)}{(\mu^2 - v^2)}[\alpha U - \beta U^2 + \gamma U^3] =$$

$$= \frac{(n-2)}{(n-1)}\frac{1}{X}Y^2 + \frac{(n-1)}{(\mu^2 - v^2)}[\alpha X - \beta X^2 + \gamma X^3]. \quad (15b)$$

Further, let us introduce another new variable η such that

$$d\xi = X\, d\eta\, .$$

Then, Eqs.(15) yield

$$\frac{dX}{d\eta} = XY, \frac{dY}{d\eta} = \frac{n-2}{n-1}Y^2 + \frac{n-1}{\mu^2 - v^2}[\alpha X^2 - \beta X^3 + \gamma X^4]. \quad (16)$$

We suppose that $X(\eta)$ and $Y(\eta)$ are nontrivial solutions of Eq. (16) and $q(X(\eta), Y(\eta)) = \sum_{j=0}^{m} a_j(X)\, Y^j(X)$ is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$q\big(X(\eta), Y(\eta)\big) = \sum_{j=0}^{m} a_j(X)\, Y^j(X) = 0, \quad (17)$$

where $a_j(X)\ (j = 0, 1, 2, 3, \dots m-1, m)$ are polynomials in X and $a_m \neq 0$.

Here, Eq.(17) is called the first integral to the system of Eqs.(16).

By division theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X, Y]$ such that

$$\frac{dq}{d\eta} = \frac{dq}{dX}\frac{dX}{d\eta} + \frac{dq}{dY}\frac{dY}{d\eta} =$$

$$= \{g(X) + h(X)Y\} \sum_{j=0}^{m} a_j(X)\, Y^j(X). \quad (18)$$

Now, using Eqs.(16), (17) and (18) we write

$$\sum_{j=0}^{m} a_j'(X)\, X\, Y^{j+1} + \sum_{j=0}^{m} j a_j(X)\, Y^{j-1}\left\{\frac{n-2}{n-1}Y^2 + \frac{n-1}{\mu^2 - v^2}[\alpha X^2 - \beta X^3 + \gamma X^4]\right\} =$$

$$= \sum_{j=0}^{m} g(X)\, a_j(X)\, Y^j + \sum_{j=0}^{m} h(X)\, a_j(X)\, Y^{j+1}.$$

From the above equation, equating the coefficients of $Y^j\ (j = m+1, m, \dots 3, 2, 1, 0)$ from both sides, we obtain

$$X\, a_m'(X) = h(X)\, a_m(X) - m\frac{(n-2)}{(n-1)}a_m(X), \quad (19a)$$

$$X\, a_{m-1}'(X) = g(X)\, a_m(X) + h(X)\, a_{m-1}(X) - \frac{(m-1)(n-2)}{n-1}a_{m-1}(X), \quad (19b)$$

$$X\, a_2'(X) = g(X)\, a_3(X) + h(X)\, a_2(X) - \frac{2(n-2)}{n-1}a_2(X) - \frac{4(n-1)}{\mu^2 - v^2}a_4(X)[\alpha X^2 - \beta X^3 + \gamma X^4], \quad (19c)$$

$$X\, a_1'(X) = g(X)\, a_2(X) + h(X)\, a_1(X) - \frac{n-2}{n-1}a_1(X) - \frac{3(n-1)}{\mu^2 - v^2}a_3(X)[\alpha X^2 - \beta X^3 + \gamma X^4], \quad (19d)$$

$$X\, a_0'(X) = g(X)\, a_1(X) + h(X)\, a_0(X) - \frac{2(n-1)}{\mu^2 - v^2}a_2(X)[\alpha X^2 - \beta X^3 + \gamma X^4], \quad (19e)$$

$$a_1(X)\frac{n-1}{\mu^2-\nu^2}\,[\alpha X^2-\beta X^3+\gamma X^4]=$$

$$=g(X)\,a_0(X). \tag{19f}$$

Since $a_m(X)$ is a polynomial in *X*, we deduce from Eq.(19*a*) that $h(X)=\frac{m(n-2)}{n-1}$.

