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Abdigaliyeva A.N.

Faculty of Mechanics and Mathematics,
al-Farabi Kazakh National University, Almaty, Kazakhstan
e-mail: abdigalieva_an@mail.ru

Modelling of the turbulent energy decay based on the finite-difference and spectral methods

Abstract: The work deals with the modeling of turbulent energy using finite-difference and spectral methods. Simulation of the turbulent process is based on the filtered three-dimensional unsteady Navier-Stokes equations, for the closure of the main equation the dynamic model is used. The mathematical model is solved numerically, the equation of motion is solved by a finite-difference method, the equation for pressure is solved by spectral method. Also new algorithm for the numerical solution of the Poisson equation for finding pressure is developed. In the results of simulation, the change of turbulent kinetic energy over the time, the integral length scale, the change of longitudinal-transverse correlation functions are obtained, and longitudinal and transverse one-dimensional spectra are defined.

Key words: turbulent energy, finite-difference method, spectral method, Poisson equation, cyclic penta-diagonal scheme.

Introduction

Despite of the large number of works devoted to the modeling of turbulent processes in various fields, modeling of complex transitional and turbulent motions using the tools and applications of modern computer technology, new algorithms and approaches of applied mathematics remains relevant direction for scientists involved in applied research. This is explained by the fact that turbulent flow, characterized by a pronounced nonstationarity and nonlinearity of the processes, the presence of large displacements environment diverse, complex interactions, and dissipation of energy can not be accurately described mathematically. The problem of turbulence is still not solved. A study of turbulent processes necessary in connection with large number of devices, where there is turbulent phenomena and natural processes, where also dominated the chaos. From the standpoint of modern fluid mechanics, turbulence is contained a very useful information for engineering practice.

The main objective of the theory of turbulence - the study of the overall dynamics and the nature of turbulence, i.e. the study of the evolution of large-scale structures and statistical representation of the turbulent motion over the time.

In nature and technology, turbulent motion - is the most common form of the movement of liquids or gases. However, a quite universal and valid method for calculating turbulent flows does not exist. This is due to the complexity of the turbulent flow. Turbulence is caused by instability of laminar flow, and its character is determined by the geometry of the flow. Instability leads to the formation of wavy structures that can absorb energy from the main flow. As the wave is grown, the energy will be transferred to other forms of disturbances due to nonlinear effects, and cause disordered ripple, which is usually regarded as a manifestation of turbulence [1-4].

In this work, we make an attempt for solving this problem by using the large eddy simulation method. The idea is to impose in the phase space the initial condition for the field of velocities that satisfies the condition for continuity. Thus the main spectral equation can not be solved and a given initial condition phase space is translated into the physical space using a Fourier transform. The obtained field of velocities is used as the initial condition for the filtered Navier–Stokes equation. Then, the unsteady three dimensional Navier–Stokes equations are solved to simulate the degeneration of the isotropic turbulence.

The isotropic medium in turbulence undergoes a very rapid homogeneous deformation; then, all the characteristic sizes and any averaged characteristics of turbulence are constant, but variable in time. In order to determine turbulent characteristics, it is necessary to numerically model the time variation change in all the parameters and the degeneration of the isotropic turbulence at different Reynolds numbers.

Formulation of the problem

The numerical modeling of the problem is based on the solution of unsteady filtered Navier–Stokes equations with the continuity equation in the Cartesian coordinate system:

$$\begin{cases} \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i \bar{u}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\mathrm{Re}}\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}, \\ \frac{\partial \bar{u}_i}{\partial x_i} = 0, \\ \tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j, \end{cases} \quad (1)$$

where $\bar{u}_i$ – are velocity components, $\bar{p}$ – is the pressure, t – is the time, ν – is the kinematic coefficient of viscosity, $\tau_{i,j}$ – is the sub grid tensor responsible for small scale structures to be simulated, i, j – 1, 2, 3.

