On a boundedness result of non-toroidal pseudo-differential operators

Abstract. In this article, we prove boundedness results for \( \theta \)-toroidal pseudo-differential operators generated by a differentiation operator with a non-periodic boundary condition. \( \theta \)-toroidal pseudo-differential operators are a natural generalization of a toroidal one. As in the classical case, this class of operators act on a suitable test function space by weighting the Fourier transform “very well”. Standard operations as adjoints, products and commutators with \( \theta \)-toroidal pseudo-differential operators can be characterized by their \( \theta \)-toroidal symbols. For pseudo-differential operators on \( \mathbb{R}^n \), the symbol analysis is well developed. Here, we provide more complicated properties of the \( \theta \)-toroidal pseudo calculus. Namely, we introduce a Holder space induced by a differentiation operator with a non-periodic boundary condition. Finally, for the elements of this space we prove theorems on boundedness of the operators acting on the specified functional spaces. Indeed, in this paper we continue a development of the so called “non-harmonic analysis” introduced in the recent papers of the authors.

Key words: \( \theta \)-toroidal pseudo-differential operator, \( \theta \)-toroidal Holder space, \( \theta \)-Fourier transform, \( \theta \)-symbol, bounded operator.

Introduction

In [3], it was introduced an analysis generated by the differential operator

\[
L_y(x) = -i \frac{dy(x)}{dx}, \quad 0 < x < 1
\]

acting on \( L_2(0; 1) \) with the boundary condition

\[
\theta y(0) - y(1) = 0,
\]

where \( \theta \geq 1 \).

Spectrum of the operator \( L \) is

\[
\lambda_\xi = -i \ln \theta + 2\xi \pi, \quad \xi \in \mathbb{Z}
\]

System of eigenfunctions of the operator \( L \) is

\[
u_\xi(x) = \theta^{-x} e^{i2\xi \pi x}, \quad \xi \in \mathbb{Z}.
\]

For the following spectral properties of the operator \( L \) we refer to [3] and [7].

1. The system of eigenfunctions of the operator \( L \) is a Riesz basis in \( L_2(0; 1) \);
2. If function \( f \) belongs to the domain of operator \( L \), then \( f(x) \) expands to a uniformly convergent series of eigenfunctions of the operator \( L \);
3. The resolvent of the operator \( L \) is

\[
(L - \lambda I)^{-1} f(x) = \frac{e^{i\lambda x + 1}}{\Delta(\lambda)} \int_0^1 e^{-i\lambda t} f(t) dt + i e^{i\lambda x} \int_0^x e^{-i\lambda t} f(t) dt.
\]
\[ \Delta(\lambda) = \theta - e^{\lambda}. \]

**θ-toroidal pseudo-differential operators**

θ-Fourier transform and θ-toroidal Hölder spaces. Here we give a definition of θ-Fourier transform [3]. θ-Fourier transform \((f \mapsto \hat{f})\) \(C^\infty_\theta[0,1] \to S(Z)\) is given by

\[ \hat{f}(\xi) = \int_0^1 f(x) \overline{v_\xi(x)} \, dx \tag{6} \]

and its inverse \(\hat{f}^{-1}\) is given by

\[ f(x) = \sum_{\xi \in Z} \hat{f}(\xi) u_\xi(x). \tag{7} \]

**Remark 1** The functional space \(C^\infty_\theta[0,1]\) is called the space of test functions and \(S(Z)\) is space of rapidly decaying functions [3].

In what follows, we will use the following spaces from [5].

We define θ-toroidal Hölder spaces

\[ A^s(\{0,1\}, \theta) = \left\{ f : [0,1] \to C : \|f\|_A^s = \sup_{x, \xi \in [0,1]} \frac{|f(x+h) - f(x)|}{|h|^s} < \infty \right\} \tag{8} \]

\[ A^s_\theta([0,1], \theta) = \{ f \in A^s([0,1], \theta) : f(0) = 0 \} \tag{9} \]

for each \(0 < s \leq 1\). These spaces are Banach.

