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Modelling of Coupled Nonlinear Axial and Lateral Vibrations of Drill Strings

Abstract. In this work a nonlinear mathematical model of coupled axial and lateral vibrations of a drill string under the effect of a longitudinal compressing force is investigated. The drill string is modelled in the form of a rotating elastic rod. To solve the model the Bubnov-Galerkin method and numerical stiffness switching method are applied. It is shown that the coupled axial and lateral vibrations of the drill string only arise at odd frequencies in Bubnov-Galerkin's expansion. Numerical analysis of the influence of the drill string geometrical and frequency characteristics on its vibrations is carried out, and the corresponding recommendations are provided.

Key words: drill string, nonlinear model, axial and lateral vibrations, rod, the Bubnov-Galerkin method, stiffness switching method.

Introduction

The oil and gas industry is the leading industry in Kazakhstan and has a huge role to play in national economy. Since the first oil production from the Karashungul field in 1899 and so far the oil and gas industry of Kazakhstan has undergone a rapid expansion and needed continuous improvement of drilling rigs.

One of the main elements of the drilling rig is a drill string which may be subject to various types of vibrations in the process of the borehole drilling. It connects a bit in the bottom hole and drilling equipment located on the surface.

Mathematical modelling of drill string motion is highly nonlinear and fairly complicated due to the drillstring dynamics involving axial, lateral and torsional modes of vibrations [1]. Amongst them, the high-amplitude axial and lateral vibrations determined by the mode of drill string motion not only lead to the additional destruction of rocks and borehole walls and, thereby, to decrease in efficiency of drilling, but also increase risks of drilling equipment wear [2].

Nonlinear dynamic models of coupled axial and lateral vibrations of rotating drill strings taking into account external influences and contact with the borehole wall were studied by A.P. Christoforou and A.S. Yigit [3-4]. Nonlinear mathematical model of a drill string including the effects of bending and torsion and also interactions between the drill string and the well was considered by Melakhessou H. et

al. [5]. However, in the works mentioned the authors only studied the behaviour of the lower part of the drill string (the bottom hole assembly).

This work aims at studying coupled nonlinear axial and lateral vibrations of the whole drill string, presented in the form of a rotating elastic rod. Numerical analysis of vibrations and their visualization are conducted.

Nonlinear Mathematical Model

Modelling of drill string vibrations becomes much more difficult in the case when the string deformations are assumed to be finite and it is necessary to consider geometrical nonlinearity in the model.

To create the mathematical model of drill string vibrations taking into account geometrical nonlinearity the potential of elastic strain Φ constructed on the basis of the V. V. Novozhilov theory of finite strains [6] is used. As a design scheme of the drill string an elastic isotropic rod with pinned ends is accepted. Likewise, the hypothesis of plane sections is applied.

According to the theory of finite strains, the components of the strain tensor ε_{ij} nonlinearly depend on the displacement components $U(x, y, z, t)$, $V(x, y, z, t)$, $W(x, y, z, t)$ over the x -, y - and z -axes, respectively, and enable one to determine completely the rod deformation in any point of the rod.

The elastic potential Φ is expressed in terms of the components of the stress tensor σ_{ij} and the strain tensor ε_{ij} as follows:

$$\Phi = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}. \quad (1)$$

$$\Phi = \left[G \left(1 + \frac{\nu}{1-2\nu} \right) (\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + \frac{2\nu}{1-2\nu} (\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx}) + 2(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2) \right], \quad (2)$$

where $G = \frac{E}{2(1+\nu)}$ is the shear modulus, E is Young's modulus, ν is Poisson's ratio.

Elastic potential (2) is represented further in terms of the elongations e_{xx}, e_{yy}, e_{zz} , shears e_{xy}, e_{yz}, e_{zx} and angles of rotation $\omega_x, \omega_y, \omega_z$ relative to the corresponding coordinate axes, and the V.V. Novozhilov second system of simplifications is accepted [6].

