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Critical exponents of Fujita type for certain time-fractional diffusion equations

Abstract. Solutions of initial value problems for non-linear parabolic partial differential equations may not exist for all time. In other words, these solutions may blow up in some sense or other. Recently in connection with problems for some class of non-linear parabolic equations, Kaplan [1], Ito [2] and Friedman [3] gave certain sufficient conditions under which the solutions blow up in a finite time. Although their results are not identical, we can say according to them that the solutions are apt to blow up when the initial values are sufficiently large. The data at which solutions can blow up is called critical exponents of Fujita.

The present paper is devoted to research critical exponents of Fujita type for certain non-linear time-fractional diffusion equations with the nonnegative initial condition. The Riemann-Liouville derivative is used as a fractional derivative. To prove the blow up, we use the known test function method developed in papers by Mitidieri and Pohozhaev [16].

Key words: blow-up, global weak solution, critical exponents of Fujita, time-fractional diffusion equation.

Introduction

In the paper [4], Fujita considered the initial value problem:

$$\begin{cases} u_t = \Delta u + u^p, \ \text{ for } \ (x,t) \in R^N \times (0,\infty), \\ u(x,0) = a(x) \geq 0, \ \text{ for } \ x \in R^N, \end{cases} \tag{1}$$

where p is positive number, $a(x) \in L^1(R^N)$ is nonnegative and positive on some subset of R^N of positive measure and Δ denotes the Laplacian in N variables.

More precisely, he considered this problem on $R^N \times [0,T)$ for some $T \leq +\infty$. A (classical or weak solution) of equation on $R^N \times [0,T)$ for some $T < +\infty$ is called a local (in time) solution. The supremum of all such T's for which a solution exists is called the maximal time of existence, T_{max}. When $T_{\max} = +\infty$ we say the solution is global. When $T_{\max} < +\infty$, we say the solution of equation is not global (or the solution "blows up in finite time".

Let $p_c = 2/N$. Fujita proved the following assertions:

(i) if $0 < p < p_c$ and $a(x) > 0$ for some x_0, then the solution of problem (1) grows infinitely at some finite instant of time;

(ii) if $p > p_c$, then problem has a positive solution for every $t > 0$.

More exactly, for each $k > 0$, there exists a $\delta > 0$ such that problem (1) has a global solution whenever $0 \leq a(x) \leq \delta e^{-k|x|^2}$. The number p_c is referred to as the *critical exponent*. In the critical case, this problem was solved in [5] for $N = 1,2$ and in [6] for arbitrary N. It was shown that if $p = p_c$ and then there is no nonnegative global solution for any nontrivial initial data. The proof was simplified by Weissler [7].

Later, Fujita [8] extended his own results to the more general case in which $f(u)$ (the term describing the reaction) is convex and satisfies appropriate conditions (the main of which is the Osgood condition). The results obtained for problem (1) were generalized in [9] to the case of an initial-boundary value problem in a cone with the term $|x|^{\sigma} u^{p+1}$ instead of u^{p+1}; it was proved that, in this case, the critical exponent is equal to $(2+\sigma)/N$.

It was shown in [10] that the critical exponent for the porous medium equation is equal to $m-1+2/N$, where $m > (N-2)+/N$.

The equation

$$u_t = \Delta u^m + |x|^{\sigma} t^s u^{1+p},\ t > 0,\ x \in R^N,$$

with nonnegative initial data was considered in [11]. It was shown that the critical exponent for this problem is equal to $(m-1)(s-1)+(2+2s+\sigma)/N > 0$.

The following parabolic equation with the fractional power $(-\Delta)^{\beta/2}, 0<\beta<2,$ of the Laplace operator was studied in [12]:

$$u_t + (-\Delta)^{\beta/2} u = u^{1+p},\ (t,x) \in R^+ \times R^N.$$

Using Fujita's method [4], the authors [13] discussed nonnegative solutions of the equation

$$u_t + (-\Delta)^{\beta/2} u = h(t) u^{1+p},\ (t,x) \in R^+ \times R^N. \quad (2)$$

where $h(t)$ behaves as at $\sigma > -1,\ 0 < pN\beta \le \beta(1+\sigma)$. The proof given in [13] is based on the reduction of Eq. (2) to an ordinary differential equation for the mean value of u with the use of the fundamental solution [say, $P_\beta(x,\ t)$] of $L_\beta := \partial/\partial t + (-\Delta)^{\beta/2}$. Apparently, the approach of [13] cannot be used for systems of two differential equations with distinct diffusion terms unless, for example, $P_\beta(x,\ t)$ can be compared with $P_\gamma(x,\ t)$ for $\beta < \gamma$.