θ-toroidal symbol class. Suppose that \(m \in \mathbb{R}, 0 \leq \delta, \rho \leq 1\). Then the θ-toroidal symbol class \(S^m_{\delta, \rho}([0,1] \times Z)\) consists of those function \(a(x, \xi)\) which are smooth in \(x\) for all \(\xi \in Z\), and which satisfy

\[ |\Delta^{\alpha} \partial_x^\beta a(x, \xi)| \leq C_{\alpha \beta m}(\xi)^{m-\alpha + \beta} \tag{10} \]

for every \(\xi \in Z, x \in [0,1], \alpha, \beta \in \mathbb{Z}_+\), where

\[ \langle \xi \rangle := 1 + |\xi|. \]

We call \(a(x, \xi)\) a symbol [3]. The operator \(\Delta_\xi\) is the difference operator

\[ \Delta_\xi \partial(x) := \epsilon \sigma(\xi), \]

where \(\sigma(\xi) : Z \to C\).

We denote the θ-toroidal pseudo-differential operator by

\[ a(X, D)f(x) = \sum_{\xi \in Z} u_\xi(x) a(x, \xi) \hat{f}(\xi) \tag{11} \]

where \(a(x, \xi)\) is a symbol of a θ-toroidal pseudo-differential operator [5].

We can write for \(h \in T\),

\[ a(X, D)f(x + h) = \sum_{\xi \in Z} u_\xi(x) a(x + h, \xi) \int_0^1 f(y + h) \overline{v_\xi(x)} \, dy. \]

**Theorem 1. (Bernstein).** Assume that \(f \in A^s_\theta([0,1], \theta)\), for \(s > \frac{1}{2}\). Then we have \(|\hat{f}|_{L^1(Z)} \leq C_s \|f\|_{A^s}\).

**Proof.** We prove this statement by recalling a definition of the norm

\[ |\hat{f}|_{L^1(Z)} = \sum_{\xi \in Z} |\hat{f}(\xi)| = \sum_{\xi \in Z} \int_0^1 f(x) \overline{v_\xi(x)} \, dx \leq \sum_{\xi \in Z} \int_0^1 |f(x)| \, dx = \sum_{\xi \in Z} \int_0^1 \frac{|f(x) - f(0)|}{|x|^s} \, dx \leq \sum_{\xi \in Z} \sup_{x \in [0,1]} \frac{|f(x) - f(0)|}{|x|^s} \, dx \]

\[ \leq \sum_{\xi \in Z} |f|_{A^s} \int_0^1 x^s \, dx \leq \sum_{\xi \in Z} \frac{1}{s + 1} |f|_{A^s} \leq \sum_{\xi \in Z} \frac{1}{s + 1} \left( |f|_{A^s} + \sup_{x \in [0,1]} |f(x)| \right) = \left( \sum_{\xi \in Z} \frac{1}{s + 1} \right) \|f\|_{A^s} \leq C_s \|f\|_{A^s}. \]
Finally, we proved the theorem.

**Boundedness for \( \theta \)-toroidal pseudo-differential operator**

Here we prove similar theorems as in [5].

**Theorem 2.** Let \( a(X,D) = a(D) \) be a pseudo-differential operator with symbol \( a(\xi) \) depending only on the discrete variable \( \xi \). If \( a(\xi) \in L^\infty(Z) \), then

\[
|a(D)f|_{A^s} \leq |a|_{L^\infty(Z)} |f|_{A^s},
\]

for \( 0 < s \leq 1 \).

**Proof.** By the formula, we have

\[
a(X,D)f(x+h) - a(X,D)f(x) = \sum_{\xi \in Z} u_\xi(x) a(\xi) \int_0^1 (f(y+h) - f(y)) \nu_\xi(y) dy.
\]

Thus, we obtain

\[
|a(X,D)f(x+h) - a(X,D)f(x)| \leq |h|^{1-s} |a(\xi)| \int_0^1 \frac{|f(y+h) - f(y)|}{|h|^s} dy.
\]

Finally, we get

\[
|a(D)f|_{A^s} \leq \left( \sum_{\xi \in Z} |a(\xi)| \right) |f|_{A^s}.
\]

