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GLOBAL ATTRACTORS AND COMMON DIAGONAL LYAPUNOV METRICS FOR QUASILINEAR REACTION–DIFFUSION SYSTEMS

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Abstract. This paper studies autonomous uniformly parabolic reaction–diffusion systems with conormal boundary conditions and a principal part diagonal in the component index. Each diffusion tensor may depend on the full state vector. The aim is to obtain compatible global solvability, dissipativity and coefficient-perturbation estimates under explicit assumptions on an invariant rectangular state region. The analysis combines variational truncations, scalar boundary regularity, continuation in the strong trace space, matrix comparison and energy estimates. For sufficiently smooth coefficients, global strong solutions and uniform positive-time smoothing are established without a smallness restriction on the state derivatives of the diffusion tensors. The resulting global attractor is compact and attracting in the strong trace norm. Uniform entrywise bounds on the reaction Jacobians provide a computable sufficient construction of a common diagonal Lyapunov metric and an explicit dissipation margin. This application of classical matrix theory yields energy and attractor-radius estimates and criteria for convergence to a single stationary state. A nonlinear example exhibits a varying nonsymmetric Jacobian family for which the identity metric fails but the constructed metric gives an explicit decay rate. Weak–strong uniqueness and finite-time stability under coefficient perturbations are also proved. For coefficients involving normalized fractions, a common-diffusion example verifies preservation of positive total concentration. The results identify coefficient conditions and quantitative continuous estimates that can guide the analysis of multicomponent transport models and subsequent structure-preserving discretizations.

Keywords: quasilinear parabolic systems; diagonal stability; Lyapunov metric; global attractor; weak–strong uniqueness

INTRODUCTION

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with boundary of class $C^{2+\gamma}$, where $\gamma \in (0,1)$, and let

$$\alpha = (\alpha_1, \dots, \alpha_K) : \Omega \times (0,\infty) \longrightarrow \mathbb{R}^K.$$

We study the autonomous system

$$\partial_t \alpha_k = \nabla \cdot \big(D_k(x,\alpha)\nabla\alpha_k\big) + R_k(x,\alpha), \qquad k = 1,\dots,K, \tag{1}$$

subject to the conormal conditions

$$D_k(x,\alpha)\nabla\alpha_k \cdot n = 0 \quad \text{on } \partial\Omega \times (0,\infty). \tag{2}$$

Here n is the outward unit normal. The principal part is diagonal with respect to the component index, but each tensor $D_k(x,a)$ may depend on the complete state a. The components are therefore coupled through the reaction term and the coefficients of the diagonal fluxes.

Global solvability for (1) requires estimates compatible with its nonlinear conormal boundary condition and natural trace space. The local theory is provided by Amann's quasilinear parabolic framework [1, 2, 3, 4, 5] and maximal-regularity methods [6, 7]. Invariant-region theory [8, 9] motivates the inward-pointing condition. Once the state remains bounded, the componentwise diagonal principal part permits scalar boundary estimates: De Giorgi–Nash regularity [10, 11], conormal gradient estimates [12], and oblique Schauder estimates [13, 14, 15]. These are established analytical tools; the required continuation step is their successive closure with controlled coefficient norms on fixed-length time windows.

For long-time dynamics, diagonal stability and its relation to nonsingular M-matrices are classical [16, 17, 18]. Common diagonal metrics have been studied both for matrix families [19] and for interval matrices [20]. We therefore do not claim

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a new general matrix-stability principle. Theorem 14 uses a conservative comparison construction to obtain explicit weights and a common margin from the reaction derivative bounds. Its role is to make the nonlinear PDE energy estimates directly verifiable from the same coefficient data.

Contribution and relation to previous results

The specific result concerns the following combination: an invariant rectangle, full-state dependence of each diagonal diffusion tensor, nonlinear conormal compatibility, and no smallness restriction on $D_a D_k$. Theorems 9 and 11 close the scalar regularity estimates in this setting, with constants independent of the terminal time and of the initial trace norm after any fixed positive time. Theorem 21 then gives compactness and attraction in the strong trace norm. This last step uses standard attractor theory [21, 22, 23, 24]. The two-space formulation [25] follows from this stronger result and expresses its energy-topology consequences.

Within this PDE class, Theorem 14 and the segment argument connect the entrywise Jacobian bounds to an explicit nonlinear dissipation margin, the radius estimate (25), and the singleton criteria of Theorems 24 and 25. Unlike a matrix-family criterion alone, this formulation also controls the nonlinear state-dependent fluxes, boundary regularity and compact solution tails. Example 35 computes the weights and decay rate for a nonlinear nonsymmetric family on a verified invariant rectangle; the identity metric fails even at its equilibrium. These PDE conclusions and the worked example specify the scope of the contribution, rather than treating the standard regularity or M-matrix arguments themselves as new results.

The coefficient-perturbation results use Aubin–Lions–Simon compactness [26] and energy comparison. They include normalized fractions $a_k/\sum_j a_j$ on a state region whose total concentration has a fixed positive lower bound. The quantitative estimate is finite-time. Example 36 supplies one mechanism preserving the lower bound. The principal part remains diagonal in the component index throughout; genuinely nondiagonal cross-diffusion systems, such as the Shigesada–Kawasaki–Teramoto class [27, 28, 29, 30], are outside the present assumptions. We use $d \in \{2,3\}$ to cover the usual planar and spatial transport settings; no separate dimension-specific mechanism is invoked.

Organization of the paper

After specifying the coefficient classes and phase spaces, we prove positive-time smoothing and global continuation. We then construct a common diagonal metric, derive the quadratic dissipative estimate, and establish the attractor results. The subsequent sections treat weak–strong uniqueness and coefficient perturbations. The final two sections discuss the scope of the results and present the conclusions.

MATERIALS AND METHODS

This is an analytical study of (1)–(2); no empirical data or numerical approximation is used. The basic state region is $Q_M = [0,M]^K$, with coefficients regular on an open neighborhood. A local coefficient class treats normalized fractions away from zero total concentration. Throughout, local maximal regularity, scalar boundary estimates and energy identities are applied on these specified regions. Each stage of the positive-time regularity argument records the coefficient norms needed at the next stage. The explicit examples check the state constraints, matrix inequalities and resulting dynamical conclusions directly.

STRUCTURAL CLASSES AND PHASE SPACES

For $M, \delta > 0$, set

$$Q_M = [0,M]^K, \qquad Q_M^\delta = (-\delta, M+\delta)^K.$$

We write

$$\mathbb{R}^{d\times d}_{\mathrm{sym}} = \{A \in \mathbb{R}^{d\times d} : A^T = A\}$$

for the space of real symmetric $d \times d$ matrices.

