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On numerical simulations of the 1d wave equation with a distributional coefficient and source term

Abstract. In this note, we illustrate numerical experiments for the one-dimensional wave equation with δ-like (delta like) terms. Our research is connecting the theory with the numerical realisations. By using results on very weak solutions introduced by Michael Ruzhansky with his co-authors, we investigate a corresponding regularized problem. In contrast to our expectations, the experiments show that the solution of the regularized problem has a "good" behaviour. Indeed, numerical experiments show that approximation methods work well in situations where a rigorous mathematical formulation of the problem is difficult in the framework of the classical theory of distributions. The concept of very weak solutions eliminates this difficulty, giving results of correctness for equations with singular coefficients. In the framework of this approach (very weak solutions), the expected physical properties of the equation can be reconstructed, for example, the distribution profile and the decay of the solutions for large times. Finally, we give a number of illustrations.

Key words: wave equation, numerical experiment, very weak solutions, distributional coefficient, singular source term, regularized problem, decay of solutions.

Introduction

In this paper, we follow the results of the paper [4] and study the Cauchy-Dirichlet problem for the 1D-Wave Equation

$$
\begin{cases}
\partial_{tt}^2 u(t,x) - a(t)\partial_{xx}^2 u(t,x) = f(t,x), \ (t,x) \in [0,T]\times[0,1], \\
u(t,0) = 0, t \in [0,T], \\
u(t,1) = 0, t \in [0,T], \\
u(0,x) = u_0(x), x \in [0,1], \\
\partial_t u(0,x) = u_1(x), x \in [0,1].
\end{cases} \quad (1)
$$

The notion of very weak solutions has been introduced in [GR15] to analyse second order hyperbolic equations. In [3] and [5] Ruzhansky and Tokmagambetov applied it to show the well-posedness of the Landau Hamiltonian wave equations in distributional electro-magnetic fields. Also, in [2] were investigated very weak solutions for an acoustic problem of wave propagation through a discontinuous medium.

In this paper, we allow the coefficient $a(t)$ and the source term $f(t,x)$ to be distributional in t. One of the interesting cases is when $a(t) = 1 + \delta(t - t_0)$ and $f(t,x) = \delta(t - t_1)$ for some, in general, different t_0 and t_1. For more motivation, we refer to [6] – [11].

Numerical experiments

We start by regularizing $a(t)$ and $f(t,x)$ by the parameter ε, that is,

$$a_\varepsilon(t) = (a * \varphi_\varepsilon)(t),\ f(t) = (f * \varphi_\varepsilon)(t),$$

by the convolution with the mollifier $\varphi_\varepsilon(t) = {}^1/_\varepsilon\, \varphi({}^t/_\varepsilon)$, where

$$\varphi(t) = \begin{cases} \frac{1}{C} e^{\frac{1}{(t^2-1)}}, |t| \le 1, \\ 0, \quad |t| > 1. \end{cases}$$

Here C = 0.443994 so that $\int_{-1}^{1} \varphi(t)dt = 1$.

Instead of (1) consider a regularized problem

$$(2) \qquad \begin{cases} \partial_{tt}^2 u_\varepsilon(t,x) - a_\varepsilon(t)\partial_{xx}^2 u_\varepsilon(t,x) = f_\varepsilon(t,x),\ (t,x) \in [0,T] \times [0,1], \\ u_\varepsilon(t,0) = 0, t \in [0,T], \\ u_\varepsilon(t,1) = 0, t \in [0,T], \\ u_\varepsilon(0,x) = u_0(x), x \in [0,1], \\ \partial_t u_\varepsilon(0,x) = u_1(x), x \in [0,1]. \end{cases}$$

From [4] it follows that the problem (1) has a unique very weak solution. It is given by a family of functions $\{u_\varepsilon(t,x)\}_{0<\varepsilon\le1}$. For each positive $\varepsilon \le 1$, the function $u_\varepsilon(t,x)$ is a solution of the regularized problem (2) controlled by the estimate

$$||\partial_t^\alpha \partial_x^\beta u_\varepsilon(t,x)||_{L^2} \le C\varepsilon^{-L-\alpha-\beta},$$

for some C>0 and $L \ge 0$, for all $\alpha, \beta \in \mathbb{Z}_+$.

