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Informational and entropic criteria of self-similarity of fractals and chaotic signals

Abstract. Information entropy and fractal dimension of a set of physical values are usually used us quantitative characteristic of chaos. Normalization of entropy is a well-known problem. This work is devoted to develop a method to do this. In the work proposed criteria for self-similarity of information and informational entropy. We have defined normalized values of information (I_1 = 0.567) and informational entropy (I_2 = 0.806) as fixed points of probability density function of information and informational entropy. Meaning of these values is described as criteria of self-similarity of fractals and chaotic signals with different dimensions. We have shown that self-similarity occurs if normalized informational entropy S belongs to the ranges $[0,I_1)$, $[I_1,I_2)$, $[I_2,1)$, that corresponds to topological dimensions from 1 to 3 of quasi-periodic, chaotic, stochastic objects. Validity of these findings has been confirmed by calculation of entropy for hierarchical sets of well-known fractals and nonlinear maps. These criteria can be applied to a wide range of problems, where entropy is used.

Key words: information; informational entropy; fractal; chaos; self-similarity

Introduction

Rapid development of contemporary technologies leads to necessity of study of physical processes in nanocluster materials, laws typical for microwave chaotic signals, neural networks, etc. Scale invariance is the common property of such complex processes and objects. Taking into account this property we can neglect using of physical values with given dimension (for instance, dimension corresponding to length of an object). Local properties of scale invariance are self-similarity (similarity factor is equal on different variables) and self-affinity (similarity factors are different on different variables). Self-organization of matter and motion is also a common property of different processes and can be represented as transition from chaos to order in an open nonlinear and non-equilibrium systems. So, we shall consider invariant properties of such chaotic processes.

Generally, fractal dimension and informational entropy measured for physical processes are quantitative characteristics of chaos [1, 2]. Theoretical conclusions describing behavior of entropy in chaotic systems are known. According to the Prigozhin Theorem [3], the first derivative of informational entropy by time decreases to its minimum at self-organization in a system. In case energy of system is constant, total entropy of the system decreases according to the Klimontovich Theorem [4]. Results of study of entropy can be also applied for the description of processes of controlled self-organization [5].

Recent significant researches are devoted to problems of the theory of informational entropy and its applications. For example, baryon density perturbations are studied from the point of view of information theory in [6] by use of a logarithmic measure of information. In [7], information entropy is used to describe seismic vibrations. Cell entropy is normalized to entropy corresponding to radial oscillations. The paper [8] is devoted to the importance of choosing information measures for analyzing complex structures. It is noted that informational measures are simply inadequate for determining meaningful relationships among

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variables within joint probability distributions. Results of this works reflect a fundamental importance of choice a universal normalization of information entropy.

Method for accurate calculation of universally normalized entropy of a non-equilibrium system is not realized yet. The mentioned theorems do not provide answers the questions: what is the minimum value of entropy production, how entropy decreases at self-organization? Relation between entropy criterion of self-similarity and fractal dimension characterizing corresponding chaotic processes is not quite clear also. In [9], solution of such problems by normalizing the Shannon informational entropy to the Renyi entropy is suggested. However, order of the multifractal moment is determined from experiment. The purpose of this work is to search for answers to these questions without involving empirical constants.

Informational and entropic criteria of self-similarity

Concept of information is often used in such branches of science as cybernetics, genetics, sociology, etc. As usual, considered systems are open systems. Development of methods used for the description of such systems stimulates the necessity for generalization of concept of information. As usual, open systems are considered as systems exchanging with external environment by energy, matter and information.

Actually, a complex object is characterized by its main properties. Information $I(x)$ for statistical realization of a physical non-equilibrium value x is greater than zero. Let us designate probability of realization of x as $P(x)$. So, quantity of information can be expressed as

$$I(x) = -\ln P(x). \quad (1)$$

Information is a value which can be used in different areas, but Eq. (1) corresponds with all of them.