Definition 1 (Basic coefficient class)**.** *The system belongs to $\mathscr{D}_0(\Omega, K, M)$ if:*

(A1) $D_k \in C^{2+\gamma}(\overline{\Omega} \times Q_M^\delta; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, *and constants* $0 < d_* \le d^*$ *satisfy*

$$d_*|\xi|^2 \le \xi^T D_k(x,a)\xi \le d^*|\xi|^2$$

for every $(x,a,\xi)\in\overline{\Omega}\times Q_M^{\delta}\times\mathbb{R}^d$;

(A2) $R\in C^{1+\gamma}(\overline{\Omega}\times Q_M^{\delta};\mathbb{R}^K)$;

(A3) the reaction points inward on every face of Q_M:

$$R_k(x,a)\ge 0 \quad \text{if } a_k=0,$$
$$R_k(x,a)\le 0 \quad \text{if } a_k=M,$$

for all $x\in\overline{\Omega}$ *and all points* $a\in Q_M$ *on the indicated face, including its edges and vertices;*

(A4) the coefficients are autonomous in time.

Let $0<\delta_0\le KM$ and define

$$Q_{M,\delta_0}=\left\{a\in Q_M:\sum_{k=1}^K a_k\ge\delta_0\right\},$$
$$\mathscr{O}_{\delta_0}=\left\{a\in Q_M^{\delta}:\sum_{k=1}^K a_k>\delta_0/2\right\}.$$

We use the closed state neighborhood

$$K_{\delta_0}=\left\{a\in\overline{Q_M^{\delta/2}}:\sum_{k=1}^K a_k\ge 3\delta_0/4\right\}.$$

It is compact and satisfies $Q_{M,\delta_0}\subset K_{\delta_0}\Subset\mathscr{O}_{\delta_0}$.

Definition 2 (Local class at positive total concentration)**.** *The coefficients belong to* $\mathscr{D}_0^{\delta_0}(\Omega,K,M,\delta_0)$ *if, for every compact* $K'\Subset\mathscr{O}_{\delta_0}$,

$$D_k\in C^{2+\gamma}(\overline{\Omega}\times K';\mathbb{R}^{d\times d}_{\mathrm{sym}}),$$
$$R\in C^{1+\gamma}(\overline{\Omega}\times K';\mathbb{R}^K).$$

The tensors are uniformly elliptic on K_{δ_0}*, the inward-pointing conditions hold at the coordinate faces of* Q_{M,δ_0}*, and the coefficients are autonomous. Positive invariance of the lower total-concentration bound is a separate assumption.*

The regularity in this definition is understood on neighborhoods of compact subsets, with x-regularity up to $\partial\Omega$. It accommodates coefficients containing a_k/a_0, where $a_0=\sum_j a_j$, without requiring an extension through $a_0=0$.

For scalar symmetric uniformly elliptic conormal operators, ellipticity implies the complementing condition. We next state the reaction property used in the Lyapunov estimates.

Definition 3 (Diagonal dissipativity)**.** *Let* $P=\mathrm{diag}(\pi_1,\dots,\pi_K)$ *with* $\pi_k>0$ *and let* $\sigma_R>0$*. A system in* $\mathscr{D}_0$ *belongs to* $\mathscr{D}_{\mathrm{diss}}(P,\sigma_R)$ *if*

$$\big(R(x,a)-R(x,b)\big)^T P(a-b)\le-\sigma_R|a-b|^2 \quad (3)$$

for all $x\in\overline{\Omega}$ *and* $a,b\in Q_M$.

Set $\pi_{\min}=\min_k\pi_k$ and $\pi_{\max}=\max_k\pi_k$, and define

$$\|w\|_{P,L^2}^2=\int_\Omega w(x)^T Pw(x)\,\mathrm{d}x. \quad (4)$$

Then

$$\pi_{\min}\|w\|_{L^2}^2\le\|w\|_{P,L^2}^2\le\pi_{\max}\|w\|_{L^2}^2.$$

Fix $r>d+2$ and let

$$E_r=(L^r(\Omega),W^{2,r}(\Omega))_{1-1/r,r}$$
$$=B^{2-2/r}_{r,r}(\Omega),\qquad \vartheta_r=1-\frac{d+2}{r}>0.$$

For every $0\le\mu<\vartheta_r$,

$$E_r\hookrightarrow C^{1,\mu}(\overline{\Omega}),\qquad E_r\Subset C^{1,\mu}(\overline{\Omega}). \quad (5)$$

The strong phase space is

$$\mathscr{X}_r=\left\{u\in E_r^K:\ \begin{array}{l}0\le u_k\le M,\\ D_k(x,u)\nabla u_k\cdot n=0 \text{ on } \partial\Omega\end{array}\right\}, \quad (6)$$

and the energy phase space is

$$\mathscr{Y}=\{u\in L^2(\Omega)^K:0\le u_k\le M \text{ a.e.}\} \quad (7)$$

with its $L^2(\Omega)^K$ metric. For the local class, set

$$\mathscr{X}_r^{\delta_0}=\{u\in E_r^K:0\le u_k\le M,\ \textstyle\sum_k u_k\ge\delta_0,$$
$$D_k(x,u)\nabla u_k\cdot n=0 \text{ on } \partial\Omega\}. \quad (8)$$

Positive invariance of $\mathscr{X}_r^{\delta_0}$ is an additional hypothesis for the global theory of the local class.

POSITIVE-TIME SMOOTHING AND THE GLOBAL SEMIFLOW

Local theory and the invariant rectangle

Theorem 4 (Local strong solvability)**.** *Let the system belong to* $\mathscr{D}_0$*, let* $r>d+2$*, and let* $\alpha^{\mathrm{in}}\in\mathscr{X}_r$*. There is a unique maximal strong solution*

$$\alpha\in W^{1,r}_{\mathrm{loc}}([0,T_{\max});L^r(\Omega)^K)$$
$$\cap L^r_{\mathrm{loc}}([0,T_{\max});W^{2,r}(\Omega)^K)$$
$$\cap C([0,T_{\max});E_r^K).$$

It depends continuously on the initial state on compact time intervals. If $T_{\max} < \infty$, its positive-time orbit cannot be contained in a compact subset of the admissible phase manifold whose state range lies compactly inside the coefficient domain.

The same statement holds for a system in $\mathscr{D}_0^{\delta_0}$ and initial data in $\mathscr{X}_r^{\delta_0}$ while its state range remains compactly inside the open part of $\mathscr{O}_{\delta_0}$ where $D_k(x,a)$ is positive definite for every k and every $x \in \overline{\Omega}$. This open elliptic region contains K_{δ_0}.

Proof. Existence of a maximal solution and a local semiflow follows from the quasilinear theory with nonlinear boundary conditions [1, 2], in the maximal-regularity setting [5, 6, 7]. The principal symbol is

$$\operatorname{diag}\big(\xi^T D_1(x,a)\xi, \dots, \xi^T D_K(x,a)\xi\big).$$

It is normally elliptic uniformly on the relevant compact state set. Symmetry and positive definiteness give the Lopatinskii–Shapiro condition for each conormal operator. Linearization of the nonlinear boundary condition has this same principal boundary part; differentiating D_k with respect to the state contributes only terms of order zero in the perturbation at the boundary.