For simplicity, we take $a_m(X)=1$. We can find the degrees of $g(X), a_0(X), a_1(X), etc.$ by balancing of degrees in Eqs.(19). Then, we express these functions as polynomials of appropriate degrees in *X* with undetermined coefficients. Substituting such polynomials in appropriate equations in (19) and equating coefficients of like powers of *X* from both sides of the resulting equation, we can find the undetermined coefficients. Thus, we can know the exact expressions of $a_0(X), a_1(X), a_2(X), etc.$ Substitution of these expressions in Eq.(17) can yield the expression(s) for *Y*. Recalling that $Y=\frac{dX}{d\xi}$, we can find $X(\xi)$ *or* $U(\xi)$ on integration.

Then using Eqs. (2) and (4), we can arrive at $u(\xi)$ and hence at $q(x,t)$.

In a particular case, let us take $m=1$. Then, Eqs.(19) yield

$$X\,a_1'(X)=\left\{h(X)-\frac{n-2}{n-1}\right\}a_1(X), \tag{20a}$$

$$X\,a_0'(X)=g(X)\,a_1(X)+h(X)\,a_0(X)\,, \tag{20b}$$

$$a_1(X)\frac{n-1}{\mu^2-\nu^2}\,[\alpha X^2-\beta X^3+\gamma X^4]=$$

$$=g(X)\,a_0(X). \tag{20c}$$

Since $a_j(X)$ are polynomials in *X*, we deduce from Eq. (20*a*) that $a_1(X)$ is a constant and $h(X)=\frac{n-2}{n-1}$. For simplicity, we take $a_1(X)=1$. From balancing of degrees in Eqs. (20), we conclude that $\deg[g(X)]=deg[a_0(X)]=2$.

We suppose that

$$a_0(X)=A_0+A_1X+A_2\,X^2 \tag{21}$$

where A_0, A_1, A_2 ($A_2\neq 0$) are arbitrary constants to be determined.

Substituting the expressions for $a_0(X)$ and its derivative $a_0'(X)$ and also the values of $a_1(X)$ and $h(X)$ in Eq. (20*b*), we have

$$A_1X+2A_2X^2=g(X)+$$

$$+\frac{n-2}{n-1}\,[A_0+A_1X+A_2\,X^2]Or, g(X)=$$

$$=-\frac{n-2}{n-1}\,A_0+\frac{1}{n-1}\,A_1X+\frac{n}{n-1}\,A_2X^2\,. \tag{22}$$

Substituting the expressions for $a_0(X), a_1(X), g(X)$ in Eq. (20*c*), we obtain

$$\frac{n-1}{\mu^2-\nu^2}\,[\alpha X^2-\beta X^3+\gamma X^4]=\Big(-\frac{n-2}{n-1}\,A_0+$$
$$+\frac{1}{n-1}\,A_1X+\frac{n}{n-1}\,A_2X^2\Big)\,(A_0+A_1X+A_2\,X^2).$$

Equating coefficients of like powers of *X* from both sides of the above equation, we obtain

$$\frac{n-2}{n-1}\,A_0^2=0, \tag{23}$$

$$\frac{3-n}{n-1}\,A_0A_1=0, \tag{24}$$

$$\frac{2}{n-1}\,A_0A_2+\frac{1}{n-1}\,A_1^2-\frac{\alpha(n-1)}{\mu^2-\nu^2}=0, \tag{25}$$