The following spatio-temporal fractional equation

$$\begin{cases} \mathrm{D}_{0+}^{\alpha} u + (-\Delta)^{\beta/2}(u) = h(x,t) u^{1+p}, & \text{for } (x,t) \in R^N \times R^+, \\ u(x,0) = u_0(x) \ge 0, & \text{for } x \in R^N, \end{cases}$$

where $\mathrm{D}_{0+}^{\alpha}, \alpha \in (0,1)$ is the fractional derivative in the sense of Caputo, $\beta \in [1,2]$ with nonnegative initial data was considered in [14]. The critical exponent for this problem is equal to

$$1 < p < p_c = 1 + \frac{\alpha(\beta+\alpha)+\beta\rho}{\alpha N + \beta(1-\alpha)}.$$

The main goal of the present research is to obtain results on critical exponents for time - fractional diffusion equation of the form

$$u_t = \frac{\partial^2}{\partial x^2} D_{+t}^{1-\alpha} u + u^p,\ (x,t) \in R \times (0,T) = \Omega \quad (3)$$

with the initial condition

$$u(x,0) = u_0(x) \ge 0 \quad (4)$$

where $D_{+t}^{1-\alpha}$ denotes the time-derivative of arbitrary order $(0,1)$ in the sense of Riemann-Liouville.

In the case, $\alpha = 1$ the time-fractional diffusion equation (3) reduces to the usual heat equation, which is well documented in [4].

Some definitions and properties of fractional operators

Definition 1. [15]The left and right Riemann-Liouville fractional integrals I_{a+}^{α} and I_{b-}^{α} of order $\alpha \in R, (\alpha > 0)$ are given by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,\ t \in (a,b],$$

and

$$I_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds,\ t \in [a,b),$$

respectively. Here $\Gamma(\alpha)$ denotes the Euler gamma function.

Definition 2. [15] The left Riemann-Liouville fractional derivative D_{a+}^{α} of order $\alpha \in R$, $(0<\alpha<1)$ is defined by

$$D_{a+}^{\alpha} f(t)=\frac{d}{dt} I_{a+}^{1-\alpha} f(t)=$$
$$=\frac{1}{\Gamma(1-\alpha)} \int_{a}^{t}(t-s)^{-\alpha} f(s) ds,\ t \in(a, b],$$

Similarly, the right Riemann-Liouville fractional derivative D_{b-}^{α} of order $\alpha \in R$, $(0<\alpha<1)$ is defined by

$$D_{b-}^{\alpha} f(t)=\frac{d}{dt} I_{b-}^{1-\alpha} f(t)=$$
$$=\frac{1}{\Gamma(1-\alpha)} \int_{t}^{b}(s-t)^{-\alpha} f(s) ds,\ t \in[a, b).$$

Definition 3. [15] The left and the right Caputo fractional derivatives of order $\alpha \in R$, $(0<\alpha<1)$ is defined, respectively, by

$$\mathrm{D}_{a+}^{\alpha} f(t)=I_{a+}^{1-\alpha} \frac{d}{dt} f(t)=$$
$$=\frac{1}{\Gamma(1-\alpha)} \int_{a}^{t}(t-s)^{-\alpha} f'(s) ds,\ t \in(a, b],$$

and

$$\mathrm{D}_{b-}^{\alpha} f(t)=-I_{b-}^{1-\alpha} \frac{d}{dt} f(t)=$$
$$=-\frac{1}{\Gamma(1-\alpha)} \int_{t}^{b}(s-t)^{-\alpha} f'(s) ds,\ t \in[a, b).$$

requires $f'(s) \in L^{1}(0, T)$.

Definition 4. A function $u \in L_{loc}^{1}(\Omega_{T})$, $(\Omega_{T}:=(x, t) \in R \times(0, T))$ is a local weak solution to time-fractional diffusion equation on Ω_{T} such that

$$L(u, \varphi)=\int_{\Omega_{T}} u(x, t) \mathrm{D}_{T-}^{1-\alpha} \varphi_{xx}(x, t) dxdt+$$
$$+\int_{\Omega_{T}} u(x, t) \varphi_{t}(x, t) dxdt+\int_{\Omega_{T}} u^{p}(x, t) \varphi(x, t) dxdt$$

where

$$L(u, \varphi)=\int_{0}^{T} D_{t+}^{1-\alpha} u(x, t) \varphi_{x}(x, t)\Big|_{R} dt-$$
$$-\int_{0}^{T} \frac{\partial}{\partial x} D_{t+}^{1-\alpha} u(x, t) \varphi(x, t)\Big|_{R} dt-\int_{R} u_{0}(x) \varphi(x, 0) dx,$$

for any test function $\varphi(x, t) \in C_{x, t}^{2,1}(\Omega_{T})$ defined on the domain Ω_{T} with $\varphi(x, T)=0$.