To record the coefficient dependence, let $U \subset E_r^K$ be bounded, with values in one compact subset of the coefficient domain, and put

$$A_k(u)w = -\nabla \cdot (D_k(\cdot,u)\nabla w),$$
$$B_k(u)w = D_k(\cdot,u)\nabla w \cdot n.$$

The mean-value theorem and bounded second derivatives give

$$\|D_k(\cdot,u) - D_k(\cdot,v)\|_{L^\infty} \le C_U \|u-v\|_{C^0},$$
$$\|\nabla_x[D_k(\cdot,u) - D_k(\cdot,v)]\|_{L^\infty} \le C_U \|u-v\|_{C^1}.$$

Indeed, the total spatial derivative is

$$\nabla_x D_k(x,u(x)) = D_{k,x}(x,u(x)) + D_{k,a}(x,u(x))\nabla u(x),$$

and the difference is controlled by $D_{k,xa}$ and $D_{k,aa}$. Consequently,

$$\|[A_k(u) - A_k(v)]w\|_{L^r} \le C_U \|u-v\|_{C^1} \|w\|_{W^{2,r}}.$$

The trace theorem and multiplication by $C^1(\partial\Omega)$ functions also give

$$\|[B_k(u) - B_k(v)]w\|_{W^{1-1/r,r}(\partial\Omega)} \le C_U \|u-v\|_{C^1} \|w\|_{W^{2,r}}.$$

Together with $E_r \hookrightarrow C^{1,\mu}$ and the boundedness of R_a, these estimates verify the spatial local Lipschitz properties. The full anisotropic boundary and evolution theory is supplied by the cited nonlinear-boundary results. The same argument applies on a fixed compact subset of the elliptic region for the local coefficient class. That region is open by continuity of the tensors and compactness of $\overline{\Omega}$; uniform ellipticity on K_{δ_0} ensures that it contains a neighborhood of this compact state set.

We use the compact-orbit continuation consequence of this local semiflow theory, consistent with [3, 4]. A compact subset of the admissible phase manifold admits a finite covering by local existence neighborhoods. Decreasing the local times gives one positive existence time on that compact set. Restarting a trajectory in it at times approaching a finite $T_{\max}$ would extend the solution beyond $T_{\max}$. The smoothing estimate below supplies the required compactness. □

Let $\Pi_M : \mathbb{R}^K \to Q_M$ denote the componentwise projection. The following direct argument is the version of the invariant-region principle needed here; see [8, 9] for its general context.

Lemma 5 (Invariant rectangle)**.** *If $\alpha^{\mathrm{in}} \in \mathscr{X}_r$, then*

$$0 \le \alpha_k(x,t) \le M, \quad k = 1,\dots,K, \quad 0 < t < T_{\max}.$$

Proof. Fix $\delta_1 \in (0,\delta)$ and stop the solution at its first exit from $Q_M^{\delta_1}$. On this interval R is uniformly Lipschitz in a, with constant L_R. Choose convex functions $\psi_\eta^{\pm} \in C^2(\mathbb{R})$ such that

$$\psi_\eta^-(s) \longrightarrow \tfrac{1}{2}(s^-)^2,$$
$$\psi_\eta^+(s) \longrightarrow \tfrac{1}{2}((s-M)^+)^2$$

in C^1_{loc}, with $(\psi_\eta^{\pm})'' \ge 0$, $(\psi_\eta^-)' = 0$ on $[0,\infty)$ and $(\psi_\eta^+)' = 0$ on $(-\infty, M]$. Here $s^- = \max\{-s, 0\}$

and $s^+ = \max\{s,0\}$. Testing the kth equation by $(\psi_\eta^\pm)'(\alpha_k)$ is legitimate in the strong solution class. The conormal boundary term vanishes. Ellipticity and convexity make the diffusion contribution on the left nonnegative; no state derivative of D_k occurs in this divergence-form calculation.

After letting $\eta \downarrow 0$, set

$$V(t) = \tfrac{1}{2}\sum_{k=1}^{K} \left(\|\alpha_k^-(t)\|_{L^2}^2 + \|(\alpha_k(t)-M)^+\|_{L^2}^2 \right).$$

On $\{\alpha_k < 0\}$, the point $\Pi_M \alpha$ lies on the lower kth face; hence

$$-R_k(x,\alpha)\alpha_k^- \le L_R|\alpha - \Pi_M\alpha|\alpha_k^-.$$

The upper face is treated in the same way. Summation yields $V'(t) \le C(K,L_R)V(t)$ in distributions. Since $V(0) = 0$, Gronwall's inequality gives $V \equiv 0$. An exit from the expanded rectangle is therefore impossible. □

Linear boundary estimates

For $a < b$, write $Q_{a,b} = \Omega \times (a,b)$. The Hölder spaces below are parabolic. Their norms use spatial and temporal suprema and seminorms, so estimates on overlapping windows are patched by a supremum, without summing over the number of windows.

Lemma 6 (Boundary Hölder estimate by reflection)**.** *Let $v \in L^\infty(Q_{a,b}) \cap L^2(a,b;H^1(\Omega))$ be a weak solution of*

$$\partial_t v - \nabla\cdot(A(x,t)\nabla v) = g, \quad A\nabla v\cdot n = 0, \tag{9}$$

where $A = A^T$ is measurable and uniformly elliptic and $g \in L^\infty(Q_{a,b})$. There is $\beta_0 \in (0,1)$ such that, for every $\tau \in (0, b-a)$,

$$\begin{aligned} &\|v\|_{C^{\beta_0,\beta_0/2}(\overline{\Omega}\times[a+\tau,b])} \\ &\quad \le C_\tau\left(\|v\|_{L^\infty(Q_{a,b})} + \|g\|_{L^\infty(Q_{a,b})}\right). \end{aligned} \tag{10}$$

The constant depends on τ, the dimension, the domain and its boundary atlas, and the ellipticity bounds, but not on b.

Proof. Interior points are covered by De Giorgi–Nash theory [11]. Near a boundary point, after a rigid motion write $x_d = \varphi(x')$ and use the volume-preserving map

$$\Psi(y',y_d) = (y', y_d + \varphi(y')), \qquad F = D\Psi, \quad \det F = 1.$$

For $w(y,t) = v(\Psi(y),t)$, the transformed tensor is

$$\widetilde{A}(y,t) = F(y)^{-1}A(\Psi(y),t)F(y)^{-T}.$$

It remains measurable, symmetric and uniformly elliptic, with constants controlled by the fixed boundary atlas. The conormal condition becomes $\widetilde{A}\nabla_y w \cdot e_d = 0$ on $y_d = 0$.