For modeling the sub grid tensor a viscosity model is used and it is represented as:

$$\tau_{ij} - \frac{\delta_{ij}}{3}\tau_{kk} = -2\nu_T \bar{S}_{ij}, \quad (2)$$

where $\nu_T = C_S \Delta^2 \left(2\bar{S}_{ij}\bar{S}_{ij}\right)^{1/2}$ – is the turbulence viscosity; C_S – is the empirical coefficient; $\Delta = \left(\Delta_i \Delta_j \Delta_k\right)^{1/3}$ – is the width of the grid filter; $\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$ – is the value of the tensor of deformation of velocities [5].

Boundary conditions are taken as periodic in all directions. The initial values for each component are assigned as functions dependent on wave numbers in the phase space:

$$E(k) = A \times 12\pi k^m \exp^{-\frac{m}{2}\left(\frac{k}{k_0}\right)^2} \quad (3)$$

where $A = \frac{\frac{3}{2}u^2{}_0 m^{(m+1)/2}\sqrt{2}}{12\pi k_0^{m+1}(m-1)!!\sqrt{\pi}}$; $E(k)$ – energy spectrum.

For this problem a variation parameter m and the wave number k_0, which determine the type of turbulence, are chosen. In figure 1 for modeling decay of homogenous turbulence k_0 = 10 and m = 8 are taken.

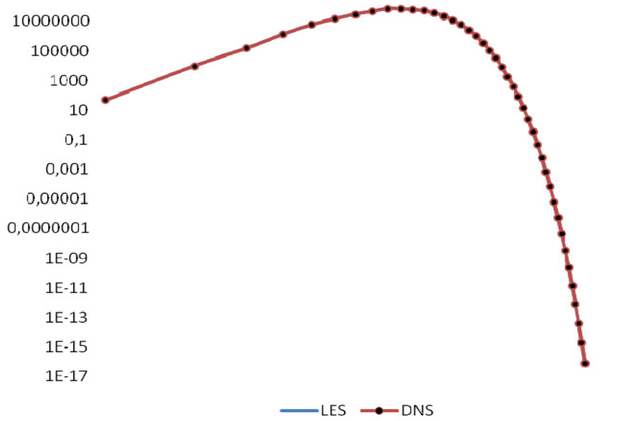

Figure 1 – Energy of the initial level of turbulence based on the fixed wave number k_0 = 10 and power of spectrum m = 8

Numerical method

For solving the Navier–Stokes equation (1), we use a splitting scheme by physical parameters that consist of three stages. At the first stage, the Navier–Stokes equation is solved, without taking pressure into account. For approximation of the convective and diffusion terms of the equation a compact scheme of a fourth order of accuracy O(t^3, h^4) is used. The intermediate field of velocity is found by the fractional step method using the cyclic penta-diagonal matrix method [5]. At the second stage the Poisson equation is solved, which is satisfies the continuity equation with considering the velocity field from the first stage. Obtained the pressure field is used at the third stage for the recallculate of the final velocity field [6].

To solve the three-dimensional Poisson equation the Fourier transform method, which consists of several steps is used. The resulting intermediate velocity field does not satisfy the continuity equation. The exact expression for the new velocity field is obtained by adding to the intermediate field the term corresponding to the pressure gradient:

$$\overline{u}_1^{n+1} = \overline{u}_1^* - \tau \frac{L_3}{L_1}\frac{\partial p}{\partial x_1}; \quad (4)$$

$$\overline{u}_2^{n+1} = \overline{u}_2^* - \tau \frac{L_3}{L_2}\frac{\partial p}{\partial x_2}; \quad \overline{u}_3^{n+1} = \overline{u}_3^* - \tau \frac{\partial p}{\partial x_3}.$$

Substituting the data in the continuity equation we get:

$$\frac{L_3}{L_1}\frac{\partial \overline{u}_1^*}{\partial x_1} + \frac{L_3}{L_2}\frac{\partial \overline{u}_2^*}{\partial x_2} + \frac{\partial \overline{u}_3^*}{\partial x_3} - \tau\left(\frac{L_3^2}{L_1^2}\frac{\partial^2 p}{\partial x_1^2} + \frac{L_3^2}{L_2^2}\frac{\partial^2 p}{\partial x_2^2} + \frac{\partial^2 p}{\partial x_3^2}\right) = 0. \quad (5)$$