**Theorem 3.** Let \( \frac{1}{2} < s < 1 \) and \( a \in S_{\delta,\rho}^m(\{0,1\} \times Z), m \geq 1 \). Then there exists \( M > 0 \) such that

\[
|a(X,D)f|_{A^s} \leq M \|f\|_{A^s}.
\]

**Proof.** By the mean value theorem, there exists \( x_h \in [0,1] \) such that

\[
u_\xi(x+h)a(x+h,\xi) - u_\xi(x)a(x,\xi) = hu_\xi(x_h)(a(x_h,\xi)\ln \theta + i2\pi \xi a(x_h,\xi) + a'(x_h, \xi)).
\]

By the Bernstein’s theorem, we have

\[
|a(X,D)f(x+h) - a(X,D)f(x)| \leq \sum_{\xi \in Z} C_1 (|\ln \theta||a(x_h,\xi)| + 2\pi |\xi||a(x_h,\xi)| + |a'(x_h, \xi)|) |\hat{f}(\xi)| \leq \sum_{\xi \in Z} C_1 (C(\xi)^m + |\xi| C(\xi)^m + C(\xi)^{m+\delta}) |\hat{f}(\xi)| \leq \sum_{\xi \in Z} C_1 C(\xi)^m + C(\xi)^{m+1} + C(\xi)^{m+\delta}) |\hat{f}(\xi)| \leq \sum_{\xi \in Z} 3CC_1 |\hat{f}(\xi)| \leq 3CC_1 C_1 \|f\|_{A^s}.
\]

Thus,

\[
|a(X,D)f|_{A^s} \leq M \|f\|_{A^s}.
\]

The next theorem gives a single sufficient condition on the symbol \( a(x,\xi) \) for the corresponding pseudo-differential operator

\[
a(X,D): A_0^\delta([0,1], \theta) \to A^\delta([0,1], \theta)
\]

to be bounded for \( 0 < s < 1 \).

**Theorem 4.** Let \( 0 < s < 1, 0 \leq \delta < \rho \leq 1 \) and \( m > 1 + \delta \). If \( a \in S_{\delta,\rho}^m \) then, the operator

\[
a(X,D): A_0^\delta([0,1], \theta) \to A^\delta([0,1], \theta)
\]

is bounded.

**Proof.** Suppose \( f \in A_0^\delta([0,1], \theta) \), we get

\[
|a(X,D)f|_{A^s} \leq \left( \sum_{\xi \in Z} |a(\xi)| \right) |f|_{A^s}.
\]
\[ a(X, D)f(x+h) - a(X, D)f(x) = \sum_{\xi \in \mathbb{Z}} u_\xi(x) \left( a(x + h, \xi) \int_0^1 f(y + h) v_\xi(y) \, dy - a(x, \xi) \int_0^1 f(y) v_\xi(y) \, dy \right) = \]
\[ = \sum_{\xi \in \mathbb{Z}} u_\xi(x) \left( a(x + h, \xi) \int_0^1 (f(y + h) - f(y)) v_\xi(y) \, dy \right) + \]
\[ + \sum_{\xi \in \mathbb{Z}} u_\xi(x) \left( (a(x + h, \xi) - a(x, \xi)) \int_0^1 f(y) v_\xi(y) \, dy \right). \]

Therefore, using the value mean theorem, we obtain

\[
\frac{|a(X, D)f(x+h) - a(X, D)f(x)|}{|h|^s} \leq \sum_{\xi \in \mathbb{Z}} \left( |a(x + h, \xi)| \int_0^1 \frac{|f(y + h) - f(y)|}{|h|^s} \, dy + \frac{|a(x + h, \xi) - a(x, \xi)|}{|h|^s} \int_0^1 |f(y)| \, dy \right) \]
\[
\leq \sum_{\xi \in \mathbb{Z}} \left( |a(x + h, \xi)| \int_0^1 \frac{|f(y + h) - f(y)|}{|h|^s} \, dy + \frac{|h| |a'(x, \xi)|}{|h|^s} \int_0^1 |f(y)| \, dy \right) \]
\[
\leq \sum_{\xi \in \mathbb{Z}} \left( C(\xi)^{-m} |f|_{A^\delta} + |h|^{1-s} C(\xi)^{-m+\delta} \int_0^1 |f(y)| \, dy \right) \leq \sum_{\xi \in \mathbb{Z}} \left( C(\xi)^{-m} |f|_{A^\delta} + |h|^{1-s} C \int_0^1 \frac{|f(y) - f(0)|}{|y|^s} \, dy \right) \]
\[
\leq |f|_{A^\delta} \left( \sum_{\xi \in \mathbb{Z}} C(\xi)^{-m+\delta} \right) \left( \xi^{-\delta} + \frac{1}{s+1} \right) \leq |f|_{A^\delta} \left( \sum_{\xi \in \mathbb{Z}} C(\xi)^{-m+\delta} \right) \left( 1 + \frac{1}{s+1} \right). \]

