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Baytulenov Zh.B.

Faculty of Mechanics and Mathematics, al-Farabi Kazakh National University, Almaty, Kazakhstan
e-mail: janibekbb@mail.ru

A modification of the method of fictitious domains for stationary model of non-Newtonian liquids

Abstract: The substantiation of one modification method of fictitious domains with continuation on junior coefficient for the stationary equations of non-Newtonian liquids is given in this work. The domain is three-dimensional and bounded. A theorem of an existence and convergence of the generalized solution of an auxiliary problem is proved. The theorem is proved by the method of a priori estimates. Inequality of imbedding theorems, Holder inequality and Young's inequality are used. The sequence of approximate solutions is constructed using the Galerkin method. The limit in the integral identity through the selected sequence. For solutions obtained uniform evaluation standards of functional spaces. An estimation of convergence of the strong solution of the approximating problem is deduced.

Key words: Fictitious domains, hydrodynamics, generalized solution, existence theorem.

Introduction

It is known, that the method of fictitious domains (MFD) is applied for a wide range of mathematical problems. The given method has been described in work [1] for the first time.

The stationary mathematical model of non-Newtonian liquids [2] will be considered in the domain $\Omega \subset R^3$:

$$\operatorname{Re}(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla P = (1 - \alpha)\Delta\mathbf{v} + \nabla \cdot \mathbf{S} + \mathbf{f}, \quad (1)$$

$$\mathbf{S} + \operatorname{We}(\mathbf{v} \cdot \nabla)\mathbf{S} = 2\alpha\mathbf{D}, \quad (2)$$

$$\operatorname{div}\mathbf{v} = 0, \quad (3)$$

with boundary conditions

$$\mathbf{v}|_B = 0, \quad (4)$$

where: $\mathbf{v} = \mathbf{v}(x)$ is a velocity vector of a liquid, $P = P(x)$ is a pressure, $\mathbf{S} = \mathbf{S}(x)$ is an elastic part of stress tensor, the tensor function $\mathbf{D} = (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)/2$ is a tensor deformation rate, $\operatorname{Re} = uL/\mu$ and $\operatorname{We} = \lambda_1 u/L$ are accordingly Reynolds's number and Vensenberg's number, $\alpha = 1 - \lambda_2/\lambda_1$ is numerical parameter, λ_1 is relaxation time, λ_2 – lag time, $0 < \lambda_2 < \lambda_1$. u, L

are the characteristic velocity and the size of model, $\mathbf{f} = \mathbf{f}(x)$ is a vector of mass forces, B is boundary of domain Ω .

The "classical" type of the method of fictitious domains with continuation by junior coefficients is proved for the described model with equations (1) – (4) in the work [3] and the following estimation of convergence rate is obtained for a strong solution:

$$\|\mathbf{v}^\varepsilon - \mathbf{v}\|_{L_2(\Omega)} + \|\mathbf{S}^\varepsilon - \mathbf{S}\|_{L_2(\Omega)} \leq C\varepsilon^{1/4}. \quad (5)$$

Further modification of the method of fictitious domains is researched. An estimation of convergence rate of this method has higher order than the estimation (5).

Main body

Thus, we will consider a modification of the method of fictitious domains (MFD) with continuation by junior coefficients for the problem (1) – (4) in the domain $D_0 \supset \Omega$:

$$\operatorname{Re}(\mathbf{v}^\varepsilon \cdot \nabla)\mathbf{v}^\varepsilon + \nabla P^\varepsilon = (1 - \alpha)\Delta\mathbf{v}^\varepsilon + \nabla \cdot \mathbf{S}^\varepsilon + \mathbf{f} - \frac{\xi(x)}{\varepsilon \|\mathbf{v}^\varepsilon\|_{L_2(D_1)}^\beta} \mathbf{v}^\varepsilon, \quad (6)$$

$$S^\varepsilon + We(v^\varepsilon \cdot \nabla)S^\varepsilon = 2\alpha D^\varepsilon, \quad (7)$$

$$\operatorname{div} v^\varepsilon = 0, \quad (8)$$

$$v^\varepsilon|_{S_1} = 0, \quad (9)$$

where: $0 < \beta < 1$, S_1 is boundary of domain D_0 , $S_1 \cap B = \emptyset$, f is continued by zero outside of domain Ω , $\varepsilon > 0$,

$$\xi(x) = \begin{cases} 0, & \text{in } \Omega \\ 1, & \text{in } D_1 = D_0 \setminus \Omega. \end{cases}$$

$$f^\varepsilon(x) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \in D_1. \end{cases}$$

If we will allow $\beta = 0$ in (6) – (9) such a problem is the "classical" variant of MFD [3]. The used spaces $H(D_0)$ and $H^{-1}(D_0)$ are described in [4].