Carrying out the transformation, we obtain the Poisson equation for the pressure field:

$$\frac{L_3^2}{L_1^2}\frac{\partial^2 p}{\partial x_1^2} + \frac{L_3^2}{L_2^2}\frac{\partial^2 p}{\partial x_2^2} + \frac{\partial^2 p}{\partial x_3^2} = \frac{1}{\tau}\left(\frac{L_3}{L_1}\frac{\partial \overline{u}_1^*}{\partial x_1} + \frac{L_3}{L_2}\frac{\partial \overline{u}_2^*}{\partial x_2} + \frac{\partial \overline{u}_3^*}{\partial x_3}\right). \quad (6)$$

The equation for pressure is approximated at the point i, j, k takes the following form:

$$\frac{P_{i+1jk} - 2P_{ijk} + P_{i-1jk}}{\Delta x^2} + \frac{P_{ij+1k} - 2P_{ijk} + P_{ij-1k}}{\Delta y^2} + \frac{P_{ijk+1} - 2P_{ijk} + P_{ijk-1}}{\Delta yz^2} = F_{ijk} \quad (7)$$

$$F_{i,j,k} = \frac{1}{\tau}\left(\frac{L_3}{L_1}\frac{\overline{u}^*_{1\,i+\frac{1}{2},j,k} - \overline{u}^*_{1\,i-\frac{1}{2},j,k}}{\Delta x_1} + \frac{L_3}{L_2}\frac{\overline{u}^*_{2\,i,j+\frac{1}{2},k} - \overline{u}^*_{2\,i,j-\frac{1}{2},k}}{\Delta x_2} + \frac{\overline{u}^*_{3\,i,j,k+\frac{1}{2}} - \overline{u}^*_{3\,i,j,k-\frac{1}{2}}}{\Delta x_3}\right) \quad (8)$$

Pressure P_{ijk} in the physical space goes into the next phase using next dysfunctional:

$$P_{ijk} = \sum_{k_3}\sum_{k_2}\sum_{k_1} \hat{p}(k_1,k_2,k_3) \times e^{I(\frac{2\pi i k_1}{N_1} + \frac{2\pi j k_2}{N_2} + \frac{2\pi k k_3}{N_3})} \quad (9)$$

$$F_{ijk} = \sum_{k_3}\sum_{k_2}\sum_{k_1} \hat{F}(k_1,k_2,k_3) \times e^{I(\frac{2\pi i k_1}{N_1} + \frac{2\pi j k_2}{N_2} + \frac{2\pi k k_3}{N_3})} \quad (10)$$

For the Poisson equation a boundary conditions are taken as periodic. For solving the Poisson equation we use spectral method in combination with Fourier transform. Substituting (9) and (10) expressions in equation (7) and performing transformation we get:

$$\widehat{p}(k_1,k_2,k_3)=\frac{\widehat{F}(k_1,k_2,k_3)}{2\left[\frac{\cos(\frac{2\pi k_1}{N_1})-1}{dx^2}+\frac{\cos(\frac{2\pi k_2}{N_2})-1}{dy^2}+\frac{\cos(\frac{2\pi k_3}{N_3})-1}{dz^2}\right]}$$

At the final stage the inverse Fourier transform is performed to obtain the solution of the Poisson equation.

Numerical results

As the result of modeling, the characteristics of the isotropic turbulence are defined. According to the semiempirical theory, the integral scale of turbulence grows with time. The calculation was performed in the area of $L_B=2\pi$ at the grid size of 128x128x128 in space, the time step is dt = 0.001, the kinematic viscosity is ν = (2π) u0 / 500, with the dimensionless parameter of Re = 500. Characteristic values of the speed, and time are taken equal: u_0 = 1, $T_0=\frac{L_B}{u_0}=1$.

