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Solvability of a two-point boundary value problem with phase and integral constraints

Abstract. New mathematical techniques for considering the complex boundary value problems to solve topical problems of natural sciences, technology, economy and ecology etc. are needed. Mathematical models of nuclear and chemical reactors management processes, control of electric power and robotic systems, economic management and others are complex boundary value problems of ordinary differential equations. Boundary value problems are called complex if besides the boundary conditions there exist the phase constraints and integral constraints on the phase coordinates of the system. The main objectives are: the necessary and sufficient conditions for the existence of solutions of boundary value problems and the methods of construction of complex solutions of boundary value problems. The aim of the work is an attempt to create a unified theory of the study of solvability of boundary value problems and the construction of a general method for solving them, based on the use of modern computer technology. The work is devoted to solving the problems of boundary value problems of nonlinear systems of ordinary differential equations. We consider the boundary value problem with boundary conditions of the given convex closed sets. The necessary and sufficient conditions for existence of a solution of the problem and construction its solution are obtained. The basis of the proposed method for solving of the boundary value problem is a possibility to reduce to a class of linear Fredholm integral equation of the first kind [1]-[9]. Necessary and sufficient condition for existence of a solution of integral equation is proved. Fredholm integral equation of the first kind belongs to the insufficiently explored problems in mathematics [11]-[22].

Key words: integral equation, solvability, construction of a solution, extreme problem, functional gradient, minimizing sequence.

Introduction

We consider and find necessary and sufficient conditions for existence of a solution of the boundary value problem

$$\dot{x} = A(t)x + \mu(t),\ t \in I = [t_0, t_1], \qquad (1)$$

$$(x(t_0) = x_0, x(t_1) = x_1) \in S \subset R^{2n}, \qquad (2)$$

at phase constraints

$$x(t) \in G(t);$$
$$G(t) = \{x \in R^n / \omega(t) \le L(t)x \le \phi(t),$$
$$t \in I\},$$

integral constraints

$$g_j(x) \le c_j, \quad j = \overline{1, m_1},$$
$$g_j(x) = c_j, \quad j = \overline{m_1 + 1, m_2},$$

$$g_j(x) = \int_{t_0}^{t_1} < a_j(t), x > dt, \qquad j = \overline{1, m_2},$$

here $A(t)$, $L(t)$ are matrixes of $n \times n, s \times n$ order, accordingly, with piece-wise continuous elements, S is the given closed set, $\omega(t)$, $\varphi(t)$, $t \in I$ are the prescribed continuous vector functions $s \times 1$, $a_j(t)$, $j = \overline{1, m_2}$ are the given piece-wise continuous vector functions of $n \times 1$ order, c_j, $j = \overline{1, m_2}$ are unknown constants, t_0, t_1 are the fixed time moments, $\mu(t) = (\mu_1(t), \dots, \mu_n(t))$ is the prescribed piece-wise continuous function, $< \cdot, \cdot >$ is a scalar production. We construct a solution of the linear system (1) with boundary conditions (2).

We represent the matrix $A(t)$ of $n \times n$ order with piecewise continuous elements as the sum $A(t) = A_1(t) + B(t)$, $t \in I$, that the matrix

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$$W(t_0,t_1)=\int_{t_0}^{t_1}\Phi(t_0,t)B(t)B^*(t)\Phi^*(t_0,t)dt$$

of $n\times n$ order be positive defined, where $\Phi(t,\tau)=\theta(t)\theta^{-1}(\tau)$, $\theta(t)$ is a fundamental matrix solutions of the linear homogeneous system $\dot\xi=A_1(t)\xi$. We note, that the matrix $\theta(t)$ is a solution of the equation $\dot\theta(t)=A_1(t)\theta(t)$, $\theta(t_0)=I_n$, where I_n is an unique matrix of $n\times n$ order. There are many options for representation the matrix $A(t)$ as the sum $A(t)=A_1(t)+B(t)$, $t\in I$:

1. The matrix $A_1(t)$ can be chosen as a constant matrix A_1 of $n\times n$ order. In this case $\theta(t)=e^{A_1t}$, $t\in I$;
2. The matrix $B(t)$ is chosen in the form $B(t)=B_1(t)P$, where $B_1(t)$ is the matrix of $n\times m$ order, P is a constant matrix of $m\times n$ order, moreover $P=(I_m,0_{m,n-m})$, where I_m is an unique matrix of $m\times m$ order, $0_{m,n-m}$ is a rectangular matrix of $m\times(n-m)$ order with zero elements.