Finally, we obtain

\[
|a(X, D)f|_{A^s} \leq |f|_{A^\delta} \left( \sum_{\xi \in \mathbb{Z}} C(\xi)^{-m+\delta} \right) \left( 1 + \frac{1}{s+1} \right). \]

So, if \(|a(\cdot, \xi)|_{A^\delta} \in L^1(\mathbb{Z})\), then \(|a(X, D)f|_{A^s} \leq M|f|_{A^\delta}\), with \(M = \sum_{\xi \in \mathbb{Z}} C_1 \left( C(\xi)^{-m+\delta} + \frac{|p(\xi)|_{A^\delta}}{s+1} \right) < \infty\). In conclusion, the operator \(a(X, D)\) will be bounded from \(A^\delta_0([0,1], \theta)\) into \(A^\delta([0,1], \theta)\). So we obtain the next result:

**Theorem 5.** Let \(s \neq 1\), \(m > 1\) and \(|a(\cdot, \xi)|_{A^\delta} \in L^1(\mathbb{Z})\). If \(a \in S^{m}_{\delta,p}\) then \(a(X, D): A^\delta_0([0,1], \theta) \to A^\delta([0,1], \theta)\) is a bounded operator.
Theorem 6. Let $0 < \varepsilon < 1$ and $k \in \mathbb{N}$ with $k > \frac{n}{2}$, let $a$ be a symbol such that $|\Delta_\xi^\varepsilon a(x, \xi)| \leq C_\alpha (\xi)^{- \frac{n}{2} - (1 - \varepsilon)|\alpha|}$ , $|\partial_\xi^\beta \xi a(x, \xi)| \leq C_\beta (\xi)^{- \frac{n}{2}}$, for $|\alpha|, |\beta| \leq k$. Then, $a(X, D)$ is a bounded operator from $L^p(T)$ into $L^p(T)$ for $2 \leq p < \infty$.

In the following theorems, we obtain Hölder boundedness using the Morrey inequality [6]: if $1 < p < \infty$ and $s = 1 - \frac{1}{p}$, then for $x, y \in \mathbb{R}$ we have

$$\frac{|u(x + h) - u(x)|}{|h|^s} \leq |u'(x)|_{L^p(\mathbb{R})}.$$ 

Function on the torus may be thought as those functions on $\mathbb{R}$ that are $1$-periodic, under these assumptions, we can use a toroidal version of Morrey inequality on $L^p(T)$.

Theorem 7. Let $0 \leq \delta < \rho \leq 1$ and $m > 1$. If $a \in S^m_{\delta, \rho}$, then $a(X, D): \Lambda^\delta_0([0,1], \theta) \to \Lambda^\rho_0([0,1], \theta)$ is a bounded operator.

Proof. The composition of the pseudo-differential operators $\frac{d}{dx}$ and $a(X, D)$ is the pseudo-differential operator $\frac{d}{dx} a(X, D)$ of degree $-m + 1 < 0$, so, $T = \frac{d}{dx} a(X, D): L^2(0,1) \to L^2(0,1)$. If $u \in \Lambda^\delta_0([0,1], \theta)$, then

$$\frac{|a(X, D)f(x+h) - a(X, D)f(x)|}{|h|^s} \leq C \|\frac{d}{dx} a(X, D)u\|_{L^2} \leq C \|\frac{d}{dx} a(X, D)\|_{(L^2, L^2)} \|u\|_{L^2}$$

