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# Estimates of the best approximations of functions with a spectrum with a given majorant

**Abstract:** Assessment of the best approximations of functions through its modulus of smoothness (direct theorems of approximation theory or theorems of Jackson type) and of modulus of smoothness through its best approximation in a given metric (inverse theorems of approximation theory or theorems similar to Bernstein's theorem) by orthogonal systems have been the subject of research of many generations of mathematicians. Earlier, inhis research MirbulatSikhov obtained non-removable direct and inverse theorems of approximation theory for various metricsfor trigonometric polynomials with arbitrary spectrum defined by the general form of the majorant function $\Lambda(t)$ under certain conditions of regularity for functions $\Lambda(t)$ .The article presents precise assessment of best approximations by trigonometric polynomials of functions of the Besovclass where the spectrum of the best approximation functions is determined by a majorant function of a special type $\Omega(t)=\prod_{j=1}^{d} t_j^{r}\left(\log\frac{1}{t_j}\right)_+^{-\gamma_j}$ . The relevance of this topic is determined by the numerical analysis, particularly with the development of computer technology. This topic is naturally linked to the further development of the tasks aiming to estimate the closest approximation of a function, approximation of inequalities of Bernstein and Nikolskiy, and of approximation theory [1-8].

**Key words:**the best approximation, assessment of the best approximation, modulus of continuity.

## Introduction

The basic concepts of approximation theory are the concepts of best approximation and the modulus of continuity that reflect constructive and structural properties of the function, respectively.In one-dimensional case, the relationship between these two fundamentally different characteristics of functions was first established by D. Jackson and S. Bernstein. This problem allows different multidimensional generalizations – depending on the approximation method and on the definition of differences used that generate the corresponding modulus of smoothness. All of these studies are related to the theorems of Jackson and Bernstein (or to the direct and inverse theorems of approximation theory) in the setup of this paper for trigonometric polynomials. The main difficulties encountered in the study of solutions to the problems of this type arethe difficulties in determining the type of the estimation needed, in figuring out how the parameters are estimated, and in showing that they can't be improved.

Let $\pi_d=[-\pi,\pi]^d - d -$ dimensional cube. Let $L^p(\pi_d)$ $(1\le p\le\infty)$ be the set of all measurable $2\pi$ periodic functions in each $d$ variable $f(x)=f(x_1,...,x_d)$ such that,

$$\|f\|_p=(2\pi)^{-d}\left(\int_{\pi_d}|f(x)|^p\,dx\right)^{\frac{1}{p}}<\infty,\ 1\le p<\infty$$

$$L_0^p(\pi_d)=\left\{f\in L^p(\pi_d):\ \int_{-\pi}^{\pi}f(x)dx_j=0(j=1,...,d)\right\}$$

 

For a subset B of the Euclidean space $R^d$ denote by $B_0$ and $B_+$ the set consisting of all elements of $x=(x_1,x_2,...,x_d)\in B$ where each component is non-negative and positive, respectively. Let $Z^d$ denote $R^d$ integer lattice. For $n\in Z_+^d$ suppose that $\|n\|_1=n_1+...+n_d$, $2^{-n}=(2^{-n_1},...,2^{-n_d})$.

For $f\in L^p(\pi_d)$ we can identify a mixed modulus of smoothness of order $k\in Z_+\equiv Z_+^1$

$$\Omega_k(f;t_1,...,t_d)_p=\sup_{\substack{|h_j|\le t_j\\ j=1,...,d}}\|\Delta_h^k f(x)\|_p\left(t\in[0,1]^d\right),$$

where $\Delta_h^k f(x)=\Delta_{h_d}^k...\Delta_{h_1}^k f(x)$ , $\Delta_{h_d}^k=\Delta_{h_j}^1\left(\Delta_{h_j}^{k-1}\right)$ ,

$$\Delta_{h_j}^1 f(x)=f(x_1,...,x_j+h_j,...,x_d)-\\ -f(x_1,...,x_j,...,x_d).$$

For these numbers $1\le p<\infty, 0<r_1\le...\le r_d$ Nikolskiy class $SH_p^{r_1,...,r_d}$ by definition consists of all functions $f\in L^p(\pi_d)$ such that for a mixed modulus of smoothness of order

$$k>r_d\ \Omega_k(f;t)_p\le\prod_{j=1}^d t_j^{r_j}$$

holds.

A more precise classification of the functions of smoothness in the metric $L^p(\pi_d)$ consists of the replacementof the function $t_j^{r_j}$ in the definitionbycommon functions such as modulus of smoothness $w_j(t_j)$.