Since the matrix $A(t)=A_1(t)+B(t)$, $t\in I$, that equation (1) is written as

$$\dot x=A_1(t)x+B(t)x+\mu(t),\ t\in I=[t_0,t_1]. \quad (3)$$

In the case of a choice $B(t)=B_1(t)P$ the equation (3) has the form

$$\dot x=A_1(t)x+B_1(t)Px+\mu(t),\ t\in I, \quad (4)$$

where $B_1(t)$ is a matrix of $n\times m$ order, Px is the vector function $m\times1$. If $m=n$, then $P=I_n$, $B(t)=B_1(t)$, $t\in I$. Without loss of generality, further we believe, that equation (4) is represented in the form (4), the matrix

$$W_1(t_0,t_1)=\int_{t_0}^{t_1}\Phi(t_0,t)B_1(t)B_1^*(t)\Phi^*(t_0,t)dt. \quad (5)$$

In addition to (4), we consider the linear control system of the form

$$\dot y=A_1(t)y+B_1(t)u(t)+\mu(t),\ t\in I, \quad (6)$$

$$\left(y(t_0)=x_0,y(t_1)=x_1\right)\in S\subset R^{2n}, \quad (7)$$

$$u(\cdot)\in L_2(I,R^m). \quad (8)$$

We note, that if $u(t)=Px(t)$, $t\in I$, then the system (6)-(8) coincides with the origin system (1), (2).

Solution of a linear control system

We use the following theorems 1 and 2. The theorems are proved in our previous works [1]-[9].

A solution of the boundary value problem relates to properties of the solutions of the following integral equation

$$Ku=\int_{t_0}^{t_1}K(t_0,t)u(t)dt=a,\ t\in I=[t_0,t_1], \quad (2.1)$$

where $K(t_0,t)=\|K_{ij}(t_0,t)\|$, $i=\overline{1,n}$, $j=\overline{1,m}$ is the known matrix of $n\times m$ order with piecewise continuous by t elements at fixed t_0, t_1, $u(\cdot)\in L_2(I,R^m)$ is the origin function, operator $K:L_2(I,R^m)\to R^n$, $a\in R^n$ is prescribed vector.

Theorem 1. *Integral equation (2.1) at any fixed $a\in R^n$ has a solution if and only if the matrix*

$$C(t_0,t_1)=\int_{t_0}^{t_1}K(t_0,t)K^*(t_0,t)dt \quad (2.2)$$

of $n\times n$ order is positive defined, where $(*)$ is a transposition sign, $t_1>t_0$.

Theorem 2. *Let $C(t_0,t_1)>0$ be a matrix. Then the general solution of integral equation (2.1) has the form*

$$u(t)=K^*(t_0,t)C^{-1}(t_0,t)a+v(t)-$$
$$-K^*(t_0,t)C^{-1}(t_0,t_1)\int_{t_0}^{t_1}K(t_0,t)v(t)dt, \quad (2.3)$$

where $t\in I$, $v(\cdot)\in L_2(I,R^m)$ is an arbitrary function, $a\in R^n$ is any vector.

Theorem 3. *Let the matrix* $W_1(t_0,t_1)$ *of* $n\times n$ *order be positive defined. Then control* $u(\cdot)\in L_2(I,R^m)$ *transfers the trajectory of system (6) from any initial point* $y(t_0)=x_0\in R^n$ *to any finite state* $y(t_1)=x_1\in R^n$ *if and only if*

$$\begin{gathered} u(t)\in U=\{u(\cdot)\in L_2(I,R^m)/\,u(t)= \\ v(t)+\lambda_1(t,x_0,x_1)+ \\ +N_1(t)z(t_1,v),t\in I,\forall v,v(\cdot)\in L_2(I,R^m)\}, \end{gathered} \tag{9}$$