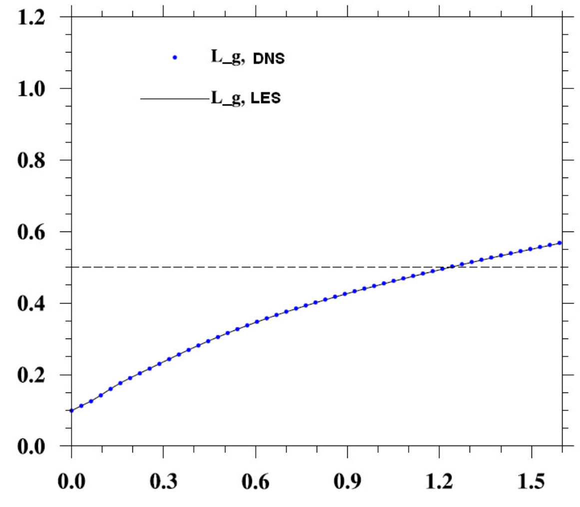

Figure 2 – The changing of integral turbulence scale, calculated at Re = 500, comparison of LES and DNS

In figure 2 results of changing of the integral scale of turbulence over the time, using dimensionless variables *Re* = 500 by simulation LES and DNS are compared. Figure 3 shows the results of the influence of the viscosity on the isotropic turbulence decay of the kinetic energy, calculated in *Re* = 500 and compared with data obtained by the LES. From Figures 2 and 3 can be seen that the integral turbulence on the expiry of the time scale is increased, while the decay of kinetic energy by the time all is rapidly approaching to zero. Also comparing of results LES and DNS has shown that during settlements is not revealed any abnormalities.

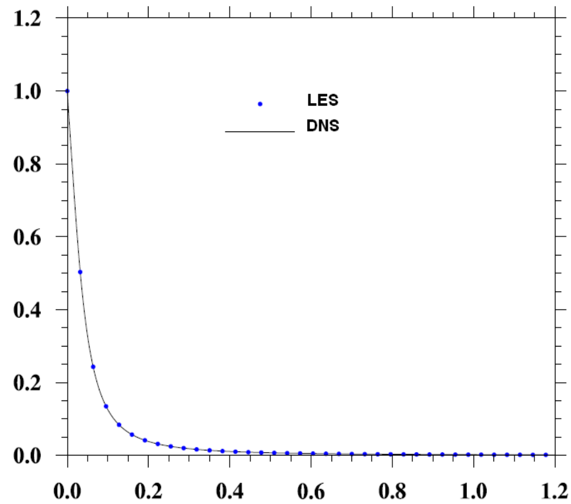

Figure 3 – The changing of turbulent kinetic energy over the time, calculated at Re = 500, comparison of LES and DNS

The correlation coefficients express the average by the volume, a correlation ratio between velocity components at various points. Figure 4 and Figure 5 shows the change of the longitudinal and transverse correlation function *f(r, t)* and *g(r, t)* by the time, calculated at Re = 500. We can see that with the growth of *r* values of functions tends to zero. Comparison of LES and DNS methods of calculation shows that don't occur any significant changes. The changing the longitudinal and transverse dimensional spectrum at different time points is possible to see in figures 6 and 7 respectively.

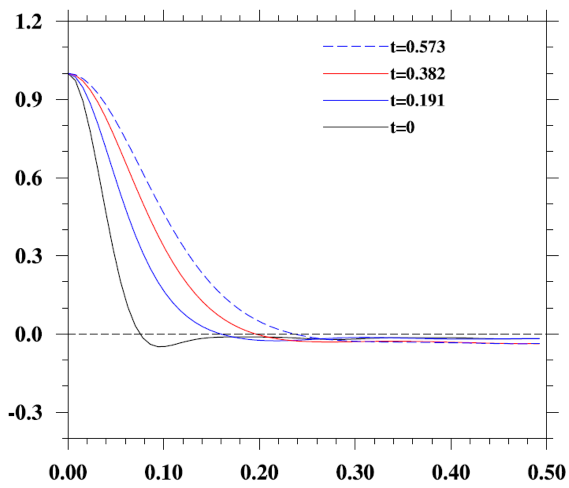

Figure 4 – The changing of longitudinal correlation function *f(r, t)* by the time, calculated at *Re* = 500:
1) *t* = 0.573; 2) *t* = 0.382; 3) *t* = 0.191; 4) *t* = 0

