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On a boundedness result of non-toroidal pseudo-differential operators

Abstract. In this article, we prove boundedness results for θ-toroidal pseudo-differential operators generated by a differentiation operator with a non-periodic boundary condition. θ-toroidal pseudo-differential operators are a natural generalization of a toroidal one. As in the classical case, this class of operators act on a suitable test function space by weighting the Fourier transform "very well". Standard operations as adjoints, products and commutators with θ-toroidal pseudo-differential operators can be characterized by their θ-toroidal symbols. For pseudo-differential operators on R^n, the symbol analysis is well developed. Here, we provide more complicated properties of the θ-toroidal pseudo calculus. Namely, we introduce a Holder space induced by a differentiation operator with a non-periodic boundary condition. Finally, for the elements of this space we prove theorems on boundedness of the operators acting on the specified functional spaces. Indeed, in this paper we continue a development of the so called "non-harmonic analysis" introduced in the recent papers of the authors.

Key words: θ-toroidal pseudo-differential operator, θ-toroidal Holder space, θ-Fourier transform, θ-symbol, bounded operator.

Introduction

In [3], it was introduced an analysis generated by the differential operator

$$Ly(x) = -i\frac{dy(x)}{dx}, \ \ 0 < x < 1 \tag{1}$$

acting on $L_2(0;1)$ with the boundary condition

$$\theta y(0) - y(1) = 0, \tag{2}$$

where $\theta \geq 1$.

Spectrum of the operator L is

$$\lambda_\xi = -i\ln\theta + 2\xi\pi, \ \ \xi \in Z \tag{3}$$

System of eigenfunctions of the operator L is

$$u_\xi(x) = \theta^x e^{i2\xi\pi x}, \ \ \xi \in Z. \tag{4}$$

and the biorthogonal system to $u_\xi(x)$ in $L_2(0;1)$ is

$$v_\xi(x) = \theta^{-x} e^{i2\xi\pi x}, \ \ \xi \in Z. \tag{5}$$

For the following spectral properties of the operator L we refer to [3] and [7].

1. The system of eigenfunctions of the operator L is a Riesz basis in $L_2(0;1)$;
2. If function f belongs to the domain of operator L, then f(x) expands to a uniformly convergent series of eigenfunctions of the operator L;
3. The resolvent of the operator L is

$$(L-\lambda I)^{-1} f(x) =$$

$$= i\frac{e^{i\lambda(x+1)}}{\Delta(\lambda)}\int_0^1 e^{-i\lambda t} f(t)dt + ie^{i\lambda x}\int_0^x e^{-i\lambda t} f(t)dt,$$

Where $\Delta(\lambda) = \theta - e^{i\lambda}$.

θ-toroidal pseudo-differential operators

θ-Fourier transform and θ -toroidal Hölder spaces. Here we give a definition of θ -Fourier transform [3]. *θ*-Fourier transform $\left(f \mapsto \hat{f}\right): C_\theta^\infty[0,1] \to S(Z)$ is given by

$$\hat{f}(\xi) := \int_0^1 f(x)\, \overline{v_\xi(x)} dx \tag{6}$$

and its inverse $\hat{f}(\xi)^{-1}$ is given by

$$f(x) = \sum_{\xi \in Z} \hat{f}(\xi) u_\xi(x). \tag{7}$$

Remark 1 The functional space $C_\theta^\infty[0,1]$ is called the space of test functions and $S(Z)$ is space of rapidly decaying functions [3].

In what follows, we will use the following spaces from [5].

We define θ-toroidal Hölder spaces

$$\Lambda^s([0,1],\theta) = \\ = \left\{ f: [0,1] \to C: |f|_{\Lambda^s} = \sup_{x,h \in [0,1]} \frac{|f(x+h) - f(x)|}{|h|^s} < \infty \right\} \tag{8}$$

$$\Lambda_0^s([0,1],\theta) = \{f \in \Lambda^s([0,1],\theta):\ f(0) = 0\} \tag{9}$$

for each $0 < s \leq 1$. These spaces are Banach.