where

$$\lambda_1(t,x_0,x_1)=B_1^*(t)\Phi^*(t_0,t)W_1^{-1}(t_0,t_1)a,$$

$$a=\Phi(t_0,t_1)x_1-x_0-\int_{t_0}^{t_1}\Phi(t_0,t)\mu(t)dt,$$

$$N_1(t)=-B_1^*(t)\Phi^*(t_0,t)W_1^{-1}(t_0,t_1)\Phi(t_0,t_1).$$

Function $z(t,v)$, $t\in I$ *is a solution of the differential equation*

$$\begin{gathered}\dot z=A_1(t)z+B_1(t)v(t),\ z(t_0)=0, \\ v(\cdot)\in L_2(I,R^m).\end{gathered} \tag{10}$$

Solution of differential equation (6) corresponding to control $u(t)\in U$ *is defined by formula*

$$\begin{gathered}y(t)=z(t,v)+\lambda_2(t,x_0,x_1)+ \\ +N_2(t)z(t_1,v)\end{gathered}, \tag{11}$$

where $t\in I$,

$$\lambda_2(t,x_0,x_1)=\Phi(t,t_0)W_1(t,t_1)W_1^{-1}(t_0,t_1)x_0+\Phi(t,t_0)W_1(t_0,t)W_1^{-1}(t_0,t_1)\Phi(t_0,t_1)x_1+$$
$$+\int_{t_0}^{t}\Phi(t,\tau)\mu(\tau)d\tau-\Phi(t,t_0)W_1(t_0,t)W_1^{-1}(t_0,t_1)\int_{t_0}^{t_1}\Phi(t_0,t)\mu(t)dt$$

$$N_2(t)=-\Phi(t,t_0)W_1(t_0,t)W_1^{-1}(t_0,t_1)\Phi(t_0,t_1),$$

$$W(t_0,t)=\int_{t_0}^{t}\Phi(t_0,\tau)B_1(\tau)B_1^*(\tau)\Phi^*(t_0,\tau)d\tau,$$

$$W(t,t_1)=W(t_0,t_1)-W(t_0,t).$$

Proof. Solution of the system (6) has the form

$$\begin{gathered}y(t)=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)u(\tau)d\tau+ \\ +\int_{t_0}^{t}\Phi(t,\tau)\mu(\tau)d\tau,\quad t\in I\end{gathered},$$

Then the control which transfers the trajectory of system (6) from initial state $x_0\in R^n$ to the state $x_1\in R^n$ is defined by condition

$$\begin{gathered}y(t_1)=x_1=\Phi(t_1,t_0)x_0+ \\ +\int_{t_0}^{t_1}\Phi(t_1,t)B_1(t)u(t)dt+\int_{t_0}^{t_1}\Phi(t_1,t)\mu(t)dt\end{gathered}.$$

This implies

$$\begin{gathered}\int_{t_0}^{t_1}\Phi(t_1,t)B_1(t)u(t)dt= \\ =x_1-\Phi(t_1,t_0)x_0-\int_{t_0}^{t_1}\Phi(t_1,t)\mu(t)dt\end{gathered}. \tag{12}$$

Since $\Phi(t_1,t)=\Phi(t_1,t_0)\Phi(t_0,t)$, $\Phi^{-1}(t_1,t_0)=\Phi(t_0,t_1)$, that expression (12) is written in the form

$$\begin{gathered}\int_{t_0}^{t_1}\Phi(t_0,t)B_1(t)u(t)dt= \\ +\Phi(t_0,t_1)x_1-x_0-\int_{t_0}^{t_1}\Phi(t_0,t)\mu(t)dt=a\end{gathered}. \tag{13}$$

Thus, the origin equation $u(\cdot)\in L_2(I,R^m)$ is a solution of the integral equation (13). Integral equation (13) can be represented as

$$Ku=\int_{t_0}^{t_1}K(t_0,t)u(t)dt=a,$$

$$K(t_0,t)=\Phi(t_0,t)B_1(t),\ t\in I.$$

As it follows from theorem 1, integral equation (13) has a solution if and only if the matrix

$$C(t_0,t_1)=\int_{t_0}^{t_1} K(t_0,t)K^*(t_0,t)dt=$$

$$=\int_{t_0}^{t_1}\Phi(t_0,t)B_1(t)B_1^*(t)\Phi^*(t_0,t)dt = W_1(t_0,t_1)$$

of $n\times n$ order is positive defined. Consequently, the set $U\neq\varnothing$, $\varnothing$ is an empty set if and only if $W_1(t_0,t_1)>0$. It means, that the system (6)-(9) is controlled.