We put $u_0(x) \equiv 0, u_1(x) \equiv 0,\ a(t) = 1 + \delta(t - t_0)$ and $f(t,x) = \delta(t - t_1)$. Then we get $a_\varepsilon(t) = 1 + \varphi_\varepsilon(t - t_0)$, $f_\varepsilon(t,x) = \varphi_\varepsilon(t - t_1)$. Finally, we have the following problem to solve numerically

$$(3) \qquad \begin{cases} \partial_{tt}^2 u_\varepsilon(t,x) - \big(1 + \varphi_\varepsilon(t - t_0)\big)\partial_{xx}^2 u_\varepsilon(t,x) = \varphi_\varepsilon(t - t_1),\ (t,x) \in [0,T] \times [0,1], \\ u_\varepsilon(t,0) = 0, t \in [0,T], \\ u_\varepsilon(t,1) = 0, t \in [0,T], \\ u_\varepsilon(0,x) = 0, x \in [0,1], \\ \partial_t u_\varepsilon(0,x) = 0, x \in [0,1]. \end{cases}$$

In the following, we demonstrate numerical simulations. All calculations are made in C++ by using the sweep method. For all simulations $\Delta t = \Delta x = 0.01$. In all computer simulations, we use Matlab R2017b. At first, we consider the case when $t_0 = t_1 = 0.2$. In Figure 1 and Figure 2, we see the decay of the solution $u_\varepsilon(t,x)$ with respect to the time t of the regularised problem (3), for $\varepsilon = 0.8$.

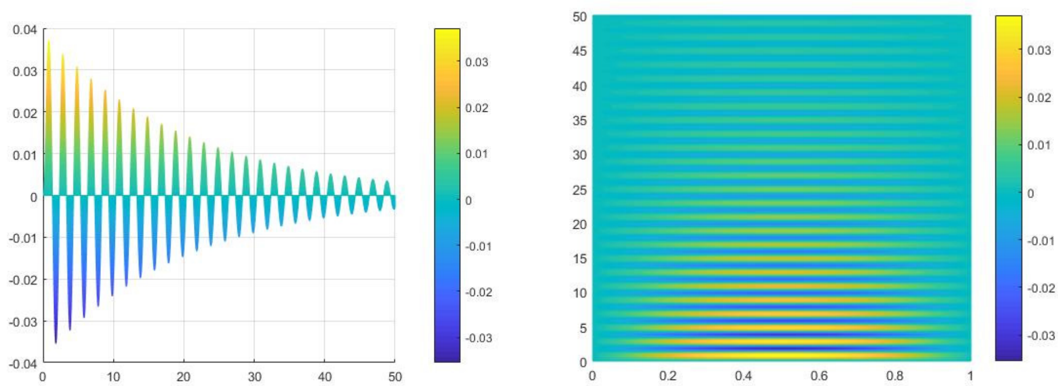

Figure 1 – In these plots, we can see the decay of the solution $u_\varepsilon(t,x)$ with respect to the time t of the regularised problem (3), for $\varepsilon = 0.8$ when $t_0 = t_1 = 0.2$. In the first plot, the time t is given by the horizontal axe, and the graphic of $\max_{x\in[0,1]} u_\varepsilon(t,x)$ is drawn. Here, we use colours to indicate the value of the solution $u_\varepsilon(t,x)$.