Let us introduce two coordinate systems: a global (fixed) system $Oxyz$ and a local one $Ox'y'z'$ which allows to consider rotation of the drill string. The Oz and Oz' axes are directed along the axis of the rod, rotation is anticlockwise.

The case when vibrations take place in the Oyz -plane is considered. Taking into account the longitudinal displacement of the rod cross-section along the z -axis and its bending along the y -axis, components of displacements are given by

$$U_0 = \frac{G(1-\nu)}{1-2\nu} \int_0^l \left(F \left(\frac{\partial w}{\partial z} \right)^2 + I_x \left(\frac{\partial^2 v}{\partial z^2} \right)^2 \right) dz + \frac{GF}{1-2\nu} \int_0^l \left(\frac{\partial w}{\partial z} \left(\frac{\partial v}{\partial z} \right)^2 + \frac{1}{2} \left(\frac{\partial v}{\partial z} \right)^4 \right) dz, \quad (5)$$

where l is the rod length, F is the cross-section area of the rod.

$$T = \frac{1}{2} \rho \int_0^l \left[F \left(\left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 + v^2 \omega^2 \right) + I_x \left(\frac{\partial^2 v}{\partial z \partial t} \right)^2 + \omega^2 (I_x + I_y) \right] dz, \quad (6)$$

where ρ is density of the material, I_x, I_y are axial inertia moments of the rod cross-section, ω is the angular speed of rod rotation.

Using the physical equations of the generalized Hooke's law for an isotropic body in the case of spatial strain [7], we obtain the following expression for the elastic potential in strains:

$$\begin{cases} V(x, y, z, t) = v(z, t), \\ W(x, y, z, t) = w(z, t) - \frac{\partial v(z, t)}{\partial z} y. \end{cases} \quad (3)$$

Transition from the global coordinate system to the local one is made by using the ratios given in [8].

To derive the mathematical model the variation Hamilton principle is applied, according to that

$$\delta J = \int_{t_1}^{t_2} \delta (T - U_0 + \Pi) dt = 0, \quad (4)$$

where U_0 is the potential energy of rod deformation, T is the kinetic energy, Π is the potential of external forces.

The potential energy of rod deformation is defined by the potential of elastic strain, and after some transformations takes the form:

The expression for kinetic energy of the rod taking into consideration the energy of its nominal rotary motion is written as

The potential of external forces Π allowing for the effect of the longitudinal compressing force $N(z, t)$ on the drill string is determined as follows:

$$\Pi = -\frac{1}{2} \int_0^l N(z,t) \left[\left(\frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 - 2 \frac{\partial w}{\partial z} \frac{\partial^2 v}{\partial z^2} y + \left(\frac{\partial^2 v}{\partial z^2} \right)^2 y^2 \right] dz. \quad (7)$$

Substituting expressions (5)-(7) into (4) and calculating the variation δJ of the action integral, the following nonlinear equations of the coupled

axial and lateral vibrations of the drill string loaded by the axial force $N(z,t)$ are obtained:

$$\begin{aligned} \rho F \frac{\partial^2 v}{\partial t^2} + EI_x \frac{\partial^4 v}{\partial z^4} - \rho I_x \frac{\partial^4 v}{\partial z^2 \partial t^2} + \frac{\partial}{\partial z} \left(N(z,t) \frac{\partial v}{\partial z} \right) - \frac{EF}{1-\nu} \left(\frac{\partial}{\partial z} \left(\frac{\partial v}{\partial z} \frac{\partial w}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial v}{\partial z} \right)^3 \right) - \rho F \omega^2 v = 0, \\ \rho F \frac{\partial^2 w}{\partial t^2} - EF \frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z} \left(N(z,t) \frac{\partial w}{\partial z} \right) - \frac{EF}{1-\nu} \frac{\partial v}{\partial z} \frac{\partial^2 v}{\partial z^2} = 0 \end{aligned} \quad (8)$$

with the boundary conditions of the form

$$\begin{aligned} v(z,t) = 0, \quad EI_x \frac{\partial^2 v(z,t)}{\partial z^2} = 0 \quad (z=0, z=l), \\ EF \frac{\partial w(z,t)}{\partial z} = 0 \quad (z=0, z=l). \end{aligned} \quad (9)$$

Mathematical model (8)-(9) generalizes the known model [9], describing nonlinear vibrations of an elastic rod without rotation and axial compression.