From theorem 2 it follows, that the general solution of the integral equation (13) has the form

$$u(t)=K^*(t_0,t)C^{-1}(t_0,t_1)a+v(t)-$$

$$-K^*(t_0,t)C^{-1}(t_0,t_1)\int_{t_0}^{t_1}K(t_0,t)v(t)dt,$$

where

$$K(t_0,t)=\Phi(t_0,t)B_1(t),$$

$$C(t_0,t_1)=W_1(t_0,t_1).$$

This implies

$$u(t)=B_1^*(t)\Phi^*(t_0,t)W_1^{-1}(t_0,t_1)a+v(t)-B_1^*(t)\Phi^*(t_0,t)\times W^{-1}(t_0,t_1)\int_{t_0}^{t_1}\Phi(t_0,t)B_1(t)v(t)\,dt,\ t\in I, \tag{14}$$

where $v(\cdot)\in L_2(I,R^m)$ is any function. We note, that solution of differential equation (10) has the form

$$z(t)=z(t,v)=\Phi(t,t_0)z(t_0)+$$
$$+\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)v(\tau)d\tau=\quad , \tag{15}$$
$$=\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)v(\tau)d\tau$$

where $z(t_0)=0$. Consequently,

$$z(t_1)=z(t_1,v)=\int_{t_0}^{t_1}\Phi(t_1,t)B_1(t)v(t)dt=$$
$$=\Phi(t_1,t_0)\int_{t_0}^{t_1}\Phi(t_0,t)B_1(t)v(t)dt. \tag{16}$$

From (14)-(16) it follows, that the origin control $u(t)=v(t)+\lambda_1(t,x_0,x_1)+N_1(t)z(t_1,v)$, $t\in I$, $\forall v,\ v(\cdot)\in L_2(I,R^m)$.

This implies proposition of the theorem that $u(t)\in U$. Finally, inclusion (9) is proved.

Let $u(t)\in U$. Then solution of differential equation (6) has the form

$$y(t)=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)[v(\tau)+\lambda_1(\tau,x_0,x_1)+N_1(\tau)z(t_1,v)]d\tau+$$
$$+\int_{t_0}^{t}\Phi(t,\tau)\mu(\tau)d\tau=\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)v(\tau)d\tau+\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)\times$$
$$\times B_1^*(\tau)\Phi^*(t_0,\tau)d\tau\, W_1^{-1}(t_0,t_1)[\Phi(t_0,t_1)x_1-x_0-\int_{t_0}^{t_1}\Phi(t_0,t)\mu(t)dt]-$$
$$-\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)B_1^*(\tau)\Phi^*(t_0,\tau)d\tau\, W_1^{-1}(t_0,t_1)\Phi(t_0,t_1)z(t_1,v)=z(t,v)+$$
$$+\lambda_2(t,x_0,x_1)+N_2(t)z(t_1,v),\ t\in I.$$

This implies representation of solution of the system (6) in the form (11). Theorem is proved.

It is easy to make sure in that

$$y(t_0)=z(t_0,v)+\lambda_2(t_0,x_0,x_1)+N_2(t_0)z(t_1,v)=x_0,$$
$$y(t_1)=z(t_1,v)+\lambda_2(t_1,x_0,x_1)+N_2(t_1)z(t_1,v)=x_1.$$

Since proposition of the theorem is valid for any $x_0 \in R^n$, $x_1 \in R^n$, that it is valid, when $(x_0, x_1) \in S \subset R^{2n}$.