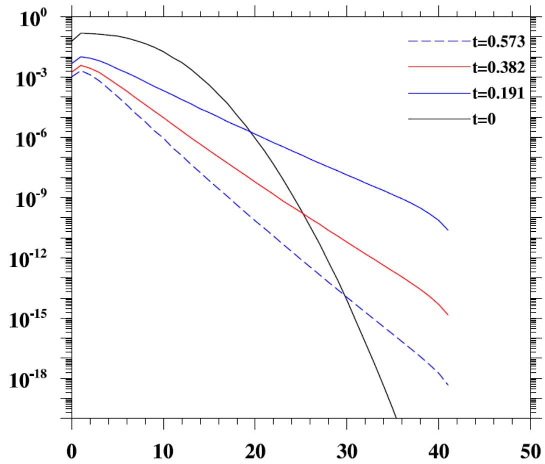

Figure 6 – The changing of a longitudinal one-dimensional spectrum over the time, calculated at *Re* = 500

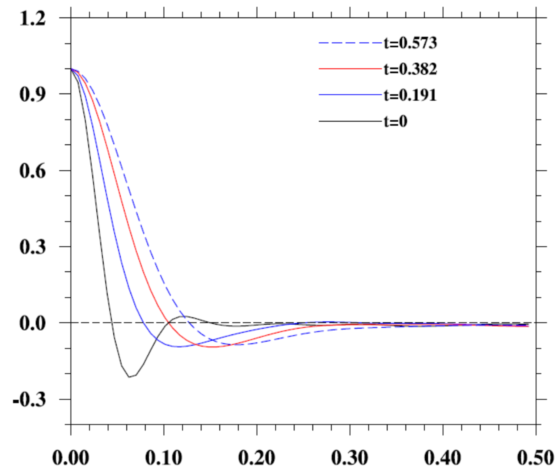

Figure 5 – The changing of transverse correlation function *g(r, t)* over the time, calculated at *Re* = 500:
1) *t* = 0.573; 2) *t* = 0.382; 3) *t* = 0.191; 4) *t* = 0

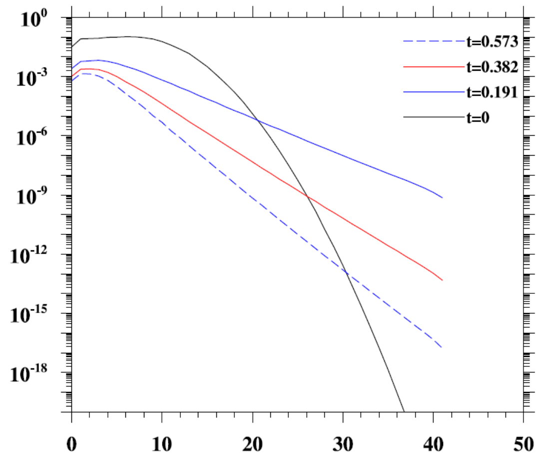

Figure 7 - The changing of a transverse one-dimensional spectrum over the time, calculated at *Re* = 500

Conclusion

In this paper, the numerical simulation of the kinematic viscosity decay the influence on homogeneous isotropic turbulence based on the finite-difference method is considered, and the comparison results with spectral method is shown. The results of numerical modeling, obtained in this paper are fully consistent with results of the authors [7].

Thus, new algorithm for the numerical solution of the Poisson equation for finding pressure is developed. Based on the constructed model the large-scale numerical simulations of isotropic turbulence by LES are carried out and comparison with results of DNS is made. All physical processes and phenomena of homogeneous turbulence are detected in the course of numerical simulation.

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