$$\frac{n+1}{n-1}\,A_1A_2+\frac{\beta(n-1)}{\mu^2-\nu^2}=0, \tag{26}$$

$$\frac{n}{n-1}\,A_2^2-\frac{\gamma(n-1)}{\mu^2-\nu^2}=0\,. \tag{27}$$

From Eq.(23), we obtain $A_0=0$ and then Eq.(25) yields

$$A_1=\mp(n-1)\sqrt{\frac{\alpha}{\mu^2-\nu^2}}\,. \tag{28}$$

Further, from Eq. (27), we have

$$A_2=\pm(n-1)\sqrt{\frac{\gamma}{n\,(\mu^2-\nu^2)}}\,. \tag{29}$$

Using these values of A_1 and A_2, Eq. (26) yields the constraint condition

$$\gamma=\frac{n\,\beta^2}{\alpha\,(n+1)^2}\,. \tag{30}$$

Hence, we write

$$A_2=\pm\frac{\beta(n-1)}{\alpha(n+1)}\sqrt{\frac{\alpha}{\mu^2-\nu^2}}\,. \tag{31}$$

Substituting the values of A_0, A_1 and A_2 in Eq. (21), we have

$$a_0(X) = \mp (n-1)\sqrt{\frac{\alpha}{\mu^2 - v^2}}\, X \pm$$

$$\pm \frac{\beta(n-1)}{\alpha(n+1)}\sqrt{\frac{\alpha}{\mu^2 - v^2}}\, X^2 . \qquad (32)$$

As $m = 1$ and $a_1(X) = 1$ in the present case, Eq. (17) yields

$$a_0(X) + Y = 0$$

$$Or, Y = \frac{dX}{d\xi} = -a_0(X) =$$

$$= \pm (n-1)\sqrt{\frac{\alpha}{\mu^2 - v^2}}\, X \mp$$

$$\mp \frac{\beta(n-1)}{\alpha(n+1)}\sqrt{\frac{\alpha}{\mu^2 - v^2}}\, X^2 . \qquad (33)$$

Integrating Eq. (33), we obtain the solutions

$$U(\xi) = X(\xi) = \pm \frac{(n+1)\alpha}{2\beta} \times$$

$$\times \left\{1 \pm \tanh\left[\frac{(n-1)}{2}\sqrt{\frac{\alpha}{\mu^2 - v^2}}\,(\xi - \xi_0)\right]\right\} \qquad (34)$$

and

$$U(\xi) = X(\xi) = \pm \frac{(n+1)\alpha}{2\beta} \times$$

$$\times \left\{1 \pm \coth\left[\frac{(n-1)}{2}\sqrt{\frac{\alpha}{\mu^2 - v^2}}\,(\xi - \xi_0)\right]\right\} \qquad (35)$$

where ξ_0 is an integration constant.

Choosing $\xi_0 = 0$ and recalling that $q(x,t) = u(\xi) = [U(\xi)]^{\frac{1}{n-1}}$ with $\xi = x - vt$, we obtain kink and anti-kink soliton solutions of Eq. (1) as

$$q(x,t) = \left[\pm \frac{(n+1)\alpha}{2\beta}\left\{1 \pm \tanh\left[\frac{(n-1)}{2}\sqrt{\frac{\alpha}{\mu^2 - v^2}}\,(x - vt)\right]\right\}\right]^{\frac{1}{n-1}} \qquad (36)$$

and

$$(x,t) = \left[\pm \frac{(n+1)\alpha}{2\beta}\left\{1 \pm \coth\left[\frac{(n-1)}{2}\sqrt{\frac{\alpha}{\mu^2 - v^2}}\,(x - vt)\right]\right\}\right]^{\frac{1}{n-1}} . \qquad (37)$$

These solutions are those obtained by *Wazwaz* [11].

One can try for solutions with $m = 2, 3, 4$ which will become complicated. Attempts for solutions with $m \geq 5$ must be dropped out as algebraic equations with degrees greater than or equal to 5 are generally not solvable.

Conclusion

In this paper, the first integral method is successfully applied in finding exact solutions of generalized Klein-Gordon equation. The performance of this method is found to be effective and reliable. The method can be applied in finding exact solutions of many nonlinear evolution equations arising in the studies of social dynamics, science and engineering. One advantage of the method is that it is applicable to both integrable as well as non-integrable systems.

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