Lemma 1. *Let* $W_1(t_0, t_1) > 0$ *be a matrix. Then the boundary value problem (1), (2) is equivalent to the following problem:*

$$\begin{aligned} &v(t) + T_1(t)x_0 + T_2(t)x_1 + \bar{\mu}(t) + \\ &+N_1(t)z(t_1, v) = Py(t) \\ &t \in I \end{aligned} \quad (17)$$

$$\dot{z} = A_1(t)z + B_1(t)v(t),\ z(t_0) = 0,\quad t \in I$$

$$v(\cdot) \in L_2(I, R^m), \quad (18)$$

$$(x_0, x_1) \in S, \quad (19)$$

where

$$T_1(t) = -B_1^*(t)\Phi^*(t_0, t)W_1^{-1}(t_0, t_1),$$

$$T_2(t) = B_1^*(t)\Phi^*(t_0, t)W^{-1}(t_0, t_1)\Phi(t_0, t_1),$$

$$\bar{\mu}(t) = -B_1^*(t)\Phi^*(t_0, t)W_1^{-1}(t_0, t_1)\int_{t_0}^{t_1}\Phi(t_0, t)\mu(t)dt,$$

$$\begin{aligned} y(t) = z(t, v) + C_1(t)x_0 + C_2(t)x_1 + \\ +f(t) + N_2(t)z(t_1, v) \end{aligned}, \quad (20)$$

$$C_1(t) = \Phi(t, t_0)W_1(t, t_1)W_1^{-1}(t_0, t_1),$$

$$C_2(t) = \Phi(t, t_0)W_1(t_0, t)W_1^{-1}(t_0, t_1)\Phi(t_0, t_1),$$

$$f(t) = \int_{t_0}^{t}\Phi(t, \tau)\mu(\tau)d\tau -$$

$$-\Phi(t, t_0)W_1(t_0, t)W_1^{-1}(t_0, t_1)\int_{t_0}^{t_1}\Phi(t_0, t)\mu(t)dt.$$

Proof of the lemma follows from theorem 3 and the equality $y(t) = x(t)$, $t \in I$, at $u(t) \in U$, $u(t) = Py(t)$, $t \in I$. It is easy to make sure that expressions (17) – (19) are equivalent to the expressions (1), (2), at $W_1(t_0, t_1) > 0$.

We consider the optimal control problem: minimize the functional

$$I(v, x_0, x_1) = \int_{t_0}^{t_1}\left|v(t) + T_1(t)x_0 + T_2(t)x_1 + \bar{\mu}(t) + N_1(t)z(t_1, v) - Py(t)\right|^2 dt \to \inf, \quad (21)$$

at conditions

$$\dot{z} = A_1(t)z + B_1(t)v(t),\ z(t_0) = 0,\ t \in I, \quad (22)$$

$$v(\cdot) \in L_2(I, R^m),\ (x_0, x_1) \in S, \quad (23)$$

where $y(t) = y(t, v, x_0, x_1)$, $t \in I$ is defined by formula (23).

We note, that:

1. Functional $I(v, x_0, x_1) \ge 0$. Consequently, functional $I(v, x_0, x_1)$ is bounded below on the set $X = L_2(I, R^m) \times S$, where $(v, x_0, x_1) \in X \subset H$, $H = L_2(I, R^m) \times R^{2n}$.

2. If $I(v_*, x_{0*}, x_{1*}) = 0$, where $(v_*, x_{0*}, x_{1*}) \in X$, is a solution of the optimization problem (21)-(23), then

$$\begin{aligned} x_*(t) = y_*(t, v_*, x_{0*}, x_{1*}) = z(t, v_*) + C_1(t)x_{0*} + \\ +C_2(t)x_{1*} + f(t) + N_2(t)z(t_1, v_*), \end{aligned}$$

$t \in I$ is solution of the boundary value problem (1), (2).