Definition 1. The functions $v^\varepsilon \in H(D_0)$, $S^\varepsilon \in L_2(D_0)$ are called the generalized solution of the problem (6) – (9) if they satisfy the following integrated identities

$$-\operatorname{Re}((v^\varepsilon \cdot \nabla)\Phi v^\varepsilon)_{L_2(D_0)} + (1-\alpha)(v^\varepsilon, \Phi)_{H(D_0)} = -(S^\varepsilon : \nabla\Phi)_{L_2(D_0)} - \frac{1}{\varepsilon \|v^\varepsilon\|_{L_2(D_1)}^\beta} (v^\varepsilon, \Phi)_{L_2(D_1)} + (f, \Phi)_{L_2(D_0)}, \quad (10)$$

$$(S^\varepsilon : \varphi)_{L_2(D_0)} - We((S^\varepsilon \cdot \nabla)\varphi, v^\varepsilon)_{L_2(D_0)} = 2\alpha(D^\varepsilon : \varphi)_{L_2(D_0)}, \quad (11)$$

$$\forall \Phi \in H(D_0), \varphi \in W_4^1(D_0).$$

Theorem 1. Let $f^\varepsilon \in H^{-1}(D_0)$. Then there is even one generalized solution of the problem (6) –

(9) and there is the next estimation for this solution:

$$\|v^\varepsilon\|_{H(D_0)}^2 + \|S^\varepsilon\|_{L_2(D_0)}^2 + \frac{1}{\varepsilon} \|v^\varepsilon\|_{L_2(D_1)}^{2-\beta} \leq C < \infty.$$

Here and in the further through universal C we will designate the constants only depending from data of the task and various constants from well-known inequalities [4], and not dependent from desired functions and parameter ε .

Proof. The approximate solution of (6) – (9) we shall search in the form of $v_N^\varepsilon(x) = \sum_{j=1}^N \alpha_j \omega_j$ where $\{\omega_j\}$ is the basis in $H(D_0)$. Let numbers α_j are from system of the equations:

$$-\operatorname{Re}((v_N^\varepsilon \cdot \nabla)\omega_j v_N^\varepsilon) + (1-\alpha)(v_N^\varepsilon, \omega_j)_{H(D_0)} = -(S_N^\varepsilon : \nabla\omega_j)_{L_2(D)} - \frac{1}{\varepsilon \|v_N^\varepsilon\|_{L_2(D_1)}^\beta} (v_N^\varepsilon, \omega_j)_{L_2(D_1)} + (f, \omega_j)_{L_2(D_0)}, \quad (12)$$

$$S_N^\varepsilon + We(v_N^\varepsilon \cdot \nabla)S_N^\varepsilon = 2\alpha D_N^\varepsilon, \quad j = 1, 2, \dots, N. \quad (13)$$

For the proof of existence of the solution of the regularizable problem (12) – (13):

$$\begin{aligned}
 & -\operatorname{Re}\left(\left(\mathbf{v}_N^{v,\varepsilon} \cdot \nabla\right) \omega_j \mathbf{v}_N^{v,\varepsilon}\right) + (1-\alpha)\left(\mathbf{v}_N^{v,\varepsilon}, \omega_j\right)_{H(D_0)} = \\
 & = -\left(S_N^{v,\varepsilon} : \nabla \omega_j\right)_{L_2(D)} - \frac{1}{\varepsilon\left\|\mathbf{v}_N^{v,\varepsilon}\right\|_{L_2(D_1)}^\beta}\left(\mathbf{v}_N^{v,\varepsilon}, \omega_j\right)_{L_2(D_1)} + \left(f, \omega_j\right)_{L_2(D_0)},
 \end{aligned} \tag{14}$$

$$\begin{aligned}
 S_N^{v,\varepsilon} + We\left(\mathbf{v}_N^{v,\varepsilon} \cdot \nabla\right) S_N^{v,\varepsilon} & = 2\alpha D_N^{v,\varepsilon} + \nu \Delta S_N^{v,\varepsilon}, \\
 \left.\frac{\partial S_N^{v,\varepsilon}}{\partial n}\right|_{S_1} & = 0.
 \end{aligned} \tag{15}$$