Property 5. Integrating fractional integral by parts

$$\int_{\Omega_{T}} I_{t+}^{\alpha} u(x, t) f(x, t) dxdt=\int_{\Omega_{T}} u(x, t) I_{T-}^{1-\alpha} f(x, t) dxdt.$$

Main results

Multiply the time-fractional diffusion equation (1) by a test function $\varphi(x, t)$, we have

$$\int_{0}^{T} \int_{R} u_{t}(x, t) \varphi(x, t) dxdt=$$
$$=\int_{0}^{T} \int_{R} \frac{\partial^{2}}{\partial x^{2}} D_{t+}^{1-\alpha} u(x, t) \varphi(x, t) dxdt+$$
$$+\int_{0}^{T} \int_{R} u^{p}(x, t) \varphi(x, t) dxdt.$$

Integrating by parts the equation (3) and note that $\varphi(x, T)=0$ we get

$$\int_{0}^{T} \int_{R} u_{t}(x, t) \varphi(x, t) dxdt=$$
$$=-\int_{R} u_{0}(x) \varphi(x, 0) dx+ \tag{5}$$
$$+\int_{0}^{T} \int_{R} u(x, t) \varphi_{t}(x, t) dxdt,$$

and

$$\begin{aligned}\int_0^T\int_R \frac{\partial^2}{\partial x^2} D_{t+}^{1-\alpha} u(x,t)\varphi(x,t)\,dxdt = \\ = \int_0^T \frac{\partial}{\partial x} D_{t+}^{1-\alpha} u(x,t)\varphi(x,t)\Big|_R dt - \\ -\int_0^T D_{t+}^{1-\alpha} u(x,t)\varphi_x(x,t)\Big|_R dt + \\ +\int_0^T\int_R D_{t+}^{1-\alpha} u(x,t)\varphi_{xx}(x,t)\,dxdt\end{aligned} \tag{6}$$

By *Property 5* the last part of the (6), can be written as

$$\begin{aligned}\int_0^T\int_R D_{t+}^{1-\alpha} u(x,t)\varphi_{xx}(x,t)\,dxdt = \\ = \int_0^T\int_R u(x,t)\mathrm{D}_{T-}^{1-\alpha}\varphi_{xx}(x,t)\,dxdt\end{aligned}$$

Obviously, we can write the equation (3)-(4) in the following form

$$\begin{aligned}L(u,\varphi) = \int_{\Omega_T} u(x,t)\mathrm{D}_{T-}^{1-\alpha}\varphi_{xx}(x,t)dxdt + \\ +\int_{\Omega_T} u(x,t)\varphi_t(x,t)dxdt + \\ +\int_{\Omega_T} u^p(x,t)\varphi(x,t)dxdt\end{aligned} \tag{7}$$

where

$$L(u,\varphi) = \int_0^T D_{t+}^{1-\alpha} u(x,t)\varphi_x(x,t)\Big|_R dt -$$

$$-\int_0^T \frac{\partial}{\partial x} D_{t+}^{1-\alpha} u(x,t)\varphi(x,t)\Big|_R dt - \int_R u_0(x)\varphi(x,0)dx,$$

Theorem 1. Let $p > 1$. If

$$1 < p \le p_c = 1 + \frac{2}{\alpha},$$

then problem (3) admits no global weak nonnegative solutions other than the trivial one.

Proof. The proof proceeds by contradiction. Suppose that u is a nontrivial nonnegative solution which exists globally in time. That is u exists in $(0,T^*)$ for any arbitrary $T^* > 0$. Let T, R and θ be positive real numbers such that $0 < TR^{2/\theta} < T^*$.

Let $\Phi(z)$ be a smooth nonincreasing function such that

$$\Phi(z) = \begin{cases} 1 \text{ if } z \le 1 \\ 0 \text{ if } z \ge 2 \end{cases}$$

and $0 \le \Phi(z) \le 1$.