Necessary and sufficient condition for existence of a solution of the two-point boundary value problem

Theorem 4. *Let* $W_1(t_0, t_1) > 0$ *be a matrix. In order to the boundary value problem (1), (2) has a solution, necessary and sufficiently, that the value* $I(v_*, x_{0*}, x_{1*}) = 0$, *where* $(v_*, x_{0*}, x_{1*}) \in X$ *is a solution of the optimization problem (21) – (23).*

Proof. *Necessity.* Let boundary value problem (1), (2) has a solution. We show, that the value

$I(v_*, x_{0*}, x_{1*}) = 0$. Let $x(t; t_0, x_{0*}, x_{1*})$, $t \in I$, $(x_{0*}, x_{1*}) \in S$ be a solution of differential equation (1). As it follows from lemma 1, boundary value problem (1), (2) is equivalent to the problem (17) – (19). Hence,

$$v_*(t) + T_1(t)x_{0*} + T_2(t)x_{1*} + \\ + \bar{\mu}(t) + N_1(t)z(t_1, v_*) = Py_*(t), \\ t \in I, \quad (24)$$

$$\dot{z}(t, v_*) = A_1(t)z(t, v_*) + B_1(t)v_*(t), \; z(t_0) = 0,$$

$$t \in I, \; v_*(\cdot) \in L_2(I, R^m), \quad (25)$$

$$y_*(t) = z(t, v_*) + C_1(t)x_{0*} + C_2(t)x_{1*} + \\ + f(t) + N_2(t)z(t_1, v_*), \\ t \in I$$

where $(x_{0*}, x_{1*}) \in S$, $u(t) \in U$, $u(t) = Py_*(t)$, $t \in I$, $y_*(t) = x(t; t_0, x_{0*}, x_{1*})$, $t \in I$.

Then

$$I(v_*, x_{0*}, x_{1*}) = \int_{t_0}^{t_1} \left| v_*(t) + T_1(t)x_{0*} + T_2(t)x_{1*} + \bar{\mu}(t) + N_1(t)z(t_1, v_*) - Py_*(t) \right|^2 dt = 0,$$

by identities (24), (25). Necessity is proved.

Sufficiency. Let $I(v_*, x_{0*}, x_{1*}) = 0$ be the value. We show, that boundary value problem (1), (2) has a solution. In fact, the value $I(v_*, x_{0*}, x_{1*}) = 0$ if and only if the equality

$$v_*(t) + \lambda_1(t, x_{0*}, x_{1*}) + N_1(t)z(t_1, v_*) = \\ = Py(t, v_*, x_{0*}, x_{1*})$$

is held, where

$$y(t, v_*, x_{0*}, x_{1*}) = z(t, v_*) + \lambda_2(t, x_{0*}, x_{1*}) + \\ + N_2(t)z(t_1, v_*), \\ t \in I.$$

Function $y(t, v_*, x_{0*}, x_{1*})$, $t \in I$ is solution of differential equation (6), at conditions (7), (8). Consequently,

$$\dot{y}(t, v_*, x_{0*}, x_{1*}) = A_1(t)y(t, v_*, x_{0*}, x_{1*}) + \\ + B_1(t)u_*(t) + \mu(t) = A_1(t)y(t, v_*, x_{0*}, x_{1*}) + \\ + B_1(t)Py(t, v_*, x_{0*}, x_{1*}) + \mu(t),$$

where $u_*(t) = v_*(t) + \lambda_1(t, x_{0*}, x_{1*}) + N_1(t)z(t_1, v_*)$, $y(t_0) = x_{0*}$, $y(t_1) = x_{1*}$, $(x_{0*}, x_{1*}) \in S$, $u_*(t) \in U$. This implies, that $y(t, v_*, x_{0*}, x_{1*}) = x(t; t_0, x_{0*}, x_{1*})$, $t \in I$ is solution of the boundary value problem (1), (2). Sufficiency is proved. Theorem is proved.

As it follows from theorem 4, if the value $I(v_*, x_{0*}, x_{1*}) > 0$, then the boundary value problem (1), (2) has not solution. Thus, for constructing of a solution of boundary value problem (1), (2) necessary to solve optimization problem (21) – (23).