Operating also as in [3], for solution (14) – (15) it is possible to get an estimation:

$$\left\|\mathbf{v}_N^{v,\varepsilon}\right\|_{H(D_0)}^2 + \left\|S_N^{v,\varepsilon}\right\|_{L_2(D_0)}^2 + \nu\left\|\nabla S_N^{v,\varepsilon}\right\|_{L_2(D_0)}^2 + \frac{1}{\varepsilon}\left\|\mathbf{v}_N^{v,\varepsilon}\right\|_{L_2(D_1)}^{2-\beta} \leq C\|f\|_{H^{-1}(D_0)}^2. \tag{16}$$

By virtue of an estimation (16) and Brayer’s lemmas, it is similar [3], existence the theorem for regularized problems (14) – (15) is easy proved.

In an estimation (16) the constant C does not depend from ε, ν . Hence, from sequences $\left\{\mathbf{v}_N^{v,\varepsilon}\right\}, \left\{S_N^{v,\varepsilon}\right\}$ it is possible to allocate the subsequences for which at $\nu \rightarrow 0$ we have correlation:

$\mathbf{v}_N^{v,\varepsilon} \rightarrow \mathbf{v}_N^\varepsilon$ weakly in $H(D_0)$, $\mathbf{v}_N^{v,\varepsilon} \rightarrow \mathbf{v}_N^\varepsilon$ is strongly in $L_4(D_0)$,

$S_N^\varepsilon \rightarrow S_N^\varepsilon$ weakly in $L_2(D_0)$, $\sqrt{\nu} S_N^\varepsilon \rightarrow 0$ weakly in $W_2^1(D_0)$.

Further in (14), multiplying 2-nd equation by $\phi \in W_2^1(D_0)$ and integrating in D_0 , we pass to a limit on the selected sequences and as a result we shall get

$$\begin{aligned}
 & -\operatorname{Re}\left(\left(\mathbf{v}_N^\varepsilon \cdot \nabla\right) \omega_j \mathbf{v}_N^\varepsilon\right)_{D_0} + (1-\alpha)\left(\mathbf{v}_N^\varepsilon, \omega_j\right)_{H(D_0)} = \\
 & = -\left(S_N^\varepsilon : \nabla \omega_j\right)_{L_2(D_0)} - \frac{1}{\varepsilon\left\|\mathbf{v}_N^\varepsilon\right\|_{L_2(D_1)}^\beta}\left(\mathbf{v}_N^\varepsilon, \omega_j\right)_{L_2(D_1)} + \left(f, \omega_j\right)_{L_2(D_0)}, \\
 & \left(S_N^\varepsilon : \varphi\right)_{L_2(D_0)} - We\left(\left(S_N^\varepsilon \cdot \nabla\right) \varphi, \mathbf{v}_N^\varepsilon\right)_{L_2(D_0)} = 2\alpha\left(D_N^\varepsilon : \varphi\right)_{L_2(D_0)}.
 \end{aligned}$$

Hence, we have (12) – (13). Further we shall multiply (12), (13) by ξ_j, S_N^ε accordingly, we

shall integrate over domain D_0 and as a result we shall get

$$\begin{aligned}
 (1-\alpha)\left\|\mathbf{v}_N^\varepsilon\right\|_{H(D_0)}^2 + \frac{1}{\varepsilon}\left\|\mathbf{v}_N^\varepsilon\right\|_{L_2(D_1)}^{2-\beta} & = -\left(S_N^\varepsilon : \nabla \mathbf{v}_N^\varepsilon\right)_{L_2(D_0)} + \left(f, \mathbf{v}_N^\varepsilon\right)_{L_2(D_0)}, \\
 \left\|S_N^\varepsilon\right\|^2 & = 2\alpha\left(D_N^\varepsilon : S_N^\varepsilon\right)_{L_2(D_0)}.
 \end{aligned}$$