The Bubnov-Galerkin Method

To obtain the solution of model (8)-(9), we apply the Bubnov-Galerkin method that allows to successfully analyze the behaviour of drill strings applied to oil production in vertical and deviated holes [10].

Unlike [11], here multimode approximation of the solution is considered. The compressing load is supposed to be constant and distributed along the rod length, $N(z,t) = N$.

Then, according to the Bubnov-Galerkin method, the lateral displacement $v(z,t)$ in the Oyz -plane and the longitudinal one $w(z,t)$ along the z -axis can be expanded into the following series:

$$v(z,t) = \sum_{i=1}^n f_i(t) \sin\left(\frac{i\pi z}{l}\right), \quad (10)$$

$$w(z,t) = \sum_{i=1}^n g_i(t) \cos\left(\frac{i\pi z}{l}\right). \quad (11)$$

Basis functions $\sin\left(\frac{i\pi z}{l}\right)$ and $\cos\left(\frac{i\pi z}{l}\right)$ were

chosen so that they met boundary conditions (9).

Realization of the Bubnov-Galerkin method and the further numerical solution are conducted in the Wolfram Mathematica 10.0 symbolic mathematical computation program.

To begin with, coupled axial and lateral vibrations of the rod on the main frequency are considered. Substituting functions of displacements (10)-(11) at $n=1$ in equations of motion (8) and requiring, according to the Bubnov-Galerkin method, that the orthogonality condition on the basis functions be satisfied, the following nonlinear system of second order ordinary differential equations relative to new functions $f_1(t)$, $g_1(t)$ is obtained:

$$\begin{aligned} a_1 f_1''(t) + a_2 f_1(t) + a_3 f_1(t) g_1(t) + a_4 f_1^3(t) = 0, \\ b_1 g_1''(t) + b_2 g_1(t) + b_3 f_1^2(t) = 0. \end{aligned} \quad (12)$$

with the coefficients

$$a_1 = \frac{\rho}{2l}(Fl^2 + I_x\pi^2), \quad a_2 = \frac{1}{2l^3}(EI_x\pi^4 - N\pi^2l^2 - \rho F\omega^2l^4),$$

$$a_3 = -\frac{2EF\pi^2}{3l^2(1-\nu)}, \quad a_4 = \frac{3EF\pi^4}{8l^3(1-\nu)},$$

$$b_1 = \frac{\rho Fl}{2}, \quad b_2 = \frac{\pi^2}{2l}(EF - N), \quad b_3 = -a_3.$$

The solution of system of equations (12) is found numerically. For this purpose the stiffness switching method, involving an eighth order explicit Runge-Kutta method and a linearly implicit Euler method, is utilized.

Use of two numerical methods is caused by the fact that the studied equations are stiff. The stiffness switching method, in turn, is highly efficient compared to other numerical methods while solving stiff problems [12].

For numerical computations the following values of the steel drill string parameters are used: $E = 2.1 \times 10^5 \text{ MPa}$, $\rho = 7800 \text{ kg/m}^3$, $\nu = 0.28$, outer

diameter of the drill string $D = 0.2 \text{ m}$, inner diameter $d = 0.12 \text{ m}$, $F = 2.01 \times 10^{-2} \text{ m}^2$, $I_x = 6.84 \times 10^{-5} \text{ m}^4$, $l = 500 \text{ m}$, $\omega = 0.083 \text{ rad/s}$, $N = 2.2 \times 10^3 \text{ H}$.