Let $S = \mathrm{diag}(1,\dots,1,-1)$. Extend w and the transformed source evenly, and use

$$\widetilde{A}_e(y',y_d,t) = S\widetilde{A}(y',-y_d,t)S \quad (y_d < 0).$$

Adding the weak formulations over the upper and reflected lower half-balls shows that the even extension solves an interior divergence-form equation. The zero conormal trace eliminates an interface source. The interior oscillation estimate thus gives a Hölder bound up to the boundary; see also [10] for local estimates for boundary value problems.

The reflected tensor may jump across the flat boundary. This is harmless for the present measurable-coefficient estimate. The higher regularity estimates below are applied directly to the original conormal and oblique problems.

Set $\ell_0 = \min\{\tau/8, 1\}$. For each $t \in [a+\tau, b]$, apply the local estimates on $(t-2\ell_0, t)$ and retain $(t-\ell_0, t)$. All windows have the same length and the spatial covering is finite. Close pairs lie in a common enlarged window; distant-pair Hölder quotients are controlled by the L^∞ norm. This proves (10), with no dependence on the number or position of the windows. □

Lemma 7 (Boundary gradient regularity)**.** *Assume the hypotheses of Lemma 6 and, in addition,*

$$A, g \in C^{\beta_0,\beta_0/2}(\overline{\Omega}\times[a+\tau/4, b]).$$

Put

$$\Lambda_A = \|A\|_{C^{\beta_0,\beta_0/2}(\overline{\Omega}\times[a+\tau/4,b])}.$$

There is $\beta_1 \in (0, \min\{\beta_0, \gamma\})$ *such that*

$$\|v\|_{C^{1+\beta_1,(1+\beta_1)/2}(\overline{\Omega}\times[a+\tau/2,b])} \le C_7 \left(\|v\|_{L^\infty(Q_{a,b})} + \|g\|_{C^{\beta_0,\beta_0/2}(\overline{\Omega}\times[a+\tau/4,b])} \right). \tag{11}$$

Here C_7 *depends on the dimension, the domain and its boundary atlas, the ellipticity bounds,* τ*,* β_0*,* β_1 *and* Λ_A*. In particular,* C_7 *is independent of* b *for coefficient families with a common bound on* Λ_A*.*

Proof. Let $\ell = \min\{\tau/16, 1\}$. For $t \in [a+\tau/2, b]$, set $I_t = (t-2\ell, t)$ and choose a smooth temporal cutoff χ_t that vanishes near $t-2\ell$, equals one on $[t-\ell, t]$, and has derivative bounded by C/ℓ. The function $z_t = \chi_t v$ solves a conormal problem on $\Omega \times I_t$, with zero initial value and source

$$h_t = \chi_t g + \chi_t' v.$$

The initial and conormal data are compatible. The coefficient Hölder norm is bounded by Λ_A, and the source is bounded by $C_\tau N$, where

$$N = \|v\|_{L^\infty(Q_{a,b})} + \|g\|_{C^{\beta_0,\beta_0/2}(\overline{\Omega}\times[a+\tau/4,b])}.$$

For $N = 0$ the assertion is immediate. Otherwise apply the conormal gradient estimate in [12, Theorem 1.1] to z_t/N. In that theorem the flux is $A(x,s)p$, the lower-order term is h_t/N, and both boundary and initial data vanish. Its structural hypotheses follow from ellipticity and the space-time Hölder bound on A; the bounded source satisfies its growth condition. Choose β_1 smaller than the exponent supplied by that theorem and smaller than β_0, γ.

All cylinders have length 2ℓ and all normalized data have common bounds. Hence the resulting constant has exactly the dependence stated above. On the inner window $z_t = v$. Patching by the close-pair and distant-pair argument used in Lemma 6 gives (11) on the whole time layer. □

For the next estimate, take coefficient norms on the fixed compact state neighborhood $\overline{Q_M^{\delta/2}}$. Denote

$$N_D = \max_k \|D_k\|_{C^{2+\gamma}(\overline{\Omega}\times\overline{Q_M^{\delta/2}})},$$
$$N_R = \|R\|_{C^{1+\gamma}(\overline{\Omega}\times\overline{Q_M^{\delta/2}})}.$$

These are finite by the coefficient assumptions. In the local class the corresponding norms are taken on K_{δ_0}.

Lemma 8 (Schauder estimate for the oblique problem)**.** *Let* α *be a strong solution of a system in* $\mathscr{D}_0$*, with values in* Q_M*, and suppose*

$$B_1 := \|\alpha\|_{C^{1+\beta_1,(1+\beta_1)/2}(\overline{\Omega}\times[a+\tau/2,b])^K} < \infty.$$

For every $0 < \beta_2 < \min\{\beta_1, \gamma\}$*,*

$$\|\alpha\|_{C^{2+\beta_2,1+\beta_2/2}(\overline{\Omega}\times[a+\tau,b])^K} \le C_8(\tau, B_1). \tag{12}$$

In addition to τ *and* B_1*,* C_8 *depends only on* d, K, M, Ω*, the ellipticity bounds, the indicated exponents,* N_D *and* N_R*. It is independent of* b *when these quantities are fixed. For every* $1 < q < \infty$ *and every* $[s, s+1] \subset [a+\tau, b]$*,*

$$\|\partial_t \alpha\|_{L^q(s,s+1;L^q)^K} + \|\alpha\|_{L^q(s,s+1;W^{2,q})^K} \le C_{q,\tau,B_1}, \tag{13}$$

with the same dependencies and an additional dependence on q*.*

Proof. Let $\ell = \min\{\tau/16, 1\}$ and $I_t = (t-2\ell, t)$ for $t \in [a+\tau, b]$. Set $A_k(x,s) = D_k(x, \alpha(x,s))$. On each cylinder the kth equation has the nondivergence form

$$\begin{aligned} \partial_t \alpha_k - A_k^{ij} \partial_{ij} \alpha_k &= R_k(x,\alpha) + (\partial_i A_k^{ij}) \partial_j \alpha_k \\ &=: F_k, \end{aligned} \tag{14}$$

where spatial indices are summed and

$$\partial_i A_k^{ij} = \partial_{x_i} D_k^{ij}(x,\alpha) + \sum_{m=1}^{K} \partial_{a_m} D_k^{ij}(x,\alpha) \partial_i \alpha_m.$$

The derivative on the left is the total spatial derivative of the composed coefficient. Composition and product estimates bound the $C^{\beta_2,\beta_2/2}$ norms of A_k and F_k in terms of N_D, N_R, B_1 and the structural data.