By definition, mutual information transmitted through a communication channel with characteristic $x = x(t)$ is determined by difference between Shannon one-dimensional entropy and conditional entropy [10] as

$$I(x;y) = S(x) - S(x \mid y), \quad (2)$$

where $y(t)$ is a characteristic of receiver. Unconditional Shannon entropy is defined as

$$S(x) = -\sum_{i=1}^{N} P(x_i) \ln P(x_i), \quad (3)$$

where $P(x_i)$ is expectancy of hitting of variable x in the i^{th} cell with relative size δ. Conditional entropy $S(x|y)$ can be written as

$$S(x/y) = -\sum_{i=1}^{N}\sum_{j=1}^{M} P(x_i, y_j) \ln P(x_i \mid y_j), \quad (4)$$

where $P(x_i|y_j)$ is conditional probability. Mutual information is nonzero only in presence of correlations between quantities $x(t)$, $y(t)$. For the description of dynamic systems, we can accept $y(t) = x'(t)$, i.e. the derivative of $x(t)$ is considered as a second variable.

Instead of one-dimensional Shannon entropy $S(x)$ we can use two-dimensional summarized entropy $S(x,y)$ and rewrite Eq. (2) as

$$I(x \mid y) = S(x,y) - S(x \mid y) > 0,$$

$$S(x,y) = -\sum_{i=1}^{N}\sum_{j=1}^{M} P(x_i, y_j) \ln P(x_i, y_j), \quad (5)$$

where $P(x_i, y_i)$ is probability of hitting into cells with equal sides δ of phase space (x, y). We use the designation $I(x|y)$ to emphasize the role of conditions in definition of information, but not of correlations as in Eq. (2).

Using of $S(x,y)$ instead of $S(x)$ provides better condition for positivity of conditional information $I(x|y) > 0$, because $S(x,y) > S(x)$ always. In this meaning, Eq. (5) corresponds to greater noise immunity than Eq. (2). Normalizing of two-dimensional information and entropy to the value of summarized entropy, from Eq. (2) we have

$$\tilde{I}(x \mid y) + \tilde{S}(x \mid y) = 1,$$

$$\tilde{I}(x \mid y) = I(x \mid y) / S(x,y),$$

$$\tilde{S}(x \mid y) = S(x \mid y) / S(x,y). \quad (6)$$

This is a convenient ratio for analysis of probabilistic processes written as a law of conservation of conditional information and entropy. It is from Eq. (4), (5) follows that informational entropy is the average value of information. Therefore, we shall use Eq. (2) as the basic definition of information.

According to Eq. (3) entropy calculated via probability density $f(x) = dP(x)/dx$ tends to infinity if x is a continuous value. For definition of scale-invariant regularities we must use a new approach at the description of informational phenomena. Because of this fact we can consider information as a defining independent variable. Information can be used for the description of statistical processes. So, we try to describe new properties of information with taking into account that information is a scale-invariance value.

Therefore, according to Eq. (1) probability of realization of information $P(I)$ can be described as

$$P(I) = e^{-I}. \quad (7)$$

Mathematical expression for probability $P(I)$ and probability density $f(I)$ can be written as

$$0 \le P(I) \le 1, \quad 0 \le I \le \infty;$$

$$\int_0^\infty f(I)dI = 1, \quad P(I) = \int_I^\infty f(I)dI. \quad (8)$$

From Eq. (7), (8) follows that $f(I) = P(I) = e^{-I}$. It means the equality of probability function $P(I)$ and probability density function of information $f(I)$. Information calculated via Eq. (1) is a scale-invariant value. So a law of distribution is the same for both whole object and its part. Informational entropy $S(I)$ of distribution of information is a mean value of information in an ensemble:

$$S(I) = \int_I^\infty If(I)dI = (1+I)e^{-I}. \quad (9)$$

Let us take into account that information can be normalized to unit. So, $1 \ge S \ge 0$ at $0 \le I \le \infty$. We obtained a finite value of entropy of a continuous set by introducing a measure. We accepted a probability density of information as a measure and as a result we get Eq. (9). This result is valid for information of any nature (social, cyber, genetic, etc.) and for different methods (mutual, conditional) for its determination.

We use information $I(f(I))$ and entropy $S(I)$ as characteristic functions. Fixed points of I ($f(I)$ and $S(I)$) can be described by the following mathematical expressions [11]:

$$I = f(I), \quad I = e^{-I}, \quad I = I_1 = 0.567, \quad (10)$$

$$S(I) = I, \quad (1+I)e^{-I} = I, \quad I = I_2 = 0.806. \quad (11)$$

I_2 is the minimum value of normalized multidimensional entropy achieved at transition to self-similarity. Normalized entropy of chaotic objects in a three-dimensional space (x,y,z) can be defined as