In figures 1-2 lateral and axial vibrations of the drill string with the functions of displacements $v(z,t)$ and $w(z,t)$, taken in the first approximation by the Bubnov-Galerkin method, at the given parameters of the mechanical system are shown. As can be seen from the following graphs, amplitude of the lateral vibrations is a few orders of magnitude greater than the one of the axial vibrations.

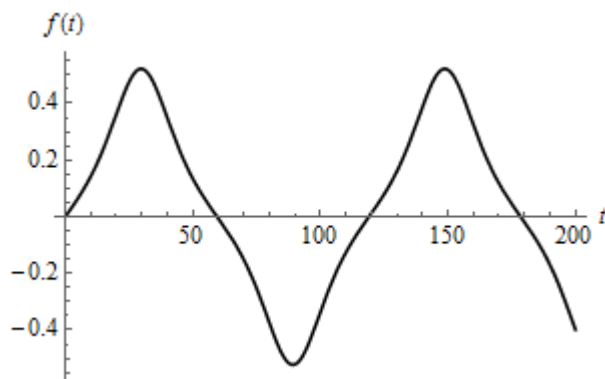


Figure 1 – Lateral vibrations of the drill string on the main frequency

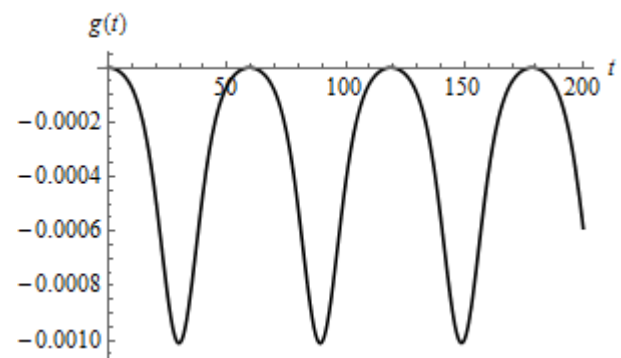


Figure 2 – Axial vibrations of the drill string on the main frequency

Also the cases of three-mode and five-mode approximations for the functions $v(z,t)$ and $w(z,t)$

by series (10), (11), i.e. at $n=3$ (Figs. 3-4) and $n=5$ (Figs. 5-6), are considered.