θ -toroidal symbol class. Suppose that $m \in R, 0 \leq \delta, \rho \leq 1$. Then the θ-toroidal symbol class $S_{\delta,\rho}^m([0,1] \times Z)$ consists of those function $a(x,\xi)$ which are smooth in x for all $\xi \in Z$, and which satisfy

$$\left|\Delta_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{a\alpha\beta m} \langle\xi\rangle^{m-\rho\alpha+\delta\beta} \tag{10}$$

for every $\xi \in Z, x \in [0,1], \alpha, \beta \in Z_+$, where

$$\langle\xi\rangle := 1 + |\xi|.$$

We call $a(x,\xi)$ a symbol [3]. The operator Δ_ξ is the difference operator

$$\Delta_\xi\ \hat{\sigma}(\xi) := \widehat{e\sigma}(\xi),$$

where $\hat{\sigma}(\xi): Z \to C$.

We denote the θ -toroidal pseudo-differential operator by

$$a(X,D)f(x) = \sum_{\xi \in Z} u_\xi(x)\, a(x,\xi) \hat{f}(\xi) \tag{11}$$

where $a(x,\xi)$ is a symbol of a θ-toroidal pseudo-differential operator [5].

We can write for $h \in T$,

$$a(X,D)f(x+h) = \\ = \sum_{\xi \in Z} u_\xi(x)\, a(x+h,\xi) \int_0^1 f(y+h) \overline{v_\xi(x)} dy.$$

Theorem 1. (Bernstein). Assume that $f \in \Lambda_0^s([0,1],\theta)$, for $s > \frac{1}{2}$. Then we have $\left|\hat{f}\right|_{L^1(Z)} \leq C_s \|f\|_{\Lambda^s}$.

Proof. We prove this statement by recalling a definition of the norm

$$\left|\hat{f}\right|_{L^1(Z)} = \sum_{\xi \in Z} \left|\hat{f}(\xi)\right| = \sum_{\xi \in Z} \left| \int_0^1 f(x) \overline{v_\xi(x)} dx \right| \leq \sum_{\xi \in Z} \int_0^1 |f(x)| dx =$$

$$= \sum_{\xi \in Z} \int_0^1 \frac{|f(x) - f(0)|}{|x|^s} |x|^s dx \leq \sum_{\xi \in Z} \int_0^1 \sup_{x \in [0,1]} \frac{|f(x) - f(0)|}{|x|^s} |x|^s\, dx$$

$$\leq \sum_{\xi \in Z} |f|_{\Lambda^s} \int_0^1 x^s dx \leq \sum_{\xi \in Z} \frac{1}{s+1} |f|_{\Lambda^s} \leq \sum_{\xi \in Z} \frac{1}{s+1} \left(|f|_{\Lambda^s} + \sup_{x \in [0,1]} |f(x)| \right) = \left(\sum_{\xi \in Z} \frac{1}{s+1} \right) \|f\|_{\Lambda^s} \leq C_s \|f\|_{\Lambda^s}.$$

Finally, we proved the theorem.

Boundedness for θ -toroidal pseudo-differential operator

Here we prove similiar theorems as in [5].

Theorem 2. Let $a(X,D)=a(D)$ be a pseudo-differential operator with symbol $a(\xi)$ depending only on the discrete variable ξ. If $a(\xi)\in L^1(Z)$, then

$$|a(D)f|_{\Lambda^s}\le |a|_{L^1(Z)}|f|_{\Lambda^s},$$

for $0<s\le 1$.

Proof. By the formula, we have

$$a(X,D)f(x+h)-a(X,D)f(x)=$$

$$=\sum_{\xi\in Z}u_\xi(x)\,a(\xi)\int_0^1 (f(y+h)-f(y))\overline{v_\xi(y)}dy.$$

Thus, we obtain

$$\frac{|a(X,D)f(x+h)-a(X,D)f(x)|}{|h|^s}\le$$

$$\le\sum_{\xi\in Z}\left(|a(\xi)|\int_0^1\frac{|f(y+h)-f(y)|}{|h|^s}dy\right).$$