If $f\in L^p(\pi_d)$ , then let $E_G(f)_p$ denote the best approximation (on $L^p$) of the function $f$ with polynomials from $T(G)$, where $G$ – is a finite set of points $Z^d$, and

$$T(G)=\left\{t(x): t(x)=\sum_{n\in G}c_n e^{i(n,x)}\right\}$$

The spectrum $G$ will be definedon $[0,1]^d$ by a continuous function $\Lambda(t)=\Lambda(t_1,...,t_d)$ that is non-decreasing in each variable and such that $\Lambda(t)>0$ and $\Lambda(t)=0$ depending on whether $\prod_{j=1}^d t_j>0$ or $\prod_{j=1}^d t_j=0$ ·

By $SB_{q,\theta}^{\Omega}(1\le q<\infty, 0<\theta\le\infty)$ let's denote the space of functions $f\in L_0^q(0,1)^d$ , for which the semi normis bounded as is shown below

$$\|f\|_{SB_{q,\theta}^{\Omega}}=\left(\int_{[0,\pi]^d}\left[\Omega_k(f;t)_q/\Omega(t)\right]^\theta\prod_{j=1}^d t_j^{-1}dt\right)^{\frac{1}{\theta}}\le 1$$

For $1<r<l,\gamma_j$ and $t_j\ge 0, j=1,...,d$ let's define the function $\Omega(t)$ as follows: if $t_j>0, j=1,...,d$, then

$$\Omega(t)=\prod_{j=1}^d t_j^r\left(\log\frac{1}{t_j}\right)_+^{-\gamma_j}$$

$$\text{if}\prod_{j=1}^d t_j=0,\ \text{then}\ \Omega(t)=0,$$

$$\left(\log\frac{1}{t_j}\right)_+=\max\left\{\log_2\frac{1}{t_j};1\right\}$$

Further, without loss of generality let's assume that $\gamma_1\le...\le\gamma_d$.

Consider the sets

$$\chi(N)=\left\{s=(s_1,...,s_d): s_j\in Z_+, j=1,...,d, \prod_{j=1}^d 2^{rs_j}s_j^{\gamma_j}\le N\right\},$$

$$Q(N) = \bigcup_{s\in\chi(N)} \rho(s),$$

where

$$\rho(s) = \{k = (k_1,...,k_d) : k_j \in Z, 2^{s_j - 1} \le |k_j| < 2^{s_j}, j = 1,...,d\}$$

Also, consider

$$\Gamma(N,\Omega) = \left\{k = (k_1,...,k_d) : |k_j| \in N, j = 1,...,d, \Omega\left(\frac{1}{|k_1|},...,\frac{1}{|k_d|}\right) \ge \frac{1}{N}\right\},$$

i.e.

$$\Gamma(N,\Omega) = \left\{k = (k_1,...,k_d) : |k_j| \in Z_+, j = 1,...,d, \prod_{j=1}^{d} |k_j|^r (\log|k_j|)_+^{\gamma_j} \le N\right\}$$

$$\Gamma^\perp(N,\Omega) = Z_+^s \setminus \Gamma(N,\Omega)$$

For functions of mixed modulus of smoothness let's assume that

$$\theta(\Omega, N) = \Gamma(\Omega, 2^l N) - \Gamma(\Omega, N),$$

i.e.

$$\theta(N) = \left\{s = (s_1,...,s_d) : s_j \in Z_+, j = 1,...,d, N < \prod_{j=1}^{d} 2^{rs_j} s_j^{\gamma_j} \le 2^l N\right\}.$$

$A(N) << B(N)$ means that there exists $C > 0$ that does not depend on $N$ such that $A(N) \le CB(N)$.

The functions of one variable $\varphi(\tau) \ge 0$ will satisfy condition $(S)$ if $\varphi(x)/\tau^\alpha$ is almost increasing for some $0 < \alpha < 1$, i.e. there is a number $C > 0$ that does not depend on $\tau_1$ and $\tau_2$, such that

$$\frac{\varphi(\tau_1)}{\tau_1^\alpha} \le C\frac{\varphi(\tau_2)}{\tau_2^\alpha}, \; 0 < \tau_1 \le \tau_2 \le 1.$$

To prove our main results we need the following results.

**Lemma 1** [2]. Let $\Lambda(t)$ satisfy $(S)$. Then for $0 < p < \infty$

$$\sum_{n\in\Gamma^\perp(N,\Lambda)} (\Lambda(2^{-n}))^p << \sum_{n\in\theta(N,\Lambda)} (\Lambda(2^{-n}))^p$$

**Lemma 2** [2]. Let $\Lambda(t)$ satisfy $(S)$ for $0 < \alpha < 1$ such that $\alpha > \beta > 0$. Then for $0 < p < \infty$

$$\sum_{n\in\Gamma^\perp(N,\Lambda)} \left(\Lambda(2^{-n}) 2^{\|n\|_1 \beta}\right)^p << \sum_{n\in\theta(N,\Lambda)} \left(\Lambda(2^{-n}) 2^{\|n\|_1 \beta}\right)^p$$