The boundary condition is $b_k \cdot \nabla \alpha_k = 0$, where $b_k = A_k n$ and $b_k \cdot n \ge d_*$. Since $n \in C^{1+\gamma}(\partial\Omega)$, the oblique coefficient satisfies

$$\|A_k\|_{C^{\beta_2,\beta_2/2}(\overline{\Omega}\times I_t)} + \|F_k\|_{C^{\beta_2,\beta_2/2}(\overline{\Omega}\times I_t)} + \|b_k\|_{C^{1+\beta_2,(1+\beta_2)/2}(\partial\Omega\times I_t)} \le \Phi(N_D, N_R, B_1),$$

where Φ is a finite bound depending also on the fixed structural data. This records the boundary coefficient norm required by the oblique Schauder estimate.

Multiply α_k by a time cutoff that vanishes near the initial end of I_t. The new source is $\chi_t F_k + \chi_t' \alpha_k$, the initial datum is zero, and the homogeneous oblique condition is unchanged. Compatibility is automatic. The Schauder estimate [15, Chapter V, Theorem 5.18] applies with a constant controlled by the displayed norms and the cutoff length. Equivalently, one can solve the cutoff linear problem in the Schauder class and identify that solution with the cutoff strong solution by uniqueness. Patching the inner windows proves (12). Integration over a unit interval gives (13). □

Uniform smoothing and global continuation

Theorem 9 (Uniform smoothing away from the initial time)**.** *Let the system belong to $\mathscr{D}_0$ and let α be its maximal strong solution. There are $\beta_0, \beta_1, \beta_2 > 0$ such that, for every $\tau > 0$ and $T \in (\tau, T_{\max})$,*

$$\|\alpha\|_{C^{\beta_0,\beta_0/2}(\overline{\Omega}\times[\tau/4,T])^K} \le C_{0,\tau}, \tag{15}$$

$$\|\alpha\|_{C^{1+\beta_1,(1+\beta_1)/2}(\overline{\Omega}\times[\tau/2,T])^K} \le C_{1,\tau}, \tag{16}$$

$$\|\alpha\|_{C^{2+\beta_2,1+\beta_2/2}(\overline{\Omega}\times[\tau,T])^K} \le C_{2,\tau}. \tag{17}$$

These constants depend on τ, the structural data and the fixed coefficient norms N_D, N_R, but not on T or $\|\alpha^{\mathrm{in}}\|_{E_r^K}$. For the local class, the same estimates hold while the trajectory takes values in Q_{M,δ_0}, with the coefficient norms taken on K_{δ_0}. In particular,

$$\sup_{t\in[\tau,T]} \|\alpha(t)\|_{E_r^K} \le C_{r,\tau}. \tag{18}$$

Proof. Lemma 5 gives $0 \le \alpha_k \le M$. Each component solves (9) with

$$A_k(x,t) = D_k(x,\alpha(x,t)),$$
$$g_k(x,t) = R_k(x,\alpha(x,t)).$$

The reaction is uniformly bounded on the fixed compact state set. Lemma 6 therefore gives (15), with a bound $B_0(\tau)$ depending only on the structural data and τ.

We spell out the dependence of the next steps. On $[\tau/4, T]$, the composition estimate gives

$$\max_k \big(\|A_k\|_{C^{\beta_0,\beta_0/2}} + \|g_k\|_{C^{\beta_0,\beta_0/2}}\big) \le C(N_D, N_R)\big(1 + B_0(\tau)\big).$$

Thus both Λ_A and the source norm in Lemma 7 are already bounded independently of T and of the initial trace norm. Applying that lemma gives (16), with a bound $B_1(\tau)$ having the same independence. Lemma 8 is then applied with this controlled value of $B_1(\tau)$; every quantity entering its Schauder constant is fixed by the preceding two stages. This proves (17) without a circular regularity assumption.

The windows have one fixed length at each stage, so the patching introduces no additional dependence on T. The embeddings

$$C^{2+\beta_2}(\overline{\Omega}) \hookrightarrow W^{2,r}(\Omega) \hookrightarrow E_r$$

give (18). The local class is treated on the same compact K_{δ_0} throughout. □

Remark 10. *No smallness assumption is imposed on $D_a D_k$. Large state derivatives increase the constants through the coefficient norms, but do not invalidate the scalar regularity argument. Independence of the terminal time is distinct from independence of these norms.*

Theorem 11 (Global continuation)**.** *Every strong solution of a system in $\mathscr{D}_0$ with initial state in $\mathscr{X}_r$ exists for all $t \ge 0$ and defines a semiflow*

$$S(t) : \mathscr{X}_r \longrightarrow \mathscr{X}_r.$$

For a system in $\mathscr{D}_0^{\delta_0}$, the same conclusion holds on $\mathscr{X}_r^{\delta_0}$ if that space is positively invariant.

Proof. Suppose $T_{\max} < \infty$. The invariant rectangle keeps the range in $Q_M \Subset Q_M^\delta$; in the local class, the assumed positive invariance keeps it in $Q_{M,\delta_0} \Subset \mathscr{O}_{\delta_0}$. Apply Theorem 9 with $\tau = T_{\max}/4$. The tail for $t \ge T_{\max}/2$ is uniformly bounded in $C^{2+\beta_2}(\overline{\Omega})^K$ and hence relatively compact in E_r^K.

Its E_r closure retains the state bounds and the conormal compatibility conditions because $E_r \hookrightarrow C^1$. The closure is therefore a compact subset of the admissible phase manifold. The compact-orbit continuation criterion in Theorem 4 contradicts finite $T_{\max}$. □

Corollary 12 (Positive-time absorbing set)**.** *For every $\tau > 0$,*

$$\sup_{u \in \mathscr{X}_r} \|S(\tau)u\|_{E_r^K} \le C_{r,\tau}.$$

Moreover, $S(\tau)\mathscr{X}_r$ is relatively compact in $C^{1,\mu}(\overline{\Omega})^K$ for every $0 \le \mu < \vartheta_r$.

Proof. The norm bound follows from (18); compactness follows from (5). The stronger positive-time bound (17) also supplies relative compactness in E_r^K when it is needed below. □

A COMMON DIAGONAL METRIC FOR THE REACTION JACOBIANS

Define

$$m_k = \inf_{\substack{x \in \overline{\Omega} \\ a \in Q_M}} [-\partial_{a_k} R_k(x,a)],$$
$$b_{kj} = \sup_{\substack{x \in \overline{\Omega} \\ a \in Q_M}} \left|\partial_{a_j} R_k(x,a)\right|, \quad j \ne k,$$

and

$$\delta_k = m_k - \frac{1}{2}\sum_{j \ne k} b_{kj}. \tag{19}$$

If every $\delta_k > 0$, define the nonnegative matrix H by

$$H_{kj} = \begin{cases} b_{jk}/(2\delta_k), & j \ne k, \\ 0, & j = k. \end{cases} \tag{20}$$

The set

$$\mathscr{J}_R = \{D_a R(x,a) : x \in \overline{\Omega},\ a \in Q_M\}$$

is a compact family of Jacobians. Its entries satisfy uniform interval bounds; the family itself need not be an interval box, since different entries can depend on the same variables.

