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The nonlocal nonlinear Schrödinger and Maxwell – Bloch equation

Abstract. In this paper, the nonlocal nonlinear Schrödinger and Maxwell – Bloch equations are introduced. A particular case of this system, namely the Schrödinger equation, is integrable by the inverse scattering method as shown in the work of M. Ablowitz and Z. Musslimani. Following their idea, we prove the integrability of the nonlocal nonlinear Schrödinger and Maxwell – Bloch equation using its Lax pairs. Also the Darboux transformations are constructed, and soliton solutions are obtained from different "seed" solutions using them. One-fold, two-fold and N-fold determinant representations are obtained by this transformation. Moreover, soliton and solitons-like solutions, such as dynamic and topological soliton, periodic, domain walls, kink, lamp, bright and dark solitons, bright and dark rogue waves, bright and dark positons, etc., of this equation are built. In future papers, we will investigate the conservation laws of the nonlocal nonlinear Schrödinger and Maxwell – Bloch equation using the Lax pair.

Key words: nonlocal nonlinear Schrödinger and Maxwell – Bloch equation, Lax representation, Darboux transformation.

Introduction

It is known that the nature of nonlinear real systems is one of the basic notions in modern science. Nonlinearity is the property which is applied in almost all fields of science. A nonlinear phenomenon is usually modeled in terms of nonlinear ordinary and/or differential equations in particular cases. Most of such nonlinear differential equations (NDE) are completely integrable. That is they accept some classes of interesting exact solutions, such as solitons, dromions, destructive waves, semilaritons, kink, etc. They draw both mathematical and physical interest. Investigations of solitons, positons, brakes, dromions, "destructive waves" have become one of the interesting and highly active areas for research in modern science and technology during the last several decades. In particular, many of the completely integrable NDEs have already been established and investigated [1-8].

Among such integrable nonlinear systems, the Schrödinger and Maxwell – Bloch equations have a

crucial role. The Schrödinger and Maxwell – Bloch equations describe solitons in fibers with resonance and erbium systems and have a (1+1) – dimension [9]. Using the Darboux transformation, the (1+1)-dimensional Schrödinger and Maxwell – Bloch equations were analyzed in [9], where soliton and periodic solutions were constructed from various "seeds".

Recently, a (2+1) – dimensional Schrödinger and Maxwell – Bloch equation was introduced in [10]. In this section, our aim is to build a Darboux transformation for (2+1) – dimensional Schrödinger and Maxwell – Bloch equation and to obtain soliton solutions. It is well known that the Darboux transformation is an effective way to get different solutions of integrable equations [11]. For instance destructive waves and positon solutions were constructed using the Darboux transformation for one and two Hirota-Maxwell-Bloch equations in [11], [12]. The authors found soliton and positon solutions via the Darboux transformation as a representation of the determinant for the inhomogeneous Hirota-Maxwell-Bloch equation [12], [13].

In this work we consider one of the generalizations of the nonlinear Schrödinger equation, namely the (1+1)-dimensional nonlocal nonlinear Schrödinger and Maxwell – Bloch equation. This equation is a generalization of the (1+1) – dimensional nonlocal nonlinear Schrödinger equation, which was investigated in the papers of M.J. Ablowitz and Ziad H. Musslimani, Li-Yuan Ma, Zuo-Nong Zhu, T. A. Gadzhimuradov A. M. Agalarov et all. [13-16].

A Lax pair of the (1+1)-dimensional nonlocal focusing nonlinear Schrödinger – Maxwell – Bloch equation

In this section we study the nonlocal nonlinear Schrödinger and Maxwell – Bloch equation, which reads as

$$iq_t(x,t) + q_{xx}(x,t) + 2q(x,t)q^*(-x,t)q(x,t) - 2p(x,t) = 0, \quad (1)$$

$$iq_t^*(-x,t) + q_{xx}^*(-x,t) - 2q^*(-x,t)q(x,t)q^*(-x,t) + 2p^*(-x,t) = 0, \quad (2)$$