Lemma 2. *Let* $W_1(t_0, t_1) > 0$ *be a matrix, function*

$$F_0(q, t) = \left| v + T_1(t)x_0 + T_2(t)x_1 + \bar{\mu}(t) + N_1(t)z(t_1) - Py(t, v, x_0, x_1) \right|^2, \quad (26)$$

where

$$y(t, v, x_0, x_1) = z + C_1(t)x_0 + C_2(t)x_1 + f(t) + N_2(t)z(t_1), \; q = (v, x_0, x_1, z, z(t_1)) \in R^m \times R^n \times R^n \times R^n \times R^n.$$

Then the partial derivatives

$$\frac{\partial F_0(q, t)}{\partial v} = [2v + T_1(t)x_0 + T_2(t)x_1 + \bar{\mu}(t) + N_1(t)z(t_1) - Py], \quad (27)$$

$$\frac{\partial F_0(q,t)}{\partial x_0}=[2T_1^*(t)-2C_1^*(t)P^*][v+T_1(t)x_0+T_2(t)x_1+\overline{\mu}(t)+N_1(t)z(t_1)-Py], \qquad (28)$$

$$\frac{\partial F_0(q,t)}{\partial x_1}=[2T_2^*(t)-2C_2^*(t)P^*][v+T_1(t)x_0+T_2(t)x_1+\overline{\mu}(t)+N_1(t)z(t_1)-Py], \qquad (29)$$

$$\frac{\partial F_0(q,t)}{\partial z}=-2P^*(t)[v+T_1(t)x_0+T_2(t)x_1+\overline{\mu}(t)+N_1(t)z(t_1)-Py], \qquad (30)$$

$$\frac{\partial F_0(q,t)}{\partial z(t_1)}=[2N_1^*(t)-2N_2^*(t)P^*][v+T_1(t)x_0+T_2(t)x_1+\overline{\mu}(t)+N_1(t)z(t_1)-Py]. \qquad (31)$$

Formulas (27) – (31) can be obtained by directly differentiating the function $F_0(q,t)$ by variable q.

Lemma 3. *Let $W_1(t_0,t_1)>0$ be a matrix, set S be convex. Then:*

1) Functional (21) is convex, at conditions (22), (23);

2) Derivative

$$3)\quad \frac{\partial F_0(q,t)}{\partial q}=\left(\frac{\partial F_0(q,t)}{\partial v},\frac{\partial F_0(q,t)}{\partial x_0},\frac{\partial F_0(q,t)}{\partial x_1},\frac{\partial F_0(q,t)}{\partial z},\frac{\partial F_0(q,t)}{\partial z(t_1)}\right)$$

4) satisfies to the Lipschitz condition $\left\|\frac{\partial F_0(q+\Delta q,t)}{\partial q}-\frac{\partial F_0(q,t)}{\partial q}\right\|\le L|\Delta q|$, $\forall q$, $q+\Delta q\in R^{m+4n}$, *where* $L=const>0$.

Proof. It is easy to make sure that

$$F_0(q,t)=q^*E^*(t)E(t)q+2q^*E^*(t)[\overline{\mu}(t)-Pf(t)]+[\overline{\mu}(t)-Pf(t)]^*[\overline{\mu}(t)-Pf(t)],\quad t\in I,$$

where E is a matrix of $m\times 4n$ order. Then $\frac{\partial^2F_0(q,t)}{\partial q^2}=2E^*(t)E(t)\ge 0$ for any t, $t\in I$. Consequently, function $F_0(q,t)$ is convex with respect to q, i.e.

$$F_0(\alpha q^1+(1-\alpha)q^2,t)\le \alpha F_0(q^1,t)+(1-\alpha)F_0(q^2,t),$$
$$\forall q^1,\ q^2\in R^{m+4n},\ \forall\alpha,\ \alpha\in[0,1]. \qquad (32)$$

For any $v_1(\cdot)\in L_2(I,R^m)$, $v_2(\cdot)\in L_2(I,R^m)$, and at all $\alpha>0$, $\alpha\in[0,1]$ the solution of differential equation (23) under $v_\alpha(t)=\alpha v_1(t)+(1-\alpha)v_2(t)$, $t\in I$ possesses by the property

$$z(t,v_\alpha)=\alpha z(t,v_1)+(1-\alpha)z(t,v_2),\ t\in I. \qquad (33)$$

In fact,

$$z(t,v_\alpha)=\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)v_\alpha(\tau)d\tau=\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)\left[\alpha v_1(\tau)+(1-\alpha)v_2(\tau)\right]d\tau=$$

$$=\alpha\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)v_1(\tau)d\tau+(1-\alpha)\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)v_2(\tau)d\tau=\alpha z(t,v_1)+(1-\alpha)z(t,v_2),\ t\in I.$$