$$\tilde{S}(x,y,z) = S(x,y,z)/(S(x)+S(y)+S(z)), \quad (12)$$

because its maximal value is equal to sum of entropies of components. Therefore,

$$I_2 \le \tilde{S}(x,y,z) < 1. \quad (13)$$

I_1 is conveniently defined for conditional information of a geometric object as

$$I(y/x) = S(y,x) - S(y/x) = \\ = S(x) + S(y/x) - S(y/x) = S(x),$$

$$I(x/y) = S(y). \quad (14)$$

Self-similar value of information (I_1) is the minimal value of normalized one-dimensional entropy of two-dimensional object:

$$I_1 \le \tilde{S}(x) < I_2, \quad \tilde{S}(x) = S(x)/S(x,y). \quad (15)$$

Transition to chaos and to statistical regularities in one-dimensional case is characterized by the range

$$0 < \tilde{S}(\delta) < I_1, \quad \tilde{S}(\delta) = S(\delta)/\ln(1/\delta), \quad (16)$$

where $S(\delta)$ is normalized entropy of Shannon segmentation by the relative scale of measurement δ. It is known from the theory of multifractals [12] that $S(\delta)$ is also an information dimension of a set containing a measure. Therefore, we can use I_1 and I_2 as boundaries for separation of fractional parts of dimensions characterizing self-similar sets with topological dimensions $d = 1$ ($[0,I_1)$), $d = 2$ ($[I_1,I_2)$), $d= 3$ ($[I_2,1)$).

Results and Discussion

It is necessary to verify validity of criteria of self-similarity for information I_1 and entropy I_2 in natural phenomena. Fractal objects and processes are characterized by a universal property of scale invariance. However, the main characteristic of such objects which is fractal dimension cannot be unambiguously defined from experimental data, because results of application of different methods (cellular, inner, etc. dimensions) are noticeably different. Therefore, in the beginning, we shall use models of 17 geometric fractals with known dimensions from 1.26 to 2.0 [12,13].

All studied prefractals (hierarchical generations) with number of iterations equal to n contained the same number of points $N = 2^{18}$ and inscribed in squares with the same dimensions. Information entropy of a prefractal was determined via probabilities of appearing of points in square cells with relative scale $\delta = 10^{-3}$.

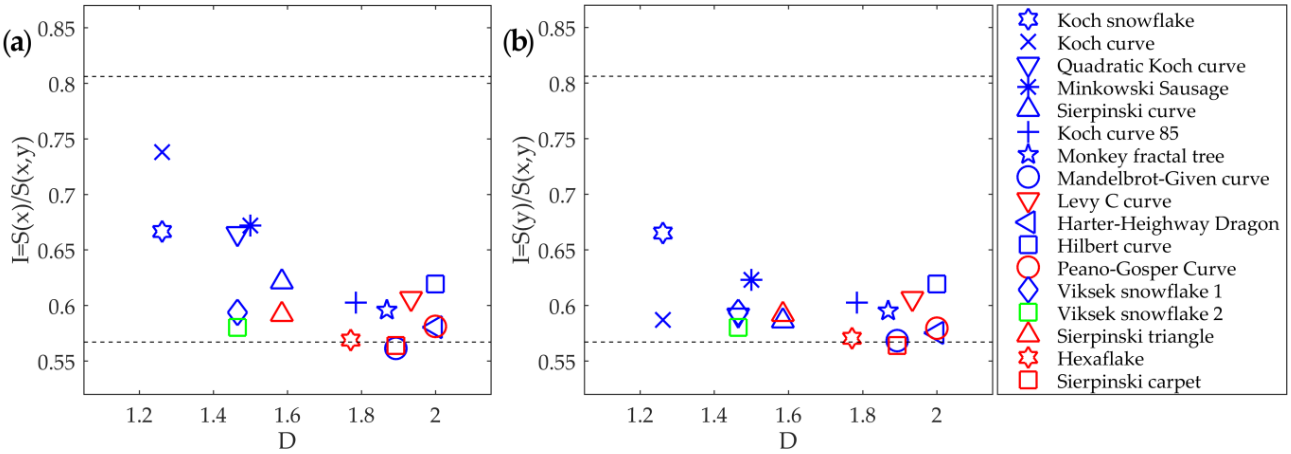

Figure 1 – Normalized information entropy (conditional information) $\widetilde{S}(x)$ (**a**) and $\widetilde{S}(y)$ (**b**) of fractals with dimensions D.