$$p_x(x,t) = 2[q(x,t)\eta(x,t) - i\omega p(-x,t)], \quad (3)$$

$$p_x^*(-x,t) = 2[q^*(-x,t)\eta(x,t) - i\omega p^*(-x,t)], \quad (4)$$

$$\eta_x(x,t) = q(x,t)p^*(-x,t) - q^*(-x,t)p(x,t), \quad (5)$$

where * denotes complex conjugation and q, p are complex functions of the real variables x and t . The equations (1)-(2) are called the nonlocal nonlinear Schrödinger equations and they were introduced by M. Ablowitz and Z. Musslimani in [10]. The nonlocal nonlinear Schrödinger equation is obtained from a new and simple reduction of the well-known AKNS system. It admits a Lax pair and an infinite number of conservation laws. The IST for decaying data is developed and a one breathing soliton solution is constructed. The IST requires different scattering data symmetries than the classical NLS equation. A nonlocal NLS hierarchy and novel nonlocal Painlevé type equations are also obtained.

Now consider equations (1) – (5) so called Schrödinger and Maxwell – Bloch equation. The system of equations (1) - (5) admits a Lax pair representation and possesses an infinite number of conservation laws; hence, it is integrable. Using the inverse scattering method, corresponding to rapidly decaying initial data, one can linearize the equation and obtain solutions to the system of equations (1)-(5) including pure soliton solutions. Some of the essential properties of the nonlocal NLS equation are derived from the classical NLS equation where the nonlocal nonlinear term $q^*(-x,t), p^*(-x,t)$ are replaced by $q^*(x,t), p^*(x,t)$. Infact, we note that the system of equations (1) – (2) and the classical NLS share the symmetry that when $x \rightarrow -x, t \rightarrow -t$ and a complex conjugate is taken, then the equation stays invariant [16].

Corresponding Lax representation for the nonlocal nonlinear Schrödinger – Maxwell – Bloch equation (1) – (5) can be written as follows

$$\psi_x = A\psi, \quad (6)$$

$$\psi_t = B\psi, \quad (7)$$

where $\psi = \begin{pmatrix} \psi_1(x,t,\lambda) \\ \psi_2(x,t,\lambda) \end{pmatrix}$ is the eigenfunction corresponding to λ and A, B are 2×2 matrices given by

$$A = -i\lambda\sigma_3 + A_0,$$

$$B = \lambda^2 B_2 + \lambda B_1 + B_0 + \frac{1}{\lambda - \omega} B_{-1}.$$

Here λ is the complex eigenvalue constant, σ_3 is the Pauli matrix and $A_0, B_2, B_1, B_0, B_{-1}$ are 2×2 matrices as follows

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0 & q(x,t) \\ -q^*(-x,t) & 0 \end{pmatrix},$$

$$B_2 = -2i\sigma_3 = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}, \quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (10)$$

$$B_1 = \begin{pmatrix} 0 & 2q(x,t) \\ -2q^*(-x,t) & 0 \end{pmatrix},$$

$$B_0 = i \begin{pmatrix} q(x,t)q^*(-x,t) & q_x(x,t) \\ -q_x^*(-x,t) & -q(x,t)q^*(-x,t) \end{pmatrix},$$

$$B_{-1} = \begin{pmatrix} \eta(x,t) & -p(x,t) \\ p^*(-x,t) & -\eta(x,t) \end{pmatrix}.$$

So the Lax representation of the integrable nonlocal nonlinear Schrödinger – Maxwell – Bloch equation is given by A and B . It can be seen that from the compatibility condition from $\psi_{xt} = \psi_{tx}$ we can obtain the zero curvature equation of the system of equations (6) and (7).

$$A_t - B_x + AB - BA = 0.$$

In the next section, one-fold Darboux transformation of the nonlocal nonlinear Schrödinger – Maxwell–Bloch equation will be obtained.

The one-fold Darboux transformation for the nonlocal nonlinear Schrödinger and Maxwell – Bloch equation

Using the Darboux transformation for the AKNS system [11], consider the transformation of the linear system of equations (6)-(7)

$$\psi^{[1]} = T\psi = (\lambda I - M)\psi. \quad (8)$$

The function $\psi^{[1]}$ is assumed to satisfy

$$\begin{aligned} \psi_x^{[1]} &= A^{[1]}\psi^{[1]}, \\ \psi_t^{[1]} &= B^{[1]}\psi^{[1]} \end{aligned} \quad (9)$$

where $A^{[1]}$ and $B^{[1]}$ are matrices depending on $q^{[1]}(x,t)$, $q^{*[1]}(-x,t)$ and λ . Here M and I are matrices of the forms