Definition 13 (Spectral subclass)**.** *A system in $\mathscr{D}_0$ belongs to $\mathscr{D}_{\mathrm{sp}}$ if $\delta_k > 0$ for every k and $\rho(H) < 1$.*

Theorem 14 (Simultaneous M-matrix criterion)**.** *Let the system belong to $\mathscr{D}_{\mathrm{sp}}$. For any $s \in (0,\infty)^K$, set*

$$\pi = (I - H)^{-1} s, \qquad P = \operatorname{diag}(\pi_1, \dots, \pi_K). \tag{21}$$

Then every $\pi_k > 0$, and, for every $J \in \mathscr{J}_R$,

$$\frac{PJ + J^T P}{2} \le -\sigma_R I, \qquad \sigma_R = \min_k \delta_k s_k > 0. \tag{22}$$

Consequently, the system belongs to $\mathscr{D}_{\mathrm{diss}}(P, \sigma_R)$.

Proof. Since $H \ge 0$ and $\rho(H) < 1$, the matrix $I - H$ is a nonsingular M-matrix and

$$(I - H)^{-1} = \sum_{n=0}^{\infty} H^n \ge 0.$$

In particular, $\pi \ge s > 0$ and $\pi - H\pi = s$. For $z \in \mathbb{R}^K$ and $J \in \mathscr{J}_R$, the entrywise bounds imply

$$z^T \frac{PJ + J^T P}{2} z = z^T P J z \le -\sum_k \pi_k m_k z_k^2 + \sum_{k \ne j} \pi_k b_{kj} \left|z_k z_j\right|.$$

Using $\left|z_k z_j\right| \le (z_k^2 + z_j^2)/2$ gives

$$z^T \frac{PJ + J^T P}{2} z \le -\sum_k \left(\pi_k \delta_k - \frac{1}{2}\sum_{j \ne k} \pi_j b_{jk}\right) z_k^2.$$

By (20),

$$\frac{1}{2}\sum_{j \ne k} \pi_j b_{jk} = \delta_k (H\pi)_k.$$

Thus the coefficient of z_k^2 on the right is $\delta_k(\pi_k - (H\pi)_k) = \delta_k s_k$, proving (22). For $a, b \in Q_M$, convexity of Q_M yields

$$R(x,a) - R(x,b) = \int_0^1 D_a R(x, b + \lambda(a-b))(a-b)\, \mathrm{d}\lambda.$$

Integrating (22) along this segment gives (3). □

Remark 15. *The inverse-positivity argument is the classical nonsingular M-matrix construction*

[16, 17, 18]. Common diagonal Lyapunov inequalities for matrix families, including interval families, have also been studied previously [20, 19]. Theorem 14 supplies a particular sufficient construction from the bounds m_k, b_{kj}: one linear system gives the weights π and the explicit common margin σ_R, uniformly in both x and a. The inequalities

$$\pi_k m_k > \frac{1}{2}\sum_{j\neq k}(\pi_k b_{kj} + \pi_j b_{jk})$$

express strict diagonal dominance of the symmetric comparison matrix used in the proof. This is a sufficient test, not a characterization of simultaneous diagonal stability. Absolute upper bounds can lose information about signs and correlations. The additional diagonal-dominance requirement can also be restrictive. For example, consider the single matrix

$$J = \begin{pmatrix} -1 & 3 \\ 1/10 & -1 \end{pmatrix}, \qquad P = \operatorname{diag}(1, 30).$$

Then

$$-\frac{PJ + J^T P}{2} = \begin{pmatrix} 1 & -3 \\ -3 & 30 \end{pmatrix} > 0,$$

since its leading principal minors are 1 and 21. Nevertheless, (19) gives $\delta_1 = 1 - 3/2 < 0$, so Theorem 14 does not apply. This is a matrix-level illustration of sufficiency, not an example satisfying the PDE state-invariance assumptions.

The Jacobians need not be Metzler matrices. Multiplying P by a positive scalar multiplies both σ_R and $\pi_{\max}$ by the same factor, so the decay rate $\sigma_R/\pi_{\max}$ is unchanged.

Example 16 (Two components)**.** *For $K = 2$,*

$$H = \begin{pmatrix} 0 & b_{21}/(2\delta_1) \\ b_{12}/(2\delta_2) & 0 \end{pmatrix}, \qquad \rho(H) = \sqrt{\frac{b_{12}b_{21}}{4\delta_1\delta_2}}.$$

Thus $\rho(H) < 1$ is equivalent to $b_{12}b_{21} < 4\delta_1\delta_2$.

Proposition 17 (Transition and dissipative reaction terms)**.** *Suppose*

$$R_k(x,a) = \sum_{j=1}^{K} q_{jk}(x)a_j - S_k(x,a_k) + C_k(x,a),$$

where $q_{jk} \geq 0$ for $j \neq k$, every row sum of $Q = (q_{jk})$ is zero, and

$$\partial_s S_k(x,s) \geq s_k^* > 0, \qquad \left|\partial_{a_j} C_k(x,a)\right| \leq c_{kj}.$$

If, for every k,

$$s_k^* + \inf_x \sum_{j\neq k} q_{kj}(x) > c_{kk} + \frac{1}{2}\sum_{j\neq k}\left(\sup_x q_{jk}(x) + c_{kj}\right),$$

then $\delta_k > 0$. If, in addition, $\rho(H) < 1$, then Theorem 14 applies.

Proof. The zero row sums give $-q_{kk} = \sum_{j\neq k} q_{kj}$. Consequently,

$$m_k \geq s_k^* + \inf_x \sum_{j\neq k} q_{kj}(x) - c_{kk},$$
$$b_{kj} \leq \sup_x q_{jk}(x) + c_{kj}.$$

Substitution in (19) proves the claim. □

QUADRATIC DISSIPATIVE ESTIMATE

Assume that the system belongs to $\mathscr{D}_{\mathrm{diss}}(P, \sigma_R)$. For a constant $a_* \in Q_M$, define

$$V_{a_*}[u] = \frac{1}{2}\int_\Omega (u - a_*)^T P(u - a_*)\,\mathrm{d}x.$$

Theorem 18 (Quadratic estimate)**.** *Every strong solution satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{a_*}[\alpha(t)] + \pi_{\min} d_* \|\nabla\alpha(t)\|_{L^2}^2 + \frac{\sigma_R}{2}\|\alpha(t) - a_*\|_{L^2}^2 \leq \frac{\|P\|_{\mathrm{op}}^2}{2\sigma_R}\|R(\cdot, a_*)\|_{L^2}^2. \tag{23}$$

If $R(x, a_*) = 0$ for every x, then

$$\|\alpha(t) - a_*\|_{L^2}^2 \leq \frac{\pi_{\max}}{\pi_{\min}} e^{-2\sigma_R t/\pi_{\max}} \left\|\alpha^{\mathrm{in}} - a_*\right\|_{L^2}^2. \tag{24}$$