Let $\xi_1=(v_1(t),x_0^1,x_1^1)\in X$, $\xi_2=(v_2(t),x_0^2,x_1^2)\in X$. Then the point

$$\xi_\alpha=\alpha\xi_1+(1-\alpha)\xi_2=(\alpha v_1+(1-\alpha)v_2,\alpha x_0^1+(1-\alpha)x_0^2,\alpha x_1^1+(1-\alpha)x_1^2)\in X.$$

The functional value

$$I(\xi_\alpha) = \int_{t_0}^{t_1} F_0\big(\alpha v_1 + (1-\alpha)v_2, \alpha x_0^1 + (1-\alpha)x_0^2, \alpha x_1^1 + (1-\alpha)x_1^2, z\big(t, \alpha v_1 + (1-\alpha)v_2\big),$$

$$z(t_1, \alpha v_1 + (1-\alpha)v_2), t)dt = \int_{t_0}^{t_1} F_0\big(\alpha v_1 + (1-\alpha)v_2, \alpha x_0^1 + (1-\alpha)x_0^2, \alpha x_1^1 + (1-\alpha)x_2^2,$$

$$\alpha z(t, v_1) + (1-\alpha)z(t, v_2), \alpha z(t_1, v_1) + (1-\alpha)z(t_1, v_2), t\big)dt \le \alpha \int_{t_0}^{t_1} F_0(\alpha q^1(t) + (1-\alpha)q^2(t))dt \le$$

$$\le \alpha \int_{t_0}^{t_1} F_0(q^1(t), t)dt + (1-\alpha)\int_{t_0}^{t_1} F_0(q^2(t), t)dt = \alpha I(\xi_1) + (1-\alpha)I(\xi_2),\ \forall \xi_1,\ \xi_2 \in X.$$

In virtue by expressions (32), (33), where

$$q^1(t) = (v_1(t), x_0^1, x_1^1, z(t, v_1), z(t_1, v_1)),$$
$$q^2(t) = (v_2(t), x_0^2, x_1^2,\ z(t, v_2), z(t_1, v_2)).$$

This implies the first proposition of the lemma.

Since derivative

$$\frac{\partial F_0(q,t)}{\partial q} = 2E^*(t)E(t)q + 2E^*(t)[\overline{\mu}(t) - Pf(t)],$$

that

$$\frac{\partial F_0(q + \Delta q, t)}{\partial q} - \frac{\partial F_0(q,t)}{\partial q} = 2E^*(t)E(t)\Delta q,$$

where $\Delta q = (\Delta v, \Delta x_0, \Delta x_1, \Delta z, \Delta z(t_1))$, $E^*(t)E(t)$ is the matrix of $(m+4n)\times(m+4n)$ order with piecewise continuous elements. Then

$$\left\| \frac{\partial F_0(q + \Delta q, t)}{\partial q} - \frac{\partial F_0(q,t)}{\partial q} \right\| \le L|\Delta q|,$$

where $L = \sup_{t_0 \le t \le t_1} \|E^*(t)E(t)\| > 0$. Lemma is proved.

Conclusion

Scientific novelty of the results is that the origin problems are reduced to the corresponding Fredholm integral equations of the first kind. Necessary and sufficient conditions for existence of a solution of the integral equations are proved by theorem. It is shown, that the boundary problem of a linear system of ordinary differential equations can be reduced to the corresponding initial optimal control problems. From the solutions of initial value problems of optimal control can be obtained the following solutions: boundary value problems with constraints, boundary problems with a parameter, construction of periodic solutions of autonomous systems. The basis of the proposed methods for solving boundary value problems with different constraints is a possibility of reducing these problems to a class of linear Fredholm integral equation of the first kind. Fredholm integral equation of the first kind belongs to the insufficiently explored problems in mathematics. Therefore, fundamental research on integral equations and solution on their basis of boundary value problems of linear ordinary differential equations is a new promising direction in mathematics.

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