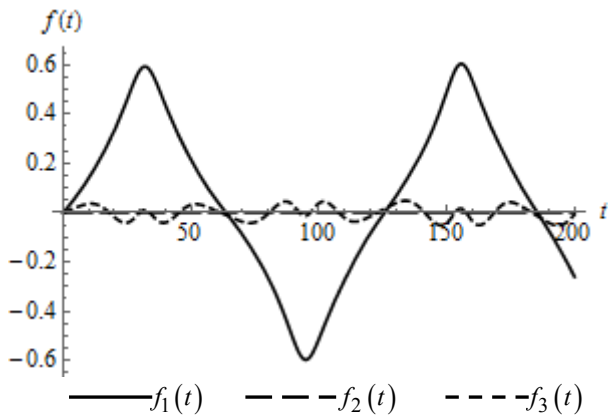


Figure 3 – Lateral vibrations of the drill string on the first three frequencies

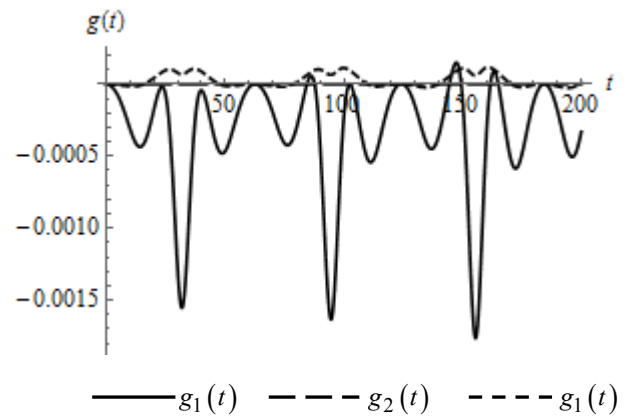


Figure 4 – Axial vibrations of the drill string on the first three frequencies

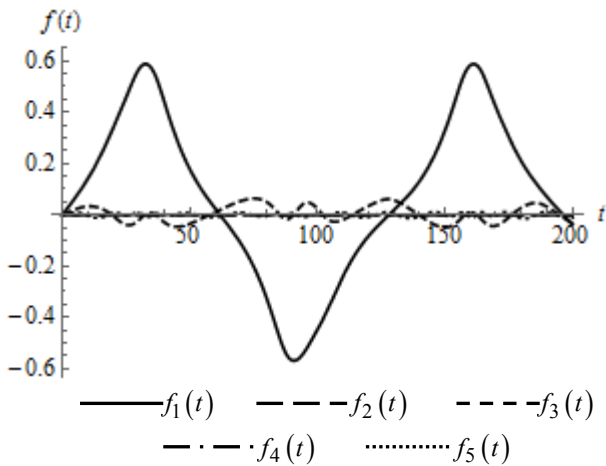


Figure 5 – Lateral vibrations of the drill string on the first five frequencies

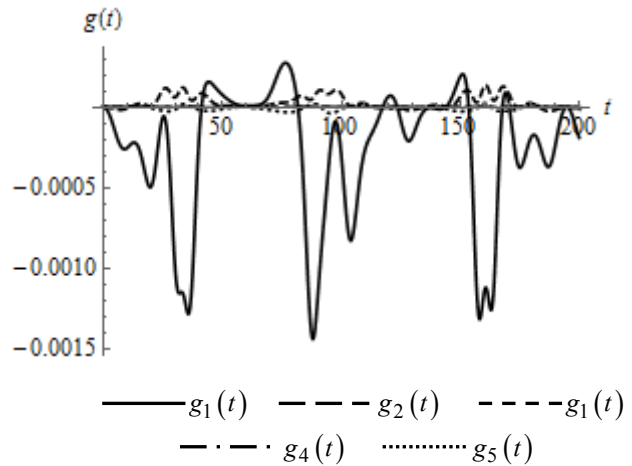


Figure 6 – Axial vibrations of the drill string on the first five frequencies

As clearly demonstrated in figures 3-4, taking into account more vibration modes allows to describe more precisely the oscillatory process of the drill string. However, it worth noting that both the lateral and axial vibrations take place only on the first and third frequencies whilst there are no vibrations on the second frequency.

Figures 5-6 show that the lateral and axial vibrations of the drill string arise only on the first, third and fifth frequencies for the case of $n=5$. On the second and fourth frequencies oscillatory process is not observed.

It follows from the above that when studying coupled nonlinear axial and lateral vibrations of the drill string, the contribution to the oscillatory

process is only made by odd modes, whereas vibrations on even frequencies do not appear.

Thus, the displacement functions $v(z,t)$ and $w(z,t)$ by the Bubnov-Galerkin method can be presented in the following form:

$$v(z,t) = \sum_{i=1}^n f_i(t) \sin\left(\frac{(2i-1)\pi z}{l}\right), \quad (13)$$

$$w(z,t) = \sum_{i=1}^n g_i(t) \cos\left(\frac{(2i-1)\pi z}{l}\right). \quad (14)$$

These representations of the solutions will give the opportunity to carry out the full analysis of axial and lateral vibrations of the drill string without additional expenses of machine time for calculation of missing harmonics.

Numerical Results and Discussions

Successful performing the drilling operations in many respects depends on a right choice of parameters influencing the motion of a drill string. Such parameters are length of the drill string, its radius and thickness of walls, rigidity, angular speed of rotation, axial compressing loading, etc.

To estimate the influence these parameters have on the axial and lateral displacements of the drill string cross-section points, nonlinear model (8)-(9) with the functions of displacements $v(z,t)$ and

$w(z,t)$ in the second approximations, determined from (13)-(14), is considered.

To begin with, we study coupled axial and lateral vibrations of the drill string, modelled in the form of the rotating isotropic rod of symmetric cross-section, at various values of the drill string length, namely at $l = 250m$ and $l = 500m$. The other parameters are left unchanged.