For $\frac{1}{2} \leq s < 1$, there exists $2 \leq p < \infty$ such that $s = 1 - \frac{1}{p}$. Applying Theorem 6 to the symbol $i2\pi \xi a(x, \xi)$ we obtain $L^p(0,1)$- boundedness for the operator $\frac{d}{dx} a(X, D)$. If $u \in \Lambda^\delta_0([0,1], \theta)$, then

$$\|a(X, D)f\|_{\Lambda^\delta_0([0,1], \theta)} \leq \max \left\{ \sup_{x \in [0,1]} |a(x, \cdot)|_{L^1(\mathbb{Z})}, C \|\frac{d}{dx} a(X, D)\|_{(L^2, L^2)} \left( \frac{1}{2} \right)^{\frac{1}{2}} \right\} \|f\|_{\Lambda^\delta_0([0,1], \theta)}$$

Finally,

$$\|a(X, D)f\|_{\Lambda^\delta_0([0,1], \theta)} \leq \max \left\{ \sup_{x \in [0,1]} |a(x, \cdot)|_{L^1(\mathbb{Z})}, C \|\frac{d}{dx} a(X, D)\|_{(L^2, L^2)} \left( \frac{1}{2} \right)^{\frac{1}{2}} \right\} \|f\|_{\Lambda^\delta_0([0,1], \theta)}$$

Theorem 8. Let $0 < \varepsilon < 1$ and $a(x, \xi)$ be a symbol such that $|\Delta_\xi^\varepsilon a(x, \xi)| \leq C_\alpha (\xi)^{- \frac{n}{2} - (1 - \varepsilon)|\alpha|}$ , $|\partial_\xi^\beta \xi a(x, \xi)| \leq C_\beta (\xi)^{- \frac{n}{2}}$, for $0 \leq |\alpha|, |\beta| \leq 1$ . If $\frac{1}{2} < s < 1$ , then $a(X, D): \Lambda^\delta_0([0,1], \theta) \to \Lambda^\rho([0,1], \theta)$ is a bounded linear operator.

Proof. If $\frac{1}{2} \leq s < 1$, there exists $2 \leq p < \infty$ such that $s = 1 - \frac{1}{p}$. Applying Theorem 6 to the symbol $i2\pi \xi a(x, \xi)$ we obtain $L^p(0,1)$- boundedness for the operator $\frac{d}{dx} a(X, D)$. If $u \in \Lambda^\delta_0([0,1], \theta)$, then

$$\frac{|a(X, D)u(x+h) - a(X, D)u(x)|}{|h|^s} \leq C \|\frac{d}{dx} a(X, D)u\|_{L^p} \leq C \|\frac{d}{dx} a(X, D)\|_{(L^p, L^p)} \|u\|_{L^p}$$

$$= C \|\frac{d}{dx} a(X, D)\|_{(L^p, L^p)} \left( \frac{1}{p} \right)^{\frac{1}{p}} \|u\|_{\Lambda^\delta_0}.$$
Hence,

\[ |a(X,D)u|_{A^{s}} \leq C \left\| \frac{d}{dx} a(X,D) \right\|_{(L^p, L^p)} \left( \frac{1}{p} \right)^{\frac{1}{p}} |u|_{A^{s}} \]

Now, since \( |\xi a(x,\xi)| \leq C \langle \xi \rangle^{-2} \), we get

\[ |a(x,\xi)| \leq C \langle \xi \rangle^{-2} |\xi|^{-1}, \xi \neq 0. \]

Hence, we obtain

\[ M = \sup_{x \in [0,1]} |a(x,\cdot)|_{L^1(\mathbb{Z})} < \infty \]

Therefore

\[ \|a(X,D)f\|_{A^{s}([0,1],\theta)} \leq \max \left\{ M, C \left\| \frac{d}{dx} p(X,D) \right\|_{(L^p, L^p)} \left( \frac{1}{p} \right)^{\frac{1}{p}} \|f\|_{A^{s}([0,1],\theta)} \right\}. \]

Acknowledgement. The authors were supported by the Ministry of Education and Science of the Republic of Kazakhstan, MESRK Grant AP05130994.

References