Finally, we get

$$|a(D)f|_{\Lambda^s}\le\left(\sum_{\xi\in Z}|a(\xi)|\right)|f|_{\Lambda^s}.$$

Theorem 3. Let $\frac{1}{2}<s<1$ and $a\in S^{-m}_{\delta,\rho}([0,1]\times Z), m\ge 1$. Then there exists $M>0$ such that

$$|a(X,D)f|_{\Lambda^s}\le M\|f\|_{\Lambda^s}.$$

Proof. By the mean value theorem, there exists $x_h\in[0,1]$ such that

$$u_\xi(x+h)a(x+h,\xi)-u_\xi(x)a(x,\xi)=$$

$$=hu_\xi(x_h)(a(x_h,\xi)\ln\theta+i2\pi\xi a(x_h,\xi)+$$

$$+a'(x_h,\xi)).$$

By the Bernstein's theorem, we have

$$\frac{|a(X,D)f(x+h)-a(X,D)f(x)|}{|h|^s}\le\sum_{\xi\in Z}|h|^{1-s}|a(x_h,\xi)\ln\theta+i2\pi\xi a(x_h,\xi)+a'(x_h,\xi)|\left|\hat f(\xi)\right|\le$$

$$\le\sum_{\xi\in Z}C_1(|\ln\theta||a(x_h,\xi)|+2\pi|\xi||a(x_h,\xi)|+|a'(x_h,\xi)|)\left|\hat f(\xi)\right|\le\sum_{\xi\in Z}C_1(C\langle\xi\rangle^m+|\xi|C\langle\xi\rangle^m+C\langle\xi\rangle^{m+\delta})\left|\hat f(\xi)\right|\le$$

$$\le\sum_{\xi\in Z}C_1\left(C\langle\xi\rangle^m+C\langle\xi\rangle^{m+1}+C\langle\xi\rangle^{m+\delta}\right)\left|\hat f(\xi)\right|\le\sum_{\xi\in Z}3CC_1\left|\hat f(\xi)\right|\le 3CC_1C_s\|f\|_{\Lambda^s}.$$

Thus,

$$|a(X,D)f|_{\Lambda^s}\le M\|f\|_{\Lambda^s}.$$

The next theorem gives a single sufficient condition on the symbol $a(x,\xi)$ for the corresponding pseudo-differential operator

$$a(X,D):\Lambda^s_0([0,1],\theta)\to\Lambda^s([0,1],\theta)$$

to be bounded for $0<s<1$.

Theorem 4. Let $0<s<1, 0\le\delta<\rho\le 1$ and $m>1+\delta$. If $a\in S^{-m}_{\delta,\rho}$ then, the operator

$$a(X,D):\Lambda^s_0([0,1],\theta)\to\Lambda^s([0,1],\theta)$$

is bounded.

Proof. Suppose $f\in\Lambda^s_0([0,1],\theta)$, we get

$$a(X,D)f(x+h)-a(X,D)f(x)=$$

$$=\sum_{\xi\in Z}u_\xi(x)\left(a(x+h,\xi)\int_0^1 f(y+h)\overline{v_\xi(y)}dy-a(x,\xi)\int_0^1 f(y)\overline{v_\xi(y)}dy\right)=$$

$$=\sum_{\xi\in Z}u_\xi(x)\left(a(x+h,\xi)\int_0^1(f(y+h)-f(y))\overline{v_\xi(y)}dy\right)+$$

$$+\sum_{\xi\in Z}u_\xi(x)\left((a(x+h,\xi)-a(x,\xi))\int_0^1 f(y)\overline{v_\xi(y)}dy\right).$$

Therefore, using the value mean theorem, we obtain

$$\frac{|a(X,D)f(x+h)-a(X,D)f(x)|}{|h|^s}$$

$$\le\sum_{\xi\in Z}\left(|a(x+h,\xi)|\int_0^1\frac{|f(y+h)-f(y)|}{|h|^s}dy+\frac{|a(x+h,\xi)-a(x,\xi)|}{|h|^s}\int_0^1|f(y)|dy\right)$$

$$\le\sum_{\xi\in Z}\left(|a(x+h,\xi)|\int_0^1\frac{|f(y+h)-f(y)|}{|h|^s}dy+\frac{|h||a'(x_h,\xi)|}{|h|^s}\int_0^1|f(y)|dy\right)$$

$$\le\sum_{\xi\in Z}\left(C\langle\xi\rangle^{-m}|f|_{\Lambda_0^s}+|h|^{1-s}C\langle\xi\rangle^{-m+\delta}\int_0^1|f(y)|dy\right)\le\sum_{\xi\in Z}\left(C\langle\xi\rangle^{-m}|f|_{\Lambda_0^s}+|h|^{1-s}C\int_0^1\frac{|f(y)-f(0)|}{|y|^s}|y|^s dy\right)$$

$$\le|f|_{\Lambda_0^s}\left(\sum_{\xi\in Z}C\langle\xi\rangle^{-m+\delta}\right)\left(\langle\xi\rangle^{-\delta}+\frac{1}{s+1}\right)\le|f|_{\Lambda_0^s}\left(\sum_{\xi\in Z}C\langle\xi\rangle^{-m+\delta}\right)\left(1+\frac{1}{s+1}\right).$$