Proof. Test the kth equation by $\pi_k(\alpha_k - a_{*,k})$ and sum over k. Since a_* is constant in space, ellipticity gives the gradient term in (23). Split the reaction term as

$$\begin{aligned} &R(x,\alpha)^T P(\alpha - a_*) \\ &\quad = (R(x,\alpha) - R(x,a_*))^T P(\alpha - a_*) \\ &\qquad + R(x,a_*)^T P(\alpha - a_*). \end{aligned}$$

The first term is bounded above by $-\sigma_R |\alpha - a_*|^2$. Young's inequality gives

$$\begin{aligned} &R(x,a_*)^T P(\alpha - a_*) \\ &\quad \le \frac{\sigma_R}{2} |\alpha - a_*|^2 + \frac{\|P\|_{\mathrm{op}}^2}{2\sigma_R} |R(x,a_*)|^2, \end{aligned}$$

proving (23). If $R(\cdot, a_*) = 0$, no Young inequality is needed, and the full reaction dissipation remains:

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{a_*}[\alpha(t)] + \frac{2\sigma_R}{\pi_{\max}} V_{a_*}[\alpha(t)] \le 0.$$

Integration and the equivalence of the weighted and unweighted L^2 norms yield (24). □

Global attractor and strong attraction

Definition 19. *A set $\mathscr{A} \subset \mathscr{X}_r$ is an $(\mathscr{X}_r, \mathscr{Y})$-global attractor if it is compact in $\mathscr{Y}$, strictly invariant, and attracts every subset of $\mathscr{X}_r$ bounded in E_r^K with respect to the $\mathscr{Y}$ metric.*

Lemma 20 (Continuity on bounded sets)**.** *Let $B \subset \mathscr{X}_r$ be bounded in E_r^K. If $u_n, u \in B$ and $u_n \to u$ in $\mathscr{Y}$, then, for every $T > 0$,*

$$\sup_{0 \le t \le T} \|S(t)u_n - S(t)u\|_{L^2} \longrightarrow 0.$$

Proof. Choose p with $d + 2 < p < r$. The compact embedding $E_r \Subset E_p$ makes the closure of B in E_p^K compact. Since $E_p \hookrightarrow C^1(\overline{\Omega})$, the invariant bounds and conormal compatibility pass to this closure. The local theory at exponent p, applied on finitely many neighborhoods covering this compact set, gives a common time $t_B > 0$ and a uniform $C([0,t_B]; E_p^K)$ bound. Uniqueness identifies these solutions with the strong solutions constructed at exponent r. Thus their spatial gradients are uniformly bounded on $[0,t_B]$. Theorem 9 gives a uniform gradient bound on the remaining part of every fixed interval $[0,T]$. In particular,

$$G_{B,T} := \sup_{v \in B} \int_0^T \|\nabla S(t)v\|_{L^\infty}^2 \, \mathrm{d}t < \infty.$$

Set $w_n = S(\cdot)u_n - S(\cdot)u$. Subtracting the equations and testing by w_n, uniform ellipticity and the bounded state derivatives of D_k and R yield

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|w_n\|_{L^2}^2 + \frac{d_*}{2} \|\nabla w_n\|_{L^2}^2 \\ &\quad \le C \left(1 + \|\nabla S(t)u\|_{L^\infty}^2\right) \|w_n\|_{L^2}^2. \end{aligned}$$

Gronwall's inequality proves the assertion. □

Theorem 21 (Compactness, strong attraction and strict invariance)**.** *Every system in $\mathscr{D}_0$ has a nonempty strictly invariant global attractor $\mathscr{A} \subset \mathscr{X}_r$ compact in E_r^K. In fact,*

$$\lim_{t \to \infty} \sup_{u \in \mathscr{X}_r} \inf_{a \in \mathscr{A}} \|S(t)u - a\|_{E_r^K} = 0.$$

In particular, $\mathscr{A}$ is an $(\mathscr{X}_r, \mathscr{Y})$-global attractor and is compact in $C^{1,\mu}(\overline{\Omega})^K$ for every $0 \le \mu < \vartheta_r$.

Proof. Theorem 9 gives a uniform $C^{2+\beta_2}(\overline{\Omega})^K$ bound for all $S(t)u$ with $t \ge 1$ and $u \in \mathscr{X}_r$. Since

$$C^{2+\beta_2}(\overline{\Omega}) \Subset E_r,$$

the union of these solution tails is relatively compact in E_r^K. For $s \ge 1$, set

$$K_s = \overline{\bigcup_{t \ge s} S(t)\mathscr{X}_r}^{E_r^K}, \qquad \mathscr{A} = \bigcap_{s \ge 1} K_s.$$

These are nonempty nested compact sets in E_r^K. The state bounds and conormal compatibility pass to their limits because $E_r \hookrightarrow C^1$, hence $K_s \subset \mathscr{X}_r$. Their intersection is nonempty and compact in E_r^K. The L^2 and E_r closures of each tail coincide: any L^2-convergent sequence in a tail has an E_r-convergent subsequence with the same limit.

If the asserted strong attraction failed, some $t_n \to \infty$ and $u_n \in \mathscr{X}_r$ would satisfy

$$\inf_{a\in\mathscr{A}} \|S(t_n)u_n - a\|_{E_r^K} \geq \varepsilon > 0.$$

Tail compactness supplies an E_r-convergent subsequence. Its limit belongs to every K_s and hence to $\mathscr{A}$, a contradiction. The uniform positive-time estimate permits this argument for all of $\mathscr{X}_r$, not only its bounded subsets.

For strict invariance, choose $a_n = S(t_n)u_n \to a \in \mathscr{A}$ in E_r^K, where $t_n \to \infty$. The strong semiflow is continuous in the trace topology by the local theory and continuation. Thus $S(t)a_n \to S(t)a$ for fixed $t \geq 0$, and every tail closure contains the limit. This gives $S(t)\mathscr{A} \subset \mathscr{A}$. Conversely, for $t_n > t$, a subsequence of $b_n = S(t_n - t)u_n$ converges in E_r^K to some $b \in \mathscr{A}$. Continuity gives $S(t)b = a$, proving $S(t)\mathscr{A} = \mathscr{A}$. Finally, the continuous embeddings into $\mathscr{Y}$ and $C^{1,\mu}$ give the stated weaker-topology consequences. □

Lemma 22 (Complete trajectories)**.** *For every $a \in \mathscr{A}$, there is a map $\gamma : \mathbb{R} \to \mathscr{A}$ such that $\gamma(0) = a$ and*

$$\gamma(t+s) = S(t)\gamma(s), \qquad t \geq 0, \quad s \in \mathbb{R}.$$

Proof. Since $S(1)\mathscr{A} = \mathscr{A}$, choose successively $a_{-n} \in \mathscr{A}$ such that $S(1)a_{-(n+1)} = a_{-n}$ and $a_0 = a$. For $t \geq -n$, set $\gamma(t) = S(t+n)a_{-n}$. The definitions agree on overlapping intervals and give the required complete trajectory. □