The relation between $q^{[1]}(x,t)$, $q^{*[1]}(-x,t)$ and $A^{[1]} - B^{[1]}$ is the same as the relation between functions $q(x,t)$, $q^*(-x,t)$ and $A - B$. In order to make the system (9) invariant under the transformation (8), T must satisfy the following equations

$$T_x + TA = A^{[1]}T, \quad (11)$$

$$T_t + TB = B^{[1]}T. \quad (12)$$

The relation between $q(x,t)$, $q^*(-x,t)$ and $q^{[1]}(x,t)$, $q^{*[1]}(x,t)$ can be derived from these equations, which is actually the Darboux transformation of the nonlocal nonlinear Schrödinger and Maxwell – Bloch equation (1)-(5).

From the above identities, after simplifications and comparisons of the coefficients from equation (11), it follows that

$$\lambda^0 : M_x = A_0^{[1]}M - MA_0, \quad (13)$$

$$\lambda^1 : A_0^{[1]} = A_0 + i[M, \sigma_3], \quad (14)$$

$$\lambda^2 : iI\sigma_3 = iI\sigma_3. \quad (15)$$

Finally, from (13)-(15) we obtain solutions

$$\begin{aligned} q^{[1]} &= q - 2im_{12}, \\ q^{*[1]} &= q^* - 2im_{21}, \end{aligned} \quad (16)$$

Similarly, comparing the coefficients of λ^i of the two sides of the equation (12) gives us

$$\lambda^0 : M_t = B_0^{[1]}M - MB_0, \quad (17)$$

$$\lambda^1 : B_0 - MB_1 = B_0^{[1]} - B_1^{[1]}M, \quad (18)$$

$$\lambda^2 : B_1 - MB_2 = B_1^{[1]} - B_2^{[1]}M. \quad (19)$$

$$\lambda^3 : IB_2 = B_2^{[1]}I, \quad (20)$$

$$\frac{1}{\lambda - \omega} : \omega B_{-1} - MB_{-1} = \omega B_{-1}^{[1]} - B_{-1}^{[1]}M. \quad (21)$$

Then the system of equations (17)-(21) yields

$$\eta^{[1]} = \frac{1}{\Delta} \left\{ (\omega - m_{11})^2 + m_{12}^* (\omega - m_{11}) p - |m_{12}|^2 \eta - m_{12} (\omega - m_{11}^*) p^* \right\}, \quad (22)$$

$$p^{[1]} = -\frac{1}{\Delta} \left\{ \eta (\omega - m_{11}) m_{12} - p^* m_{12}^2 + p (\omega - m_{11})^2 \right\}, \quad (23)$$

$$p^{*[1]} = -\frac{1}{\Delta} \left\{ 2(\omega - m_{22}) m_{12}^* \mu + p m_{12}^{*2} - p^* (\omega - m_{22})^2 \right\}, \quad (24)$$

These solutions (16) and (22)-(24) imply the one-fold Darboux transformation of the nonlocal nonlinear Schrödinger and Maxwell – Bloch equation.

Now assume that

$$M = H\Lambda H^{-1}, \quad (25)$$

where

$$H = \begin{pmatrix} \psi_1(\lambda_1, x, t) & \psi_1(\lambda_2, x, t) \\ \psi_2(\lambda_1, x, t) & \psi_2(\lambda_2, x, t) \end{pmatrix} = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} \\ \psi_{2,1} & \psi_{2,2} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (26)$$

and $\det H \neq 0$, where λ_1 and λ_2 are complex constants. In order to satisfy the constraints of A_0 as mentioned above, we first note that if $\delta = +1$, then

$$\psi^+ = \psi^{-1}, \quad A_0^+ = -A_0,$$

$$\lambda_2 = -\lambda_1^*, \quad H = \begin{pmatrix} \psi_1(\lambda_1, x, t) & -\psi_2^*(\lambda_1, x, t) \\ \psi_2(\lambda_1, x, t) & \psi_1^*(\lambda_1, x, t) \end{pmatrix} = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \end{pmatrix}, \quad (27)$$

$$H^{-1} = \frac{1}{\Delta} \begin{pmatrix} \psi_1^*(\lambda_1, x, t) & \psi_2^*(\lambda_1, x, t) \\ -\psi_2(\lambda_1, x, t) & \psi_1(\lambda_1, x, t) \end{pmatrix} = \begin{pmatrix} \psi_1^* & \psi_2^* \\ -\psi_2 & \psi_1 \end{pmatrix},$$

where $\Delta = |\psi_1|^2 + |\psi_2|^2$.