Finally, we obtain

$$|a(X,D)f|_{\Lambda^s}\le|f|_{\Lambda_0^s}\left(\sum_{\xi\in Z}C\langle\xi\rangle^{-m+\delta}\right)\left(1+\frac{1}{s+1}\right).$$

Remark. It follows from the proof of Theorem 4 that

$$|a(X,D)f|_{\Lambda^s}\le|f|_{\Lambda_0^s}\sum_{\xi\in Z}\left(C\langle\xi\rangle^{-m+\delta}+\frac{|a(\cdot,\xi)|_{\Lambda_0^s}}{s+1}\right).$$

So, if $|a(\cdot,\xi)|_{\Lambda_0^s}\in L^1(Z)$, then $|a(X,D)f|_{\Lambda^s}\le M|f|_{\Lambda_0^s}$, with $M=\sum_{\xi\in Z}C_1\left(C\langle\xi\rangle^{-m+\delta}+\frac{|p(\cdot,\xi)|_{\Lambda_0^s}}{s+1}\right)<\infty$. In conclusion, the operator $a(X,D)$ will be bounded from $\Lambda_0^s([0,1],\theta)$ into $\Lambda^s([0,1],\theta)$. So we obtain the next result:

Theorem 5. Let $s\neq1$, $m>1$ and $|a(\cdot,\xi)|_{\Lambda_0^s}\in L^1(Z)$. If $a\in S_{\delta,\rho}^{-m}$ then $a(X,D):\Lambda_0^s([0,1],\theta)\to\Lambda^s([0,1],\theta)$ is a bounded operator.

Theorem 6. Let $0 < \varepsilon < 1$ and $k \in N$ with $k > \frac{n}{2}$, let a be a symbol such that $\left|\Delta_\xi^\alpha a(x,\xi)\right| \leq C_\alpha \langle\xi\rangle^{-\frac{n}{2}\varepsilon-(1-\varepsilon)|\alpha|}$, $\left|\partial_x^\beta \xi a(x,\xi)\right| \leq C_\beta \langle\xi\rangle^{-\frac{n}{2}}$, for $|\alpha|, |\beta| \leq k$. Then, $a(X,D)$ is a bounded operator from $L^p(T)$ into $L^p(T)$ for $2 \leq p < \infty$.

In the following theorems, we obtain Hölder boundedness using the Morrey inequality [6]: if $1 < p < \infty$ and $s_p = 1 - \frac{1}{p}$, then for $x, y \in R$ we have

$$\frac{|u(x+h)-u(x)|}{|h|^{s_p}} \leq |u'(x)|_{L^p(R)}.$$

Function on the torus may be thought as those functions on R that are 1-periodic, under these assumptions, we can use a toroidal version of Morrey inequality on $L^p(T)$.

Theorem 7. Let $0 \leq \delta < \rho \leq 1$ and $m > 1$. If $a \in S_{\delta,\rho}^{-m}$, then $a(X,D): \Lambda_0^{\frac{1}{2}}([0,1],\theta) \to \Lambda^{\frac{1}{2}}([0,1],\theta)$ is a bounded operator.