The rod cross-sections near its center $z = 0.49l$ and at the end $z = 0.9l$ are considered.

As can be seen from figures 7-8, the lateral vibrations with increase in length of the rod rise in approximately three times regardless of the cross-section considered. The maximum amplitudes of the vibrations are observed in the central cross-section of the rod, and become much smaller at its end, acquiring the stick-slip nature.

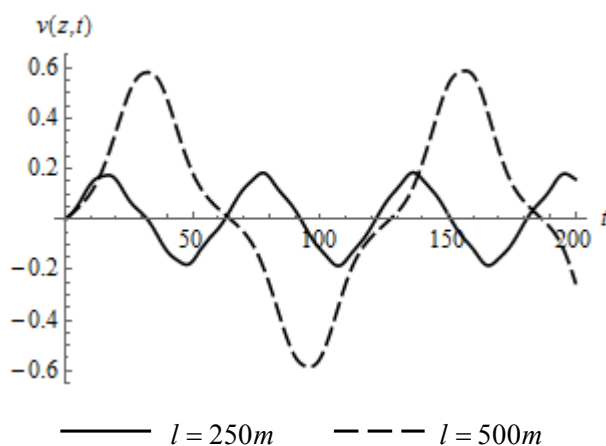


Figure 7 – Lateral vibrations of the drill strings of various lengths in 2nd approximation, $z = 0.49l$

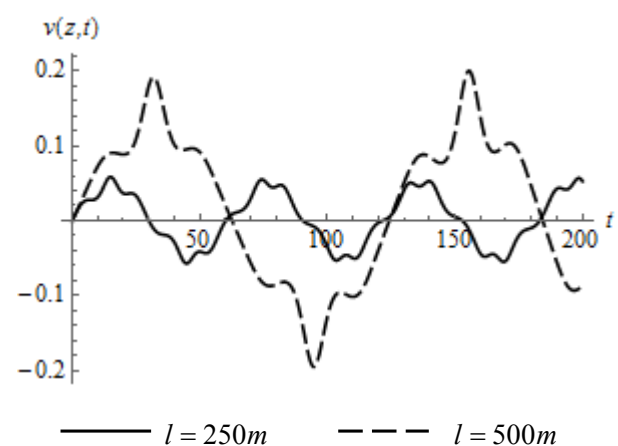


Figure 8 – Lateral vibrations of the drill strings of various lengths in 2nd approximation, $z = 0.9l$

The axial vibrations of the rod, arising due to the effect of the longitudinal compressing load, also rise while increasing its length (Figs. 9-10).

However, in this case the greatest vibrations appear at the ends of the rod, in a neighborhood of its center the axial deformations are negligible.

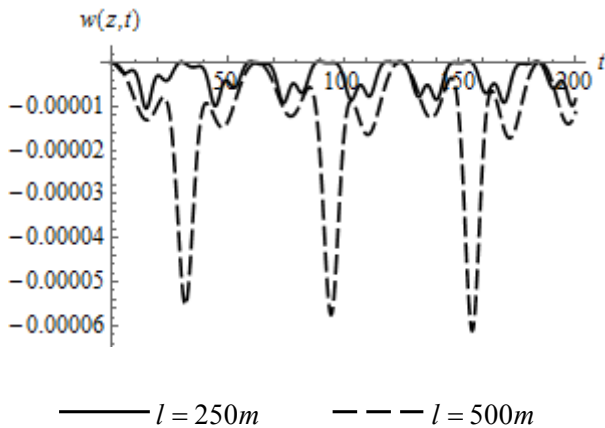


Figure 9 – Axial vibrations of the drill strings of various lengths in the second approximation, $z = 0.49l$