The test function $\varphi(x,t)$ is chosen so that

$$\begin{aligned}\int_{\Omega_T} \left|\mathrm{D}_{TR^{2/\theta}\_}^{1-\alpha}\varphi_{xx}(x,t)\right|^{p'} \varphi^{-p'/p}(x,t)\,dxdt < \\ < \infty, \int_{\Omega_T} \left|\varphi_t(x,t)\right|^{p'} \varphi^{-p'/p}(x,t)\,dxdt < \infty.\end{aligned} \tag{8}$$

To estimate the right-hand side of the Definition 4 on $\Omega_{TR^{2/\theta}}$, we write

$$\begin{aligned}\int_{\Omega_{TR^{2/\theta}}} u(x,t)\left[\mathrm{D}_{TR^{2/\theta}\_}^{1-\alpha}\varphi_{xx}(x,t)\right]dxdt = \\ = \int_{\Omega_{TR^{2/\theta}}} u(x,t)\varphi^{1/p}(x,t)\left[\mathrm{D}_{TR^{2/\theta}\_}^{1-\alpha}\varphi_{xx}(x,t)\right]\varphi^{-1/p}(x,t)\,dxdt\end{aligned}$$

Therefore, by using the $\varepsilon -$Young equality we have

$$\begin{aligned}\int_{\Omega_{TR^{2/\theta}}} u(x,t)\left[\mathrm{D}_{TR^{2/\theta}\_}^{1-\alpha}\varphi_{xx}(x,t)\right]\varphi(x,t)\,dxdt \le \\ \le \varepsilon \int_{\Omega_{TR^{2/\theta}}} \left|u(x,t)\right|^p \varphi(x,t)\,dxdt + C(\varepsilon)\int_{\Omega_{TR^{2/\theta}}} \left|\mathrm{D}_{TR^{2/\theta}\_}^{1-\alpha}\varphi_{xx}(x,t)\right|^{p'} \varphi^{-p'/p}(x,t)\,dxdt.\end{aligned}$$

Similarly,

$$\int\limits_{\Omega_{TR^{2/\theta}}} u(x,t)\varphi_t(x,t)\,dxdt \le \varepsilon \int\limits_{\Omega_{TR^{2/\theta}}} |u(x,t)|^p \varphi(x,t)\,dxdt + C(\varepsilon)\int\limits_{\Omega_{TR^{2/\theta}}} |\varphi_t(x,t)|^{p'} \varphi^{-p'/p}(x,t)\,dxdt.$$

Now, taking ε small enough, we obtain the estimate

$$\int\limits_{\Omega_{TR^{2/\theta}}} |u(x,t)|^p \varphi(x,t)\,dxdt \le C(\varepsilon)\int\limits_{\Omega_{TR^{2/\theta}}} \left\{ \left|\mathrm{D}^{1-\alpha}_{TR^{2/\theta}-}\varphi_{xx}(x,t)\right|^{p'} + |\varphi_t(x,t)|^{p'} \right\} \varphi^{-p'/p}(x,t)\,dxdt. \qquad (9)$$

We set

$$\varphi(x,t) = \Phi\left(\frac{x^2+t^\theta}{R^2}\right),$$

where $R,\theta \in Z^+$.

Let us perform the change of variables $t = \tau R^{2/\theta}$, $x = yR$ and set

$$\Omega := \left\{ (y,\tau) \in R \times \left(0, T/R^{2/\theta}\right),\ y^2 + \tau^\theta < 2 \right\},$$

$$\mu(y,\tau) = y^2 + \tau^\theta.$$

Now, we choose θ such that the right-hand sides of

$$\int\limits_{\Omega_{TR^{2/\theta}}} \left|\mathrm{D}^{1-\alpha}_{TR^{2/\theta}-}\varphi_{xx}(x,t)\right|^{p'} \varphi^{-p'/p}(x,t)\,dxdt = \int\limits_{\Omega_{TR^{2/\theta}}} \left| -\frac{1}{\Gamma(\alpha)} \int\limits_{\tau R^{2/\theta}}^{TR^{2/\theta}} (s-t)^{\alpha-1} \varphi_{sxx}(x,s)\,ds \right|^{p'} \varphi^{-p'/p}(x,t)\,dxdt =$$

$$= \int\limits_{\Omega_{TR^{2/\theta}}} \left| -\frac{1}{\Gamma(\alpha)} \int\limits_{\tau}^{T} \left(R^{2/\theta}\xi - R^{2/\theta}\tau\right)^{\alpha-1} \frac{1}{R^2 R^{2/\theta}} \left(\Phi_{\xi yy}\circ\mu\right) R^{2/\theta} d\xi \right|^{p'} (\Phi\circ\mu)^{-p'/p} R\,dy R^{2/\theta} d\tau \le$$