Proof. The composition of the pseudo-differential operators $\frac{d}{dx}$ and $a(X,D)$ is the pseudo-differential operator $\frac{d}{dx}a(X,D)$ of degree $-m+1 < 0$, so, $T = \frac{d}{dx}a(X,D): L^2(0,1) \to L^2(0,1)$. If $u \in \Lambda_0^{\frac{1}{2}}([0,1],\theta)$, then

$$\frac{|a(X,D)f(x+h)-a(X,D)f(x)|}{|h|^{s_p}} \leq C\left|\frac{d}{dx}a(X,D)u\right|_{L^2} \leq C\left\|\frac{d}{dx}a(X,D)\right\|_{(L^2,L^2)} |u|_{L^2}$$

$$= C\left\|\frac{d}{dx}a(X,D)\right\|_{(L^2,L^2)} \left(\int_0^1 \frac{|u(x)-u(0)|}{|x|^{2\left(\frac{1}{2}\right)}} |x|^{2\left(\frac{1}{2}\right)} dx\right)^{\frac{1}{2}} \leq C\left\|\frac{d}{dx}a(X,D)\right\|_{(L^2,L^2)} \left(\frac{1}{2}\right)^{\frac{1}{2}} |u|_{\Lambda_0^{\frac{1}{2}}}.$$

Finally,

$$\|a(X,D)f\|_{\Lambda_0^{\frac{1}{2}}([0,1],\theta)} \leq \max\left\{\sup_{x\in[0,1]} |a(x,\cdot)|_{L^1(Z)}, C\left\|\frac{d}{dx}a(X,D)\right\|_{(L^2,L^2)} \left(\frac{1}{2}\right)^{\frac{1}{2}}\right\} \|f\|_{\Lambda_0^{\frac{1}{2}}([0,1],\theta)}$$

Theorem 8. Let $0 < \varepsilon < 1$ and $a(x,\xi)$ be a symbol such that $\left|\Delta_\xi^\alpha \xi a(x,\xi)\right| \leq C_\alpha \langle\xi\rangle^{-\frac{\varepsilon}{2}-(1-\varepsilon)|\alpha|}$, $\left|\partial_x^\beta \xi a(x,\xi)\right| \leq C_\beta \langle\xi\rangle^{-\frac{\varepsilon}{2}}$, for $0 \leq |\alpha|, |\beta| \leq 1$. If $\frac{1}{2} < s < 1$, then $a(X,D): \Lambda_0^s([0,1],\theta) \to \Lambda^s([0,1],\theta)$ is a bounded linear operator.

Proof. If $\frac{1}{2} \leq s < 1$, there exists $2 \leq p < \infty$ such that $s = 1 - \frac{1}{p}$. Applying Theorem 6 to the symbol $i2\pi\xi a(x,\xi)$ we obtain $L^p(0,1)$- boundedness for the operator $\frac{d}{dx}a(X,D)$. If $u \in \Lambda_0^s([0,1],\theta)$, then

$$\frac{|a(X,D)u(x+h)-a(X,D)u(x)|}{|h|^s} \leq C\left|\frac{d}{dx}a(X,D)u\right|_{L^p} \leq C\left\|\frac{d}{dx}a(X,D)\right\|_{(L^p,L^p)} |u|_{L^p}$$

$$= C\left\|\frac{d}{dx}a(X,D)\right\|_{(L^p,L^p)} \left(\int_0^1 \frac{|u(x)-u(0)|}{|x|^{p^s}} |x|^{p^s} dx\right)^{\frac{1}{2}} \leq C\left\|\frac{d}{dx}a(X,D)\right\|_{(L^p,L^p)} \left(\frac{1}{p}\right)^{\frac{1}{p}} |u|_{\Lambda_0^s}.$$

Hence,

$$|a(X,D)u|_{\Lambda^s} \le C\left\|\frac{d}{dx}a(X,D)\right\|_{(L^p,L^p)}\left(\frac{1}{p}\right)^{\frac{1}{p}}|u|_{\Lambda_0^s}$$

Now, since $|\xi a(x,\xi)| \le C\langle\xi\rangle^{-\frac{\varepsilon}{2}}$, we get

$$|a(x,\xi)| \le C\langle\xi\rangle^{-\frac{\varepsilon}{2}}|\xi|^{-1}, \xi \ne 0.$$

Hence, we obtain

$$M = \sup_{x\in[0,1]}|a(x,\cdot)|_{L^1(Z)} < \infty$$

Therefore

$$\|a(X,D)f\|_{\Lambda_0^s([0,1],\theta)} \le$$
$$\le \max\left\{M, C\left\|\frac{d}{dx}p(X,D)\right\|_{(L^p,L^p)}\left(\frac{1}{p}\right)^{\frac{1}{p}}\right\}\|f\|_{\Lambda_0^s([0,1],\theta)}.$$

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