**Lemma 3** [6]. For $n \in \theta(N)$ the following holds

$$2^{\|n\|_1} \succ\prec N^{1/r} \prod_{j=1}^{d} n_j^{-\gamma_j/r}$$

**Lemma** 4 [7].The following sum

$$\sum_{n\in\theta(N)}\prod_{j=1}^{d} n_j^{-\gamma_j}, \qquad \gamma_1 \le \ldots \le \gamma_d$$

is equal to:

1) $(\log N)^{d-1-\gamma_1-\ldots-\gamma_d}$, if $\gamma_d < 1$;

2) $(\log N)^{\nu-1-\gamma_1-\ldots-\gamma_d}(\log\log N)^{\nu-1}$, if $\gamma_\nu < 1 = \gamma_{\nu+1} = \ldots = \gamma_{\nu+\mu} < \gamma_{\nu+\mu+1}$;

3) $(\log N)^{-1}(\log\log N)^{\nu-1}$, if $\gamma_1 = \ldots = \gamma_\nu = 1 < \gamma_{\nu+1}$;

4) $(\log N)^{-\nu_1}$, if $1 < \nu_1$.

Mirbulat Sikhov previously proved the following theorems:

**Theorem 1** [8]. Let $1 < p < q < \infty, 1 \le \theta \le \infty$ and $\Lambda(t)-$ be the function of mixed modulus of smoothness of order $k$. If $f(x) \in SB^{\Lambda}_{p.\theta}$ and

$$\sum_{n\in Z_+^d}\left[2^{\|n\|_1\left(\frac{1}{p}-\frac{1}{q}\right)}\Lambda\left(2^{-n}\right)\right]^{\rho} < \infty$$

where $\rho = \dfrac{\theta q}{\theta - q}$ for $q < \theta$ and $\rho = \infty$ where $q \ge \theta$, then $f(x) \in L_0^q(\pi_s)$ and

$$\|f(x)\|_q << \left(\sum_{n\in Z_+^d}\left[2^{\|n\|_1\left(\frac{1}{p}-\frac{1}{q}\right)}\Lambda\left(2^{-n}\right)\right]^{\rho}\right)^{\frac{1}{\rho}},$$

$$E_{Q(N,\Lambda)}(f)_q << \left(\sum_{n\in\Gamma^{\perp}(N,\Lambda)}\left[2^{\|n\|_1\left(\frac{1}{p}-\frac{1}{q}\right)}\Lambda\left(2^{-n}\right)\right]^{\rho}\right)^{\frac{1}{\rho}}$$

**Theorem 2** [8]. Let $1 < p < q < \infty, 1 \le \theta \le \infty$ and $\Lambda(t)-$ be the function of mixed modulus of smoothness of order $k$ .If $f(x) \in SB^{\Lambda}_{p.\theta}$, then the following inequality holds

$$E_{Q(N,\Lambda)}(f)_q << \left(\sum_{n\in\Gamma^{\perp}(N,\Lambda)}\left(\Lambda\left(2^{-n}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}$$

where $\rho = \dfrac{\theta p_0}{\theta - p_0}$ for $p_0 < \theta$ and $\rho = \infty$ for $p_0 \ge \theta$ and $p_0 = \min(p,2)$.

**Theorem 3**. Let $1 < p < q < \infty, 1 \le \theta \le \infty$, $q < \theta$, $\gamma_1 \le \ldots \le \gamma_d$ and $0 < r < 1$. If $f(x) \in SB^{\Omega}_{p.\theta}$ and

$$\sum_{n\in Z_+^d}\left[2^{\|n\|_1\left(\frac{1}{p}-\frac{1}{q}\right)}\Lambda\left(2^{-n}\right)\right]^{\rho} < \infty$$

where $\rho = \dfrac{\theta q}{\theta - q}$, then $f(x) \in L_0^q(\pi_s)$ and

$$E_{Q(N)}(f)_q << N^{\frac{1-r}{r}} \cdot \Phi(N,d,q,p,r)$$

where $\Phi(N,d,q,p,r)$ is equal to:

1) $(\log N)^{\frac{d-1-\gamma_1-\ldots-\gamma_d}{\rho}}$, if $\gamma_d < 1$;

2) $(\log N)^{\frac{\nu-1-\gamma_1-\ldots-\gamma_d}{\rho}}(\log\log N)^{\frac{\nu-1}{\rho}}$, if $\gamma_\nu < 1 = \gamma_{\nu+1} = \ldots = \gamma_{\nu+\mu} < \gamma_{\nu+\mu+1}$;

3) $(\log N)^{-\frac{1}{\rho}}(\log\log N)^{\frac{\nu-1}{\rho}}$, if $\gamma_1 = \ldots = \gamma_\nu = 1 < \gamma_{\nu+1}$;