When using the equality $S_N^\varepsilon : \nabla v_N^\varepsilon = S_N^\varepsilon : D_N^\varepsilon$, it is easy to deduce the estimation of

$$\|v_N^\varepsilon\|_{H(D_0)}^2 + \|S_N^\varepsilon\|_{L_2(D_0)}^2 + \frac{1}{\varepsilon} \|v_N^\varepsilon\|_{L_2(D_1)}^{2-\beta} \leq C. \tag{17}$$

In estimation in (17) the constant C does not depend from N, ε . Hence, from sequences $\{v_N^\varepsilon\}, \{S_N^\varepsilon\}$ it is possible to allocate subsequences for which:

$$v_N^\varepsilon \rightarrow v^\varepsilon \text{ weakly in } H(D_0),$$

$v_N^\varepsilon \rightarrow v^\varepsilon$ strongly in $L_4(D_0)$,
 $S_N^\varepsilon \rightarrow S^\varepsilon$ weakly in $L_2(D_0)$,
 is correctly at $N \rightarrow \infty$. At that $v^\varepsilon, S^\varepsilon$ are to have a place the estimation

$$\|v^\varepsilon\|_{H(D_0)}^2 + \|S^\varepsilon\|_{L_2(D_0)}^2 + \frac{1}{\varepsilon} \|v^\varepsilon\|_{L_2(D_1)}^{2-\beta} \leq C. \tag{18}$$

So, the theorem – 1 is proved.

Theorem 2. *Let the condition theorem 1 are satisfied. Then the solution of (6) – (9) at $\varepsilon \rightarrow 0$ converges to the solution of (1) – (4).*

The given theorem is proved as well as the theorem 1 on the basis of estimation (18).

Definition 2. *The strong solution of (1) – (4) is referred to the functions $v(x), S(x), P(x)$, possessing the quadratically as the summable derivatives are entering into the equations (1) – (2) and satisfying to the system (1) – (3) and to*

boundary conditions in (4) almost everywhere in Ω by adequate measure.

The concept of the strong solution of an auxiliary problem (6) – (9) is determined similarly.

Now we shall find estimations of convergence rate of the strong solution of (6) – (9) to the strong solution of (1) – (4). We assume that it is executed the following condition:

Condition 1. *Let a strong solution of (6) – (9) and of (1) – (4) exist with an estimation:*

$$\exists C_1, \infty > C_1 > 0 : \|v\|_{L_3(\Omega)} + \|\nabla S\|_{L_3(\Omega)} \leq C_1, \tag{19}$$

and numbers Re and We let satisfy inequality

$$\frac{\sqrt[3]{6}}{\alpha} (C_1 We)^2 - 2 \sqrt[4]{48} C_1 Re < 1 - \alpha. \tag{20}$$

Theorem 3. *Let conditions of theorem 1 and condition 1 are satisfied, then the estimation of*

$$\|v^\varepsilon - v\|_{L_2(\Omega)} + \|S^\varepsilon - S\|_{L_2(\Omega)} \leq C \varepsilon^{\frac{3-2\beta}{4-3\beta}}, \quad 0 < \beta < 1. \tag{21}$$

is fair.

(We shall notice, that at $\beta \rightarrow 1$ this estimation has the order ε , that it is much better than in (5)).

Proof. We shall to multiply (1) by trial function $\psi : \text{div} \psi = 0, \psi|_{S_1} = 0$ scalar in $L_2(\Omega)$:

$$\begin{aligned} & -\text{Re} \int_{\Omega} ((v \cdot \nabla) \psi) dx + (1 - \alpha) \int_{\Omega} \nabla v \cdot \nabla \psi dx - (1 - \alpha) \int_B \frac{\partial v}{\partial n} \psi dB + \\ & + \int_{\Omega} S : \nabla \psi dx - \int_B S \cdot n \cdot \psi dB + \int_B P \cdot n \cdot \psi dB - \int_{\Omega} f \psi dx = 0. \end{aligned} \tag{22}$$