From (24) use formula (27) we obtain

$$M = \frac{1}{\Delta} \begin{pmatrix} \lambda_1 |\psi_1|^2 - \lambda_1^* |\psi_2|^2 & (\lambda_1 + \lambda_1^*) \psi_1 \psi_2^* \\ (\lambda_1 + \lambda_1^*) \psi_2 \psi_1^* & \lambda_1 |\psi_2|^2 - \lambda_1^* |\psi_1|^2 \end{pmatrix}. \quad (28)$$

In the following section we give the determinant representation of the Darboux transformation for the nonlocal nonlinear Schrödinger and Maxwell – Bloch equation.

The determinant representation of Darboux transformation for the nonlocal nonlinear Schrödinger – Maxwell–Bloch equation

Here the determinant representation is obtained for the one-fold, two-fold and n-fold Darboux transformation of the (1+1)-dimensional nonlocal nonlinear Schrödinger and Maxwell – Bloch equation. The reduction condition on the

eigenfunctions are $\psi_{2,2i} = \psi_{1,2i-1}^*$, $\psi_{2,2i-1} = -\psi_{1,2i}^*$ and for the eigenvalues are $\lambda_{2i} = -\lambda_{2i-1}^*$.

The determinant representation of the one-fold Darboux transformation of the nonlocal nonlinear Schrödinger – Maxwell–Bloch equation implies the following theorem (as [12]- [14]).

Theorem 1. The one-fold Darboux transformation of the nonlocal nonlinear Schrödinger – Maxwell–Bloch equation is

$$T_1(\lambda, \lambda_1, \lambda_2) = \lambda I - M = \lambda I + t_0^{[1]} = \frac{1}{\Delta_1} \begin{pmatrix} (T_1)_{11} & (T_1)_{12} \\ (T_1)_{21} & (T_1)_{22} \end{pmatrix}, \quad (29)$$

where

$$t_0^{[1]} = \frac{1}{\Delta_1} \begin{pmatrix} \begin{vmatrix} \psi_{2,1} & \lambda_1 \psi_{1,1} \\ \psi_{2,2} & \lambda_2 \psi_{1,2} \end{vmatrix} & - \begin{vmatrix} \psi_{1,1} & \lambda_1 \psi_{1,1} \\ \psi_{1,2} & \lambda_2 \psi_{1,2} \end{vmatrix} \\ \begin{vmatrix} \psi_{2,1} & \lambda_1 \psi_{2,1} \\ \psi_{2,2} & \lambda_2 \psi_{2,2} \end{vmatrix} & - \begin{vmatrix} \psi_{1,1} & \lambda_1 \psi_{2,1} \\ \psi_{1,2} & \lambda_2 \psi_{2,2} \end{vmatrix} \end{pmatrix}, \quad \Delta_1 = \begin{vmatrix} \psi_{1,1} & \psi_{1,2} \\ \psi_{2,1} & \psi_{2,2} \end{vmatrix}, \quad (30)$$

$$(T_1)_{11} = \begin{vmatrix} 1 & 0 & \lambda \\ \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{1,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{1,2} \end{vmatrix}, \quad (T_1)_{12} = \begin{vmatrix} 0 & 1 & 0 \\ \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{1,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{1,2} \end{vmatrix}, \quad (31a)$$

$$(T_1)_{21} = \begin{vmatrix} 1 & 0 & 0 \\ \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{2,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{2,2} \end{vmatrix}, \quad (T_1)_{22} = \begin{vmatrix} 0 & 1 & \lambda \\ \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{2,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{2,2} \end{vmatrix}. \quad (31b)$$

T_1 satisfies the following equations

$$T_{1x} + T_1 A = A^{[1]} T_1, \quad (32)$$

$$T_{1t} + T_1 B = B^{[1]} T_1. \quad (33)$$

$$A_0^{[1]} = A_0 + [\sigma_3, t_0^{[1]}], \quad (34)$$

$$B_{-1}^{[1]} = T_1 |_{\lambda=-\mu} B_{-1} T_1^{-1} |_{\lambda=-\mu}. \quad (35)$$