4) $(\log N)^{-\frac{\nu_1}{\rho}}$, if $1 < \nu_1$.

where $\tau = \left(\dfrac{1}{p} - \dfrac{1}{q}\right)\rho$, $\gamma_j = \dfrac{b_j \tau}{r}$, $\gamma_1 \le \ldots \le \gamma_d$

**Proof .** By Theorem 1, Lemma 2, Lemma3, we have

$$E_{Q(N)}(f)_q << \left\{ \sum_{n\in\Gamma^{\perp}(N,\Lambda)} \left[ 2^{\|n\|_1\left(\frac{1}{p}-\frac{1}{q}\right)} \Lambda\left(2^{-n}\right) \right]^{\rho} \right\}^{\frac{1}{\rho}} << \left\{ \sum_{n\in\theta(N)} \left[ 2^{\|n\|_1\left(\frac{1}{p}-\frac{1}{q}\right)} \Lambda\left(2^{-n}\right) \right]^{\rho} \right\}$$

$$<< \frac{1}{N} \left\{ \sum_{n\in\theta(N)} 2^{\|n\|_1 \rho\left(\frac{1}{p}-\frac{1}{q}\right)} \right\}^{\frac{1}{\rho}} << \frac{1}{N} N^{\frac{1}{r}} \left( \sum_{n\in\theta(N)} \prod_{j=1}^{d} n_j^{-\gamma_j} \right)^{\frac{1}{\rho}}$$

By Lemma 4

1) $N^{\frac{1-r}{r}} (\log N)^{\frac{d-1-\gamma_1-\ldots-\gamma_d}{\rho}}$, if $\gamma_d < 1$;

2) $N^{\frac{1-r}{r}} (\log N)^{\frac{\nu-1-\gamma_1-\ldots-\gamma_d}{\rho}} (\log\log N)^{\frac{\nu-1}{\rho}}$, if $\gamma_\nu < 1 = \gamma_{\nu+1} = \ldots = \gamma_{\nu+\mu} < \gamma_{\nu+\mu+1}$;

3) $N^{\frac{1-r}{r}} (\log N)^{-\frac{1}{\rho}} (\log\log N)^{\frac{\nu-1}{\rho}}$, if $\gamma_1 = \ldots = \gamma_\nu = 1 < \gamma_{\nu+1}$;

4) $N^{\frac{1-r}{r}} (\log N)^{-\frac{\nu_1}{\rho}}$, if $1 < \nu_1$.

**Theorem 4.** Let $1 \le q < p < \infty$ or $1 < q = p < \infty$. $1 \le \theta \le \infty$ and $\Lambda(t)$ - be the function of mixed modulus of smoothness of order $k$. If $f(x) \in SB^{\Lambda}_{p.\theta}$, then the following inequality holds

$$E_{Q(N,\Lambda)}(f)_q << \frac{n^{\rho(s-1)}}{2^{nr}}$$

where $\rho = \dfrac{\theta p_0}{\theta - p_0}$ for $p_0 < \theta$ and $\rho = \infty$ for $p_0 \ge \theta$ and $p_0 = \min(p,2)$.

**Proof.** By Theorem 2 and Lemma 2

$$E_{Q(N,\Lambda)}(f)_q << E_{Q(N,\Lambda)}(f)_p << \left( \sum_{n\in\Gamma^{\perp}(N,\Lambda)} \left(\Lambda\left(2^{-n}\right)\right)^{\rho} \right)^{\frac{1}{\rho}} << \left( \sum_{n\in\theta(N)} \left(\Lambda\left(2^{-n}\right)\right)^{\rho} \right)^{\frac{1}{\rho}}$$

Taking into account that $|\theta(N)| \succ\prec (\log)^{d-1}$ (see. [7]), we have

$$E_{Q(N,\Lambda)}(f)_q << \frac{1}{N^r} |\theta(N)|^{\rho(s-1)} \le \frac{1}{2^{nr}} (n)^{\rho(s-1)}$$

$$E_{Q(N,\Lambda)}(f)_q << \frac{n^{\rho(s-1)}}{2^{nr}}.$$

**Conclusion**

In this paper we obtained estimates of the best approximations of functions of the Besovclass by trigonometric polynomials when the spectrum of the best approximations of a function is determined by a majorant function of a special type $\Omega(t) = \prod_{j=1}^{d} t_j^r \left( \log \frac{1}{t_j} \right)_+^{-\gamma_j}$.

Earlier, in his research Mirbulat Sikhov obtained non-removable direct and inverse theorems of approximation theory for various metrics for trigonometric polynomials with arbitrary spectrum defined by the general form of the major ant function $\Lambda(t)$ under certain conditions of regularity for functions $\Lambda(t)$.

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