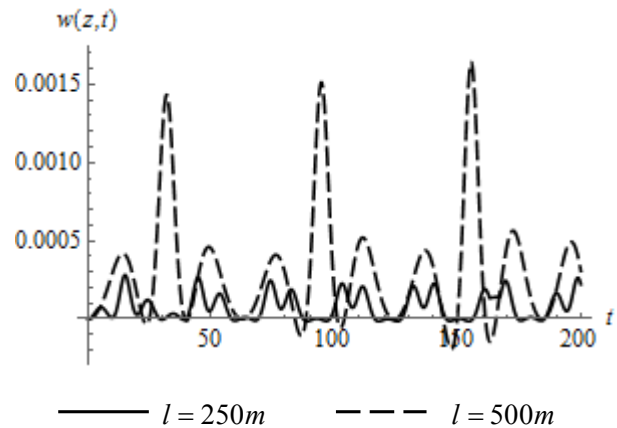


Figure 10 – Axial vibrations of the drill strings of various lengths in the second approximation, $z = 0.9l$

Also, the influence of the drill string angular speed of rotation on the arising axial and lateral vibrations is studied. The rod of length $l = 200m$ with the values of the angular speed $\omega = 0.25 \text{ rad/s}$ and $\omega = 0.5 \text{ rad/s}$ is in consideration.

Figures 11-12 show that increase in the angular speed of rotation of the rod leads to rise of amplitude of its lateral and axial vibrations as well.

Meanwhile, double increase in the angular speed ω causes double increase in the amplitude of the lateral vibrations (Fig. 11), whereas change of this characteristic from 0.25 rad/s to 0.5 rad/s results in more than 4 times rise of the amplitude of the axial ones (Fig. 12). Hence, change of the rod angular speed of rotation has a greater influence on its axial vibrations.

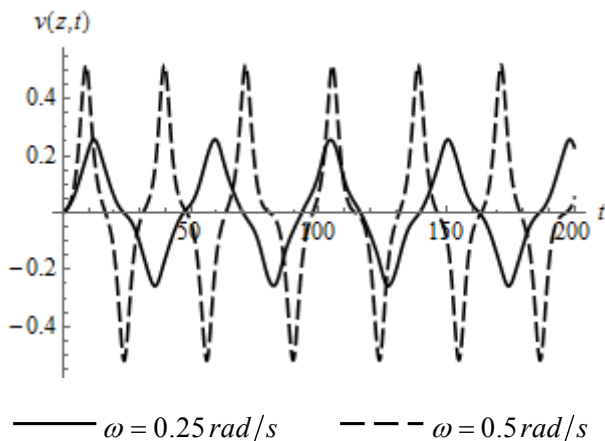


Figure 11 – Lateral vibrations of the drill string of length $l = 200m$ at different values of the angular speed of rotation

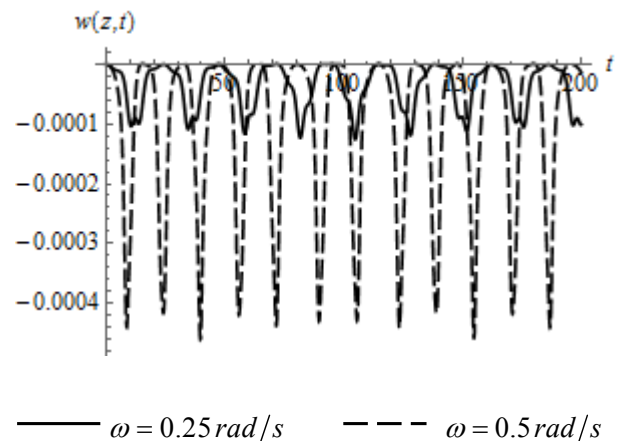


Figure 12 – Axial vibrations of the drill string of length $l = 200m$ at different values of the angular speed of rotation

The drill strings with an external diameter $D = 0.2m$ and internal diameter $d = 0.12m$ have been considered previously. Now it is investigated how much coupled axial and lateral vibrations of the

strings will change with reduction in their external diameter.