We have chosen $n = 6$, $\delta = 10^{-3}$ for all fractals for which the self-similar normalized values of entropy (12), (14), (15) can be considered independent on n, δ with an error less than one percent.

Figure 1 shows values of normalized information entropy of 17 fractals with indication of their abbreviated names corresponding to accepted in [12,13]. The normalized one-dimensional entropy (conditional information) for all fractals is less than I_2 and tends to the self-similar value I_1 with increasing dimension D_0. Choice of $S(x)$ or $S(y)$ corresponds to non-fractal ($n = 0$) length of x or y. Conditional information depends on two variables, so, the range $[I_1,I_2)$ corresponds to self-similarity of information.

To verify the existence of all self-similar entropy ranges (13), (15), (16), we use nonlinear maps with chaotic realizations. The one-dimensional logistic map and the two-dimensional Henon map are described by the Equations [14] as

$$y_{i+1} = ry_i\left(1-y_i\right), \quad (17)$$

$$\begin{cases} x_{i+1} = 1-ax_i^2+y_i, \\ y_{i+1} = bx_i, \end{cases} \quad (18)$$

where r, a, b are parameters. Let us use a new three-dimensional mapping with parameters R, R_*, γ written as

$$\begin{cases} x_{i+1} = R\left(\left|1 - \dfrac{y_i}{R_*}\right|\right)^{-\gamma}, \\ y_{i+1} = R\left(\left|1 - \dfrac{z_i}{R_*}\right|\right)^{-\gamma}, \\ z_{i+1} = R\left(\left|1 - \dfrac{x_i}{R_*}\right|\right)^{-\gamma}. \end{cases} \quad (19)$$

Map (19) follows from requirement for nonlinearity of fractal measure with components x, y, z and $\gamma = D - d$, which is the difference between fractal and topological dimensions. Types of chaotic realizations of maps (17), (18) are known [14], therefore, Figure 2 shows only chaos of values of x, y, z according to system of Eq. (19).

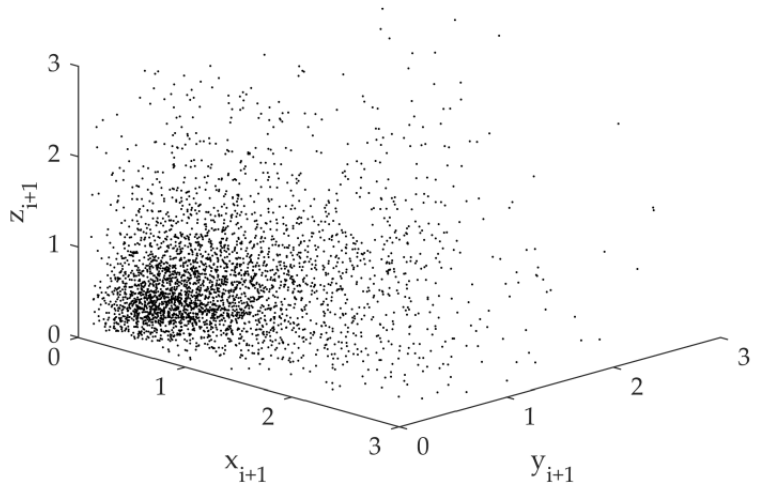

Figure 2 – Chaos of the three dimensional map (19) with parameters $R = 0.77$, $R_* = 1.1$, $\gamma = 0.2$ - 0.57, step equal to 10^{-2} and initial conditions are $x_1 = 1$, $y_1 = 1.1$, $z_1 = 0.9$. Number of samples is $N = 2^{18}$, number of iterations before attractor is $i_{max} = 10^3$.

Figure 3 presents normalized values of one-dimensional, two-dimensional, three-dimensional information entropies corresponding to Eq. (17), (18), (19). We have assumed the sum of variances of variables as a general order parameter leading to bifurcations:

$$\sigma^2 = \sum_{j=1}^{3} \sigma_j^2 ,$$

$$\sigma_j^2 = \left\langle x_j^2 \right\rangle - \left\langle x_j \right\rangle^2 ,$$

$$x_j = (x, y, z). \quad (20)$$

Depending on the map parameters, various signals can be referred as quasi-regular ($[0,I_1)$), chaotic ($[I_1,I_2)$), stochastic ($[I_2,1)$). Transitions between these modes are possible. This can be seen from the examples of bifurcation diagrams shown in Figure 4. Crowding of lines at small values of dispersion (transition to chaos) is more noticeable in three-dimensional case than in low-dimensional cases. Intermittency (alternation of order and chaos) is frequently observed in one-dimensional case than in two-dimensional and three-dimensional dynamical systems. Thus, Figures 1 and 3 clearly confirm validity of information criteria for self-similarity I_1, I_2.