Then the solutions of the system (1)-(5) are of the form

$$q^{[1]} = q - 2i \frac{(T_1)_{12}}{\Delta_1}, \quad (36)$$

$$\eta^{[1]} = \frac{(|\mu + (T_1)_{11}|^2 - |(T_1)_{12}|^2)\eta + p(T_1)_{21}(\mu + (T_1)_{11}) - p^*(T_1)_{12}(\mu + (T_1)_{22})}{W}, \quad (37)$$

$$p^{[1]} = \frac{p[(\mu + (T_1)_{11})^2 - p^*(T_1)_{12}^2] + 2\eta(T_1)_{12}(\mu + (T_1)_{11})}{W}, \quad (38)$$

$$p^{*[1]} = \frac{p[(\mu + (T_1)_{22})^2 + p^*(T_1)_{21}^2] - 2\eta(T_1)_{21}(\mu + (T_1)_{22})}{W}, \quad (39)$$

where $W = (T_1)_{11}(T_1)_{22} - (T_1)_{12}(T_1)_{21}$.

The transformation T_1 has the following property

$$T_1(\lambda, \lambda_1, \lambda_2) \Big|_{\lambda=\lambda_i} \begin{pmatrix} \psi_{1,i} \\ \psi_{2,i} \end{pmatrix} = 0, i=1,2. \quad (40)$$

Now we are ready to prove the theorem.

Proof of the Main theorem. From the expression (9), it follows that

$$M = \frac{1}{\Delta_1} \begin{pmatrix} \lambda_1 \psi_{1,1} \psi_{2,2} - \lambda_2 \psi_{1,2} \psi_{2,1} & (\lambda_2 - \lambda_1) \psi_{1,1} \psi_{1,2} \\ (\lambda_1 - \lambda_2) \psi_{2,1} \psi_{2,2} & \lambda_1 \psi_{1,2} \psi_{2,1} + \lambda_2 \psi_{1,1} \psi_{2,2} \end{pmatrix}. \quad (41)$$

From equation (29), we obtain

$$T_1(\lambda, \lambda_1, \lambda_2) = \lambda I - M = \frac{1}{\Delta_1} \begin{pmatrix} \lambda \Delta_1 - \begin{vmatrix} \psi_{2,1} & \lambda_1 \psi_{1,1} \\ \psi_{2,2} & \lambda_2 \psi_{1,2} \end{vmatrix} & - \begin{vmatrix} \psi_{1,1} & \lambda_1 \psi_{1,1} \\ \psi_{1,2} & \lambda_2 \psi_{1,2} \end{vmatrix} \\ \begin{vmatrix} \psi_{2,1} & \lambda_1 \psi_{2,1} \\ \psi_{2,2} & \lambda_2 \psi_{2,2} \end{vmatrix} & \lambda \Delta_1 + \begin{vmatrix} \psi_{1,1} & \lambda_1 \psi_{2,1} \\ \psi_{1,2} & \lambda_2 \psi_{2,2} \end{vmatrix} \end{pmatrix} \quad (42)$$

$$\lambda I + t_0^{[1]} = \lambda I - M = \frac{1}{\Delta_1} \begin{pmatrix} \lambda \Delta_1 - \begin{vmatrix} \psi_{2,1} & \lambda_1 \psi_{1,1} \\ \psi_{2,2} & \lambda_2 \psi_{1,2} \end{vmatrix} & - \begin{vmatrix} \psi_{1,1} & \lambda_1 \psi_{1,1} \\ \psi_{1,2} & \lambda_2 \psi_{1,2} \end{vmatrix} \\ \begin{vmatrix} \psi_{2,1} & \lambda_1 \psi_{2,1} \\ \psi_{2,2} & \lambda_2 \psi_{2,2} \end{vmatrix} & \lambda \Delta_1 + \begin{vmatrix} \psi_{1,1} & \lambda_1 \psi_{2,1} \\ \psi_{1,2} & \lambda_2 \psi_{2,2} \end{vmatrix} \end{pmatrix} \quad (43)$$