Further we shall to multiply (6) also by ψ scalar in $L_2(D_0)$:

$$\begin{aligned} & - \operatorname{Re} \int_{D_0} \left((\mathbf{v}^\varepsilon \cdot \nabla) \mu \mathbf{v}^\varepsilon \right) dx + (1 - \alpha) \int_{D_0} \nabla \mathbf{v}^\varepsilon \cdot \nabla \psi dx + \int_{D_0} S^\varepsilon \cdot \nabla \psi dx - \\ & - \int_{D_0} f \psi dx + \frac{1}{\varepsilon \|\mathbf{v}^\varepsilon - \mathbf{v}\|_{L_2(D_1)}^\beta} \int_{D_1} \mathbf{v}^\varepsilon \psi dx = 0. \end{aligned} \quad (23)$$

Let us designate execute and shall consider a difference (23) and (22):

$$\begin{aligned} & - \operatorname{Re} \int_{D_0} \left((\omega \cdot \nabla) \omega \mathbf{v}^\varepsilon \right) dx + (1 - \alpha) \int_{D_0} \nabla \omega \cdot \nabla \omega dx + \int_{D_0} \theta \cdot \nabla \omega dx - \\ & - \int_B P \cdot \mathbf{n} \cdot \omega dB + (1 - \alpha) \int_B \frac{\partial \mathbf{v}}{\partial n} \omega dB + \int_B S \cdot \mathbf{n} \cdot \omega dB + \frac{1}{\varepsilon} \|\omega\|_{L_2(D_1)}^{2-\beta} = 0. \end{aligned} \quad (24)$$

Now we shall to multiply (7) by trial function ϕ scalar in $L_2(D_0)$:

$$\int_{D_0} S^\varepsilon : \phi dx - We \int_{D_0} (\mathbf{v}^\varepsilon \cdot \nabla) \phi \cdot S^\varepsilon dx = 2\alpha \int_{D_0} D^\varepsilon : \phi dx, \quad (25)$$

Let us to multiply (2) by ϕ scalar in $L_2(\Omega)$:

$$\int_{\Omega} S \cdot \phi dx - We \int_{\Omega} (\mathbf{v} \cdot \nabla) \phi \cdot S dx = 2\alpha \int_{\Omega} D \cdot \phi dx, \quad (26)$$

Let us continue \mathbf{v}, S outside of Ω zero, we shall take $\phi = \theta$, then from a difference (25) and (26) we shall get:

$$\|\theta\|^2 - We \int_{D_0} (\omega \cdot \nabla) \theta \cdot S dx = 2\alpha \int_{D_0} (D^\varepsilon - D) \cdot \theta dx. \quad (27)$$

Further we shall divide in (27) by 2α and sum up (24):

$$\begin{aligned} & (1 - \alpha) \|\nabla \omega\|_{L_2(D_0)}^2 + \frac{1}{2\alpha} \|\theta\|_{L_2(D_0)}^2 + \frac{1}{\varepsilon} \|\omega\|_{L_2(D_1)}^{2-\beta} - \frac{We}{2\alpha} \int_{D_0} (\omega \cdot \nabla) \theta \cdot S dx = \\ & = \operatorname{Re} \int_{D_0} \left((\omega \cdot \nabla) \omega \mathbf{v}^\varepsilon \right) dx + (1 - \alpha) \int_B \frac{\partial \mathbf{v}}{\partial n} \omega dB + \int_B S \cdot \mathbf{n} \cdot \omega dB - \int_B P \cdot \mathbf{n} \cdot \omega dB. \end{aligned} \quad (28)$$

Let us appreciate some integrals in (28), using inequalities of Gelder and Yunga, the embedding theorems [4] and in (19):

$$\begin{aligned} & \frac{We}{2\alpha} \int_{D_0} (\omega \cdot \nabla) \theta \cdot S dx = - \frac{We}{2\alpha} \int_{D_0} (\omega \cdot \nabla) S \cdot \theta dx \leq \frac{We}{2\alpha} \|\nabla S\|_{L_3} \|\theta\|_{L_2} \|\omega\|_{L_6} \leq \\ & \leq \frac{We}{2\alpha} C_1 \|\theta\|_{L_2} \sqrt[4]{48} \|\omega_x\|_{L_2} \leq \frac{1}{4\alpha} \|\theta\|^2 + \alpha \left(\frac{We}{2\alpha} C_1 \right)^2 \sqrt[3]{48} \|\omega_x\|^2, \end{aligned}$$