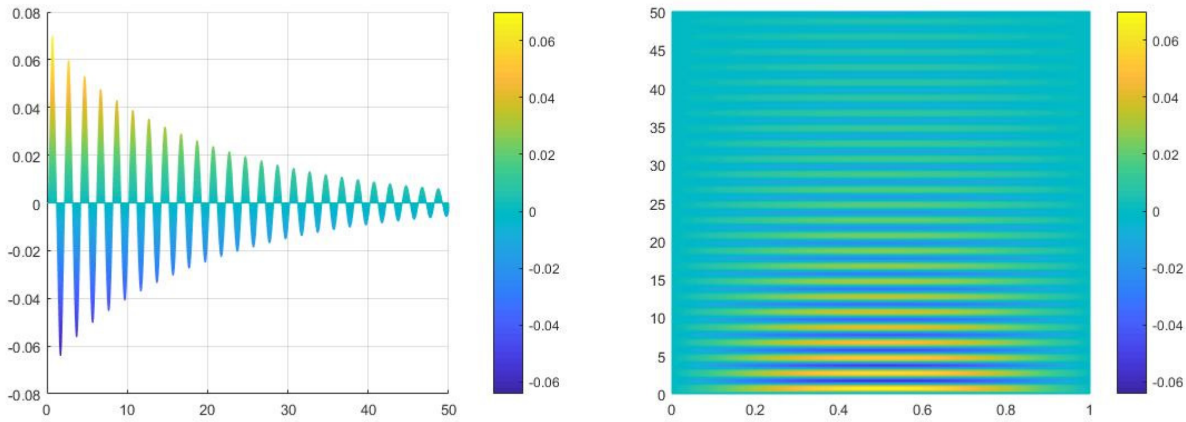

Figure 2 – In these pictures, we see the decay of the solution $u_\varepsilon(t,x)$ with respect to the time t of the regularised problem (3), for $\varepsilon = 0.01$ when $t_0 = t_1 = 0.2$.

Now, compare the solution $u_\varepsilon(t,x)$ at $t = 100$ of the regularized problem (3), for several values of ε. In Figure 3, there is given a comparison of the solution $u_\varepsilon(t,x)$ at time $t = 100$ of the regularized problem (3), for the parameter ε at $\varepsilon = 0.8, 0.5, 0.3, 0.1, 0.08, 0.05, 0.03, 0.01$.

Consider the case when t_0 and t_1 are different. Let us start with the case $t_0 < t_1$. Let $t_0 = 0.2$ and $t_1 = 10$ for $\varepsilon = 0.01$. Then for the illustrations we have Figure 4.

Now, we consider the case $t_1 < t_0$. Let $t_0 = 30$ and $t_1 = 0.2$ for $\varepsilon = 0.01$. Then for the illustrations we obtain Figure 5.

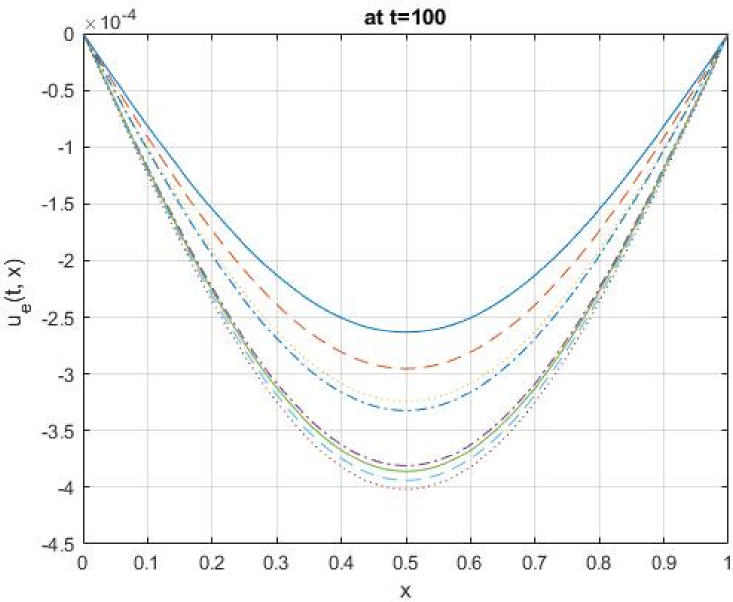

Figure 3 – Comparison of the solution $u_\varepsilon(t,x)$ at time $t = 100$ of the regularized problem (3), for the parameter ε. The graphics correspond to $u_\varepsilon(t,x)$ at $\varepsilon = 0.8, 0.5, 0.3, 0.1, 0.08, 0.05, 0.03, 0.01$ from top to bottom, respectively.