$$\le R^{2/\theta(\alpha-1)p'-2p'+2/\theta+1} \int\limits_{\Omega_{TR^{2/\theta}}} \left| -\frac{1}{\Gamma(\alpha)} \int\limits_{\tau}^{T} (\xi-\tau)^{\alpha-1} \left(\Phi_{syy}\circ\mu\right) d\xi \right|^{p'} (\Phi\circ\mu)^{-p'/p}\,dyd\tau \le$$

$$\le R^{2/\theta(\alpha-1)p'-2p'+2/\theta+1} \int\limits_{\Omega_{TR^{2/\theta}}} \left| D^{1-\alpha}_{T-}\left(\Phi_{\tau yy}\circ\mu\right) \right|^{p'} (\Phi\circ\mu)^{-p'/p}\,dyd\tau$$

and

$$\int\limits_{\Omega_{TR^{2/\theta}}} |\varphi_\tau(x,t)|^{p'} \varphi^{-p'/p}(x,t)\,dxdt \le R^{-2/\theta p'+2/\theta+1} \int\limits_{\Omega_{TR^{2/\theta}}} \left|\left(\Phi_\tau\circ\mu\right)\right|^{p'} (\Phi\circ\mu)^{-p'/p}\,dyd\tau$$

are of the same order in R. In doing so we find $\theta = \alpha$.

Then have the estimate

$$\int\limits_{\Omega_{TR^{2/\alpha}}} |u(x,t)|^{p'} \varphi(x,t)\,dxdt \le CR^\lambda, \qquad (10)$$

where

$$\lambda = \frac{2}{\theta}(\alpha-1)p' - 2p' + \frac{2}{\theta} + 1$$

and

$$C = C(\varepsilon)\int_{\Omega_{TR^{2/\theta}}}\left(\left|D_{T-}^{1-\alpha}\left(\Phi_{\tau yy}\circ\mu\right)\right|^{p'} + \left|\left(\Phi_{\tau}\circ\mu\right)\right|^{-p'}\right)\left|\left(\Phi\circ\mu\right)\right|^{-p'/p} dyd\tau.$$

If we choose $\lambda < 0$, (i.e. is $p < p_c$) and let $R \to \infty$ in (10), we obtain

$$\int_{\Omega} u^p(x,t)\,dxdt \le 0. \qquad (11)$$

This implies that $u = 0$ a.e., which is a contradiction.

In case $\lambda = 0$, (i.e. is $p = p_c$) observe that the convergence of the integral in (10) if

$$\Omega_R = \left\{(x,\ t) \in R \times (0,T) : R^2 < x^2 + t^\alpha \le 2R^2\right\}$$

then

$$\lim_{R\to\infty}\int_{\Omega_R}\left|u(x,t)\right|^p \varphi(x,t)\,dxdt = 0 \qquad (12)$$

If instead of using the $\varepsilon-$Young equality, we rather use the Hölder inequality, then instead of estimate (9), we get

$$\int_{\Omega_{TR^{2/\alpha}}}\left|u(x,t)\right|^p\varphi(x,t)\,dxdt \le L\left(\int_{\Omega_R}\left|u(x,t)\right|^p\varphi(y,\tau)\,dyd\tau\right)^{1/p} \qquad (13)$$

where

$$L := \left(\int_{\Omega_1}\left|D_{T-}^{1-\alpha}\left(\Phi_{\tau yy}\circ\mu\right)\right|^{p'}\left(\Phi\circ\mu\right)^{-p'/p} dyd\tau\right)^{p'} + \left(\int_{\Omega_1}\left|\left(\Phi_{\tau}\circ\mu\right)\right|^{p'}\left(\Phi\circ\mu\right)^{-p'/p} dyd\tau\right)^{1/p'}$$

and

$$\Omega_1 = \left\{(y,\ \tau) \in R \times \left(0, T/R^{2/\alpha}\right) : 1 < y^2 + \tau^\alpha \le 2\right\}.$$

Using (13), we obtain via (12), after passing to the limit as $R \to \infty$,

$$\int_{\Omega}\left|u(x,t)\right|^p dxdt = 0$$

This leads to $u = 0$ a.e. and completes the proof.

Conclusion

In this paper were studied Fujita-type critical exponents for certain time-fractional diffusion equations with the nonnegative initial condition. As a result, using the test function method, the critical exponents of Fujita were determined in the following form $1 < p \le p_c = 1 + \frac{2}{\alpha}$.

Consequently, by using the Fujita-type critical exponents we proved that, the problem (3) admits no global weak nonnegative solutions other than the trivial one.

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