$$\begin{aligned} \operatorname{Re} \int_{D_0} ((\omega \cdot \nabla) \omega v^\varepsilon) dx &= \operatorname{Re} \int_{D_0} (\omega \cdot \nabla) \omega v dx \leq \operatorname{Re} \|v\|_{L_3} \|\omega_x\|_{L_2} \|\omega\|_{L_6} \leq C_1 \operatorname{Re} \sqrt[3]{48} \|\omega_x\|^2, \\ \int_B \left((1-\alpha) \frac{\partial v}{\partial n} + S \cdot n - P \cdot n \right) \omega dB &\leq \left(\left\| \frac{\partial v}{\partial n} \right\|_{L_2(B)} + \|S\|_{L_2(B)} + \|P\|_{L_2(B)} \right) \|\omega\|_{L_2(B)} \leq \\ &\leq C \|\omega\|_{L_2(B)} \leq C \|\nabla \omega\|_{L_2(D_1)}^{1/2} \|\omega\|_{L_2(D_1)}^{1/2} \leq C \left(\|\omega_x\|_{L_2(D_1)} \|\omega\|_{L_2(D_1)}^{(2-\beta)/2} \right)^{1/2} \|\omega\|_{L_2(D_1)}^{\beta/4} \leq \\ &\leq C \left(\sqrt{\varepsilon} \|\omega_x\|_{L_2(D_1)}^2 + \frac{1}{\sqrt{\varepsilon}} \|\omega\|_{L_2(D_1)}^{2-\beta} \right)^{1/2} \|\omega\|_{L_2(D_1)}^{\beta/4} \leq \\ &\leq C \left(\|\omega_x\|_{L_2(D_1)}^2 + \frac{1}{\varepsilon} \|\omega\|_{L_2(D_1)}^{2-\beta} \right)^{1/2} \varepsilon^{1/4} \|\omega\|_{L_2(D_1)}^{\beta/4} \leq \\ &\leq \delta \left(\|\omega_x\|_{L_2(D_0)}^2 + \frac{1}{\varepsilon} \|\omega\|_{L_2(D_1)}^{2-\beta} \right) + C_\delta \varepsilon^{1/2} \|\omega\|_{L_2(D_1)}^{\beta/2}. \end{aligned}$$

As a result at $\delta = \frac{1-\alpha}{2}$ it can be obtained from (28)

$$\begin{aligned} \left(\frac{1-\alpha}{2} - \alpha \left(\frac{We}{2\alpha} C_1 \right)^2 \sqrt[3]{48} - C_1 \operatorname{Re} \sqrt[3]{48} \right) \|\omega_x\|_{L_2(D_0)}^2 + \frac{1}{2\alpha} \|\theta\|_{L_2(D_0)}^2 + \\ + \frac{1}{2\varepsilon} \|\omega\|_{L_2(D_1)}^{2-\beta} \leq C \varepsilon^{1/2} \|\omega\|_{L_2(D_1)}^{\beta/2}. \end{aligned} \tag{29}$$

Where we estimate the right part by Yunga 's inequality with indices $p = \frac{2(2-\beta)}{\beta}$, $p' = \frac{2(2-\beta)}{4-3\beta}$:

$$\varepsilon^{1/2} \|\omega\|_{L_2(D_1)}^{\beta/2} \leq \frac{1}{4\varepsilon} \|\omega\|_{L_2(D_1)}^{2-\beta} + C \varepsilon^{\frac{6-4\beta}{4-3\beta}}.$$

Then in view of (20) from (29) the estimation is follows:

$$\|\omega_x\|_{L_2(D_0)}^2 + \frac{1}{\varepsilon} \|\omega\|_{L_2(D_1)}^{2-\beta} + \|\theta\|_{L_2(D_0)}^2 \leq C \varepsilon^{\frac{6-4\beta}{4-3\beta}},$$

i.e. we shall get required estimation of convergence rate (21). The theorem 3 is proved.

Conclusion

A theorem of an existence and convergence of the generalized solution of an auxiliary problem is proved. An estimation of convergence of the strong solution of the approximating problem is deduced.

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