It is known that the thicker walls of a drill string, the more durable it is and more reliable drilling of a

borehole may be provided. However, if the vibrations with decrease in the thickness of the drill string walls are still within the range when the drilling process is carried out without the threat of borehole failure or breakage of the drill string, it

may promote considerable economy of the material used in manufacture of drill strings.

For comparison the drill strings with external diameters $D = 0.168m$ and $D = 0.14m$ are considered.

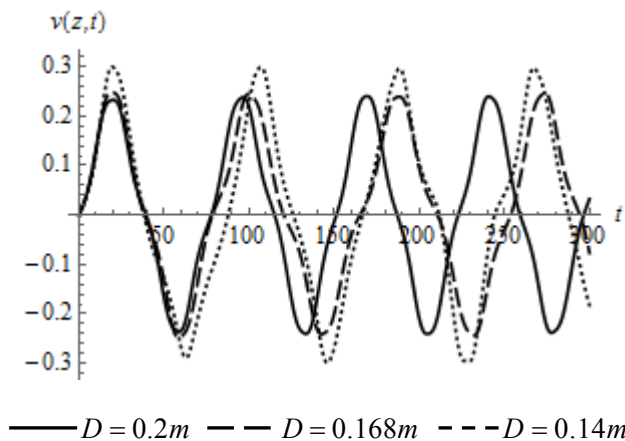


Figure 13 – Effect of the drill string walls thickness on its lateral vibrations at $l = 300m, \omega = 0.083rad/s$

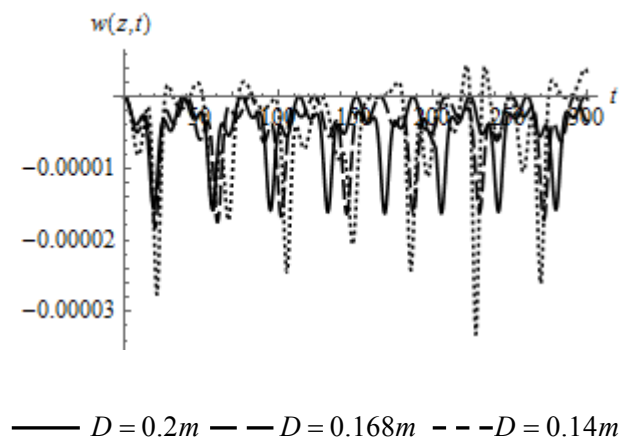


Figure 14 – Effect of the drill string walls thickness on its axial vibrations at $l = 300m, \omega = 0.083rad/s$

As can be seen from figures 13-14, decrease in the external diameter of the drill string from $0.2m$ to $0.168m$ and, consequently, the $0.016m$ decrease in the thickness of the string walls only results in a minor increase in the amplitude of the axial and lateral vibrations.

Further decrease in the external diameter to $0.14m$ causes stronger perturbation of vibrations which, however, is not critical. Thus, we might reduce the amount of material needed for production of the drill string, retaining at the same time stability of the system.

Conclusion

In this work a nonlinear mathematical model of drill string motion, presented in the form of an elastic rod, by introduction of the potential of elastic strain and application of the Hamilton principle was obtained. On the basis of this model numerical analysis of coupled axial and lateral vibrations of the rod under the effect of a constant longitudinal compressing load was carried out. The governing nonlinear system of partial differential equations by means of application of the Bubnov-Galerkin method was reduced to a system of ordinary

differential equations which was solved by the numerical stiffness switching method. It was established that the contribution to the oscillatory process is only made by odd modes of vibrations. It was shown that maximum amplitudes of the lateral vibrations corresponded to the displacement of the central cross-section of the rod, whereas the greatest perturbation of the axial vibrations was observed at the rod ends. Results of the computations at the change of thickness of the drill string walls demonstrated that the steady nature of vibrations retains and gives the opportunity to considerably save the expendable material on the manufacture of drill strings.

Due to the analysis of the drill string axial and lateral vibrations, arising during the drilling of oil and gas wells, it is possible to reduce significantly the emergence of the undesirable phenomena described in this work and to provide steady and reliable drilling.

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