and the elements of the matrix T_1 are written as

$$(T_1)_{11} = \begin{vmatrix} \psi_{2,1} & \lambda_1 \psi_{1,1} \\ \psi_{2,2} & \lambda_2 \psi_{1,2} \end{vmatrix} - \lambda \Delta_1 = \begin{vmatrix} 1 & 0 & \lambda \\ \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{1,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{1,2} \end{vmatrix},$$

$$(T_1)_{12} = - \begin{vmatrix} \psi_{1,1} & \lambda_1 \psi_{1,1} \\ \psi_{1,2} & \lambda_2 \psi_{1,2} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{1,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{1,2} \end{vmatrix},$$

$$(T_1)_{21} = \begin{vmatrix} \psi_{2,1} & \lambda_1 \psi_{2,1} \\ \psi_{2,2} & \lambda_2 \psi_{2,2} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{2,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{2,2} \end{vmatrix},$$

$$(T_1)_{22} = - \begin{vmatrix} \psi_{1,1} & \lambda_1 \psi_{2,1} \\ \psi_{1,2} & \lambda_2 \psi_{2,2} \end{vmatrix} - \lambda \Delta_1 = \begin{vmatrix} 0 & 1 & \lambda \\ \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{2,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{2,2} \end{vmatrix},$$

$$t_{0x}^{[1]} + \lambda A_0 - i\lambda^2 \sigma_3 + \lambda t_0^{[1]} \sigma_3 + t_0^{[1]} A_0 = -i\lambda^2 \sigma_3 + \lambda A_0^{[1]} - i\lambda \sigma_3 t_0^{[1]} + A_0^{[1]} t_0^{[1]}. \quad (44)$$

Comparing the coefficients of λ^i of the two sides of (44) yields

$$\lambda^0 : \quad t_{0x}^{[1]} + t_0^{[1]} A_0 = A_0^{[1]} t_0^{[1]},$$

$$\lambda^1 : \quad A_0 - i t_0^{[1]} \sigma_3 = A_0^{[1]} - i \sigma_3 t_0^{[1]},$$

$$\lambda^2 : \quad i \sigma_3 = i \sigma_3$$

Analogous theorem 1 we can formulate the next theorem.

Theorem 2. The two-fold Darboux transformation of the nonlocal nonlinear Schrödinger and Maxwell – Bloch equation is

$$T_2(\lambda, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda^2 I + \lambda t_1^{[2]} + t_0^{[2]} = \frac{1}{\Delta_2} \begin{pmatrix} (T_2)_{11} & (T_2)_{12} \\ (T_2)_{21} & (T_2)_{22} \end{pmatrix},$$

where

$$\begin{aligned}
(T_2)_{11} &= \begin{pmatrix} 1 & 0 & \lambda & 0 & \lambda^2 \\ \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{1,1} & \lambda_1 \psi_{2,1} & \lambda_1^2 \psi_{1,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{1,2} & \lambda_2 \psi_{2,2} & \lambda_2^2 \psi_{1,2} \\ \psi_{1,3} & \psi_{2,3} & \lambda_3 \psi_{1,3} & \lambda_3 \psi_{2,3} & \lambda_3^2 \psi_{1,3} \\ \psi_{1,4} & \psi_{2,4} & \lambda_4 \psi_{1,4} & \lambda_4 \psi_{2,4} & \lambda_4^2 \psi_{1,4} \end{pmatrix}, & (T_2)_{12} &= \begin{pmatrix} 0 & 1 & 0 & \lambda & 0 \\ \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{1,1} & \lambda_1 \psi_{2,1} & \lambda_1^2 \psi_{1,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{1,2} & \lambda_2 \psi_{2,2} & \lambda_2^2 \psi_{1,2} \\ \psi_{1,3} & \psi_{2,3} & \lambda_3 \psi_{1,3} & \lambda_3 \psi_{2,3} & \lambda_3^2 \psi_{1,3} \\ \psi_{1,4} & \psi_{2,4} & \lambda_4 \psi_{1,4} & \lambda_4 \psi_{2,4} & \lambda_4^2 \psi_{1,4} \end{pmatrix}, \\
(T_2)_{21} &= \begin{pmatrix} \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{1,1} & \lambda_1 \psi_{2,1} & \lambda_1^2 \psi_{1,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{1,2} & \lambda_2 \psi_{2,2} & \lambda_2^2 \psi_{1,2} \\ \psi_{1,3} & \psi_{2,3} & \lambda_3 \psi_{1,3} & \lambda_3 \psi_{2,3} & \lambda_3^2 \psi_{1,3} \\ \psi_{1,4} & \psi_{2,4} & \lambda_4 \psi_{1,4} & \lambda_4 \psi_{2,4} & \lambda_4^2 \psi_{1,4} \end{pmatrix}, & (T_2)_{22} &= \begin{pmatrix} \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{1,1} & \lambda_1 \psi_{2,1} & \lambda_1^2 \psi_{1,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{1,2} & \lambda_2 \psi_{2,2} & \lambda_2^2 \psi_{1,2} \\ \psi_{1,3} & \psi_{2,3} & \lambda_3 \psi_{1,3} & \lambda_3 \psi_{2,3} & \lambda_3^2 \psi_{1,3} \\ \psi_{1,4} & \psi_{2,4} & \lambda_4 \psi_{1,4} & \lambda_4 \psi_{2,4} & \lambda_4^2 \psi_{1,4} \end{pmatrix}, \\
\Delta_2 &= \begin{pmatrix} \psi_{1,1} & \psi_{2,1} & \lambda_1 \psi_{1,1} & \lambda_1 \psi_{2,1} \\ \psi_{1,2} & \psi_{2,2} & \lambda_2 \psi_{1,2} & \lambda_2 \psi_{2,2} \\ \psi_{1,3} & \psi_{2,3} & \lambda_3 \psi_{1,3} & \lambda_3 \psi_{2,3} \\ \psi_{1,4} & \psi_{2,4} & \lambda_4 \psi_{1,4} & \lambda_4 \psi_{2,4} \end{pmatrix}
\end{aligned}$$