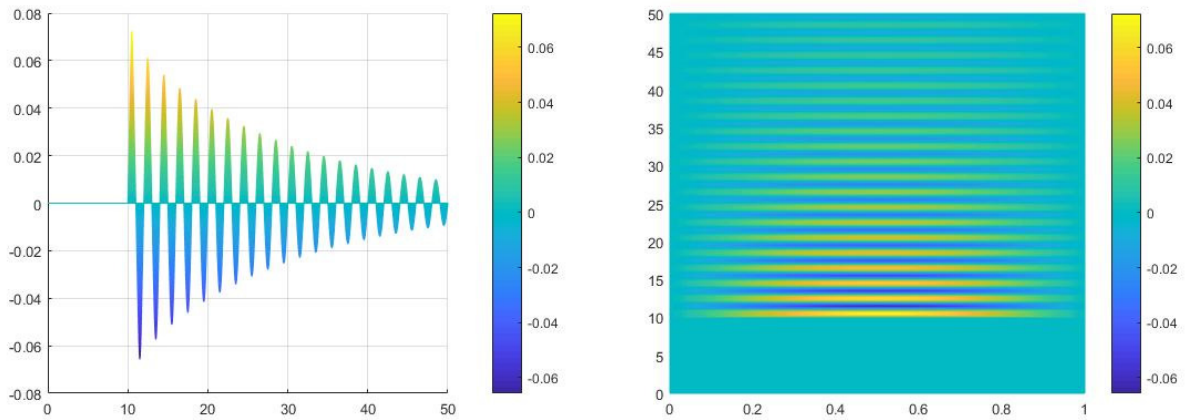

Figure 4 – In these plots, we see the decay of the solution $u_\varepsilon(t,x)$ with respect to the time t of the regularised problem (1.3), for $\varepsilon = 0.01$ when $t_0 = 0.2$ and $t_1 = 10$.

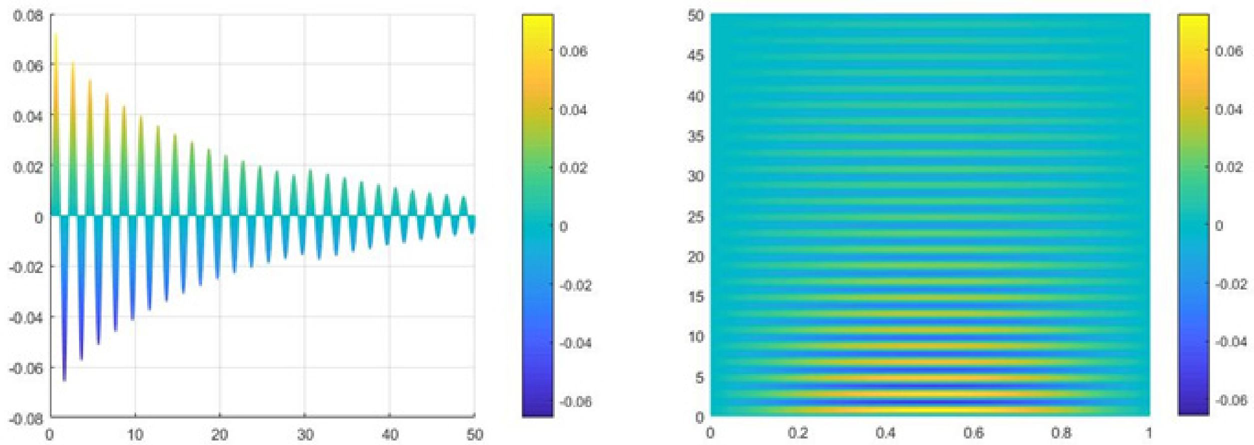

Figure 5 – In the plots, we can see the decay of the solution $u_\varepsilon(t,x)$ with respect to the time t of the regularised problem (3), for $\varepsilon = 0.01$ when $t_0 = 30$ and $t_1 = 0.2$.

Conclusion

Numerical experiments show that approximation methods work well in situations where a rigorous mathematical formulation of the problem is difficult in the framework of the classical theory of distributions. The concept of very weak solutions eliminates this difficulty, giving results of correctness for equations with singular coefficients. In the framework of this approach (very weak solutions), the expected physical properties of the equation can be reconstructed, for example, the distribution profile and the decay of the solutions for large times.

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