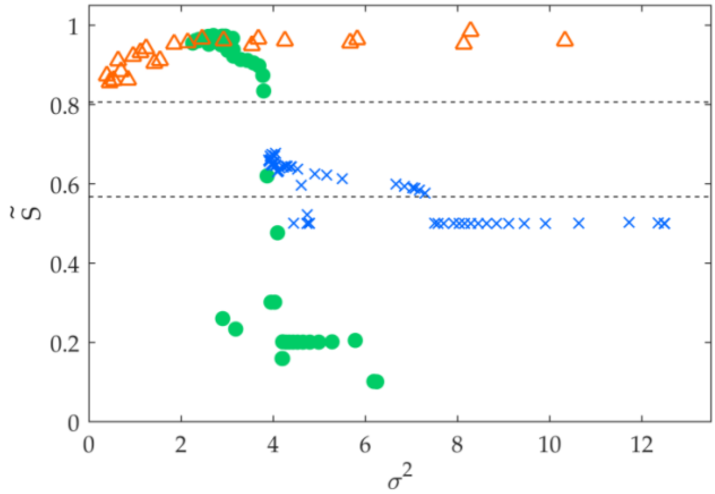

Figure 3 – Normalized information entropy of the maps (17) ● – $S(x)/\ln(1/\delta)$;
(18) × – $S(x,y)/S((x)+S(y))$; (19) Δ – $S(x,y,z)/(S(x)+S(y)+S(z))$.
Number of samples is $N = 2^{18}$, scale is $\delta = 10^{-3}$.

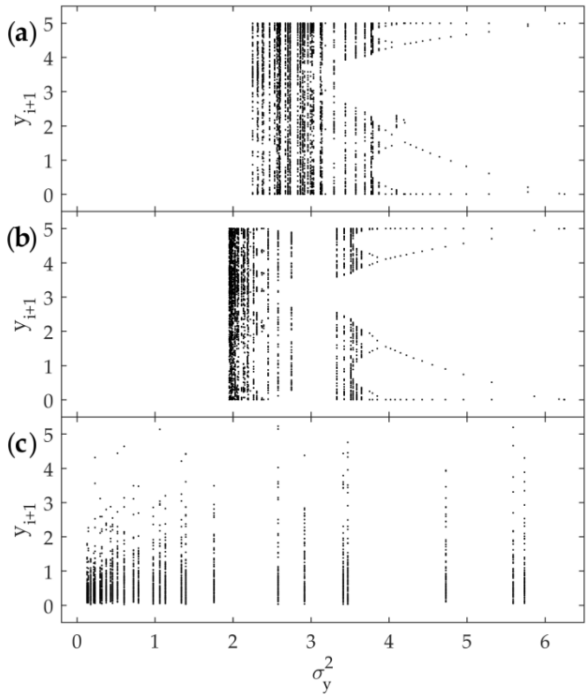

Figure 4 – Bifurcations diagrams of the maps.
(**a**) – (17), $r = 3.2$ - 4.0; (**b**) – (18), $b = 0.3$, $a = 0.5$ - 1.42; (**c**) – (19), $R = 0.77$, $R_* = 1.1$, $\gamma = 0.2$ - 0.57.
At all cases parameter step is 10^{-2}, initial value is $x_{y0} = 1$.
Total number of samples is $N = 2^{18}$, number of iterations before attractor is $i_{max} = 10^3$.

Conclusions

Informational and entropic criteria for self-similarity of fractals and chaotic signals can be applied to the quantitative analysis of phenomena with different nature. Information and entropy as measures of order and disorder have universal applicability both for natural and social phenomena. Attractors of dynamical systems, images of natural and nanotechnological objects can have a fractal structure. As usual, astrophysical, seismic, non-linear radio engineering, neural, nanoelectronic and other signals are chaotic signals. The entropic criteria established in the present work are associated with quasi-regular, chaotic, stochastic processes in considerably narrower ranges than in case of applying other known characteristics of chaos, for example, in comparison with the unit interval of difference between fractal and topological dimensions.

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