T_2 satisfies the system

$$T_{2x} + T_2 A = A^{[2]} T_2,$$

$$T_{2t} + T_2 B = B^{[2]} T_2.$$

$$A_0^{[2]} = A_0 + [\sigma_3, t_0^{[2]}],$$

$$B_{-1}^{[2]} = T_2 |_{\lambda=-\mu} B_{-1} T_2^{-1} |_{\lambda=-\mu}.$$

Then the solutions of the system (1) – (5) are given by

$$\begin{aligned}
q^{[2]} &= q - 2i \frac{(T_2)_{12}}{\Delta_2} \\
\eta^{[2]} &= \frac{(\mu + (T_2)_{11})^2 |(T_2)_{12}|^2 \eta + p (T_2)_{21} (\mu + (T_2)_{11}) - p^* (T_2)_{12} (\mu + (T_2)_{22})}{W}, \\
p^{[2]} &= \frac{p [(\mu + (T_2)_{11})^2 - p^* (T_2)_{12}^2] + 2\eta (T_2)_{12} (\mu + (T_2)_{12} (\mu + (T_2)_{11}))}{W}, \\
p^{*[2]} &= \frac{p^* ((\mu + (T_2)_{22})^2 + p (T_2)_{21}^2) - 2\eta (T_2)_{21} (\mu + (T_2)_{22})}{W},
\end{aligned}$$

where $W = (T_2)_{11} (T_2)_{22} - (T_2)_{12} (T_2)_{21}$.

The transformation T_2 has the following property

$$T_2(\lambda, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \Big|_{\lambda=\lambda_i} \begin{pmatrix} \psi_{1,i} \\ \psi_{2,i} \end{pmatrix} = 0, \quad i = 1, 2, 3, 4.$$

Conclusion

In this paper, we have obtained the DT for the nonlocal nonlinear Schrödinger and Maxwell – Bloch equation. Using the derived DT, some exact solutions including, the one-soliton solution are obtained. The determinant representations are given for one-fold, two-fold and n-fold DT for the onlocal nonlinear Schrödinger and Maxwell – Bloch equation. Using obtained results, one can also find the n-solitons, breathers and rogue wave solutions of the nonlocal nonlinear Schrödinger and Maxwell – Bloch equation. It is interesting to note that the rogue wave solutions of nonlinear equations are currently one of the most active topics in nonlinear physics and mathematics. The application of the obtained solutions in physics is an interesting subject. In particular, we hope that the presented solutions may be used in experiments or optical fibre communication. Also we will study some important generalizations of the nonlocal nonlinear Schrödinger and Maxwell – Bloch equation in future.

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