Corollary 23 (Attractor-radius estimate)**.** *If the system belongs to $\mathscr{D}_{\mathrm{diss}}(P, \sigma_R)$, then, for every $a_* \in Q_M$,*

$$\sup_{u\in\mathscr{A}} \|u - a_*\|_{L^2}^2 \leq \frac{\pi_{\max}\|P\|_{\mathrm{op}}^2}{\pi_{\min}\sigma_R^2} \|R(\cdot, a_*)\|_{L^2}^2. \quad (25)$$

Proof. Let $u = \gamma(0)$ lie on a complete trajectory in $\mathscr{A}$. Dropping the gradient term in (23) and using $V_{a_*} \leq (\pi_{\max}/2)\|\cdot - a_*\|_{L^2}^2$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{a_*}[\gamma(t)] + \frac{\sigma_R}{\pi_{\max}} V_{a_*}[\gamma(t)] \leq \frac{\|P\|_{\mathrm{op}}^2}{2\sigma_R} \|R(\cdot, a_*)\|_{L^2}^2.$$

Integrate from $-T$ to 0 and let $T \to \infty$. The values $V_{a_*}[\gamma(-T)]$ are uniformly bounded because $\mathscr{A}$ is compact in $\mathscr{Y}$. The lower weighted-norm inequality yields (25). □

Theorem 24 (Singleton attractor)**.** *Suppose that the system belongs to $\mathscr{D}_{\mathrm{diss}}(P, \sigma_R)$ and that there is a constant $a_* \in (0, M)^K$ such that $R(x, a_*) = 0$ for every $x \in \overline{\Omega}$. Then*

$$\mathscr{A} = \{a_*\},$$

and every strong solution satisfies (24). For $0 < \eta < \nu < 1$ and every $\tau > 0$, there is $C_{\eta,\nu,\tau} > 0$ such that

$$\|\alpha(t) - a_*\|_{C^\eta} \leq C_{\eta,\nu,\tau} e^{-\omega_{\eta,\nu} t}, \quad t \geq \tau,$$
$$\omega_{\eta,\nu} = \frac{\nu - \eta}{\nu + d/2} \frac{\sigma_R}{\pi_{\max}} > 0.$$

Proof. Apply (24) to a complete trajectory between times $-T$ and 0 and let $T \to \infty$. Boundedness of $\mathscr{A}$ forces the value at time zero to equal a_*. The C^η estimate follows from

$$\|f\|_{C^\eta} \leq C \|f\|_{L^2}^{\theta} \|f\|_{C^\nu}^{1-\theta}, \qquad \theta = \frac{\nu - \eta}{\nu + d/2},$$

the exponentially decaying L^2 norm, and the uniform positive-time C^ν bound of Theorem 9. The initial L^2 distance is uniformly bounded by the invariant rectangle. □

Theorem 25 (Contraction for state-independent diffusion)**.** *If $D_k = D_k(x)$ and the system belongs to $\mathscr{D}_{\mathrm{diss}}(P, \sigma_R)$, then*

$$\|S(t)u - S(t)v\|_{P,L^2}^2 \leq e^{-2\sigma_R t/\pi_{\max}} \|u - v\|_{P,L^2}^2.$$

The attractor consists of one, possibly spatially nonhomogeneous, stationary solution.

Proof. Subtracting two equations produces no coefficient-difference term in the flux. Testing by $P(S(t)u - S(t)v)$ gives the contraction estimate. Applying it to two complete trajectories between $-T$ and 0, and letting $T \to \infty$, shows that the attractor has one point. Strict invariance makes this point stationary. Any other stationary solution in $\mathscr{X}_r$ has a bounded orbit and is attracted by the same singleton, so it coincides with it. □

Corollary 26 (Eventual positivity)**.** *Under the assumptions of Theorem 24, there is $T_* > 0$ such that*

$$\alpha_k(x,t) \geq \frac{1}{2}\min_j a_{*,j}, \qquad x \in \overline{\Omega}, \quad t \geq T_*.$$

ENERGY-CLASS WEAK SOLUTIONS AND WEAK–STRONG UNIQUENESS

For the conormal problem, set

$$V = H^1(\Omega)^K, \qquad V' = (H^1(\Omega)^K)'.$$

The duality pairing is taken with $L^2(\Omega)^K$ as pivot space. Test functions in V have no prescribed zero trace on the boundary.

Definition 27. *A function*

$$u \in L^\infty(0,T;L^2(\Omega)^K) \cap L^2(0,T;V), \quad \partial_t u \in L^2(0,T;V')$$

is an energy-class weak solution if $0 \leq u_k \leq M$ almost everywhere, $u(0) = u^{\mathrm{in}}$ in $L^2(\Omega)^K$, and

$$\int_0^T \langle \partial_t u, \phi \rangle_{V',V}\, \mathrm{d}t + \sum_{k=1}^K \int_{\Omega\times(0,T)} D_k(x,u)\nabla u_k \cdot \nabla \phi_k \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega\times(0,T)} R(x,u)\cdot\phi \,\mathrm{d}x\,\mathrm{d}t \tag{26}$$

for every $\phi \in L^2(0,T;V)$. The stated energy regularity supplies the continuous L^2 representative used for the initial trace.

Theorem 28 (Weak–strong uniqueness)**.** *Let the system belong to $\mathscr{D}_{\mathrm{diss}}(P,\sigma_R)$. Let u be an energy-class weak solution and $\bar{u}$ a strong solution with the same initial state. If*

$$\nabla \bar{u} \in L^2(0,T;L^\infty(\Omega)^{K\times d}),$$

then $u = \bar{u}$ almost everywhere. For strong solutions with initial trace in $\mathscr{X}_r$, this gradient condition is automatic on every finite interval: $\bar{u} \in C([0,T];E_r^K)$ and $E_r \hookrightarrow C^1(\overline{\Omega})$.

Proof. Set $w = u - \bar{u}$. The Lions–Magenes energy identity [31] gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{P,L^2}^2 = \langle \partial_t w, Pw\rangle_{V',V}.$$

The kth flux difference equals

$$D_k(x,u)\nabla w_k + [D_k(x,u) - D_k(x,\bar{u})]\nabla \bar{u}_k.$$

Let L_D be a common state-Lipschitz constant for the tensors on $\overline{\Omega}\times Q_M$, with the operator norm on tensors. The mixed term satisfies

$$\left|\pi_k \int_\Omega [D_k(x,u) - D_k(x,\bar{u})]\nabla \bar{u}_k \cdot \nabla w_k \,\mathrm{d}x\right| \leq \frac{\pi_k d_*}{2}\|\nabla w_k\|_{L^2}^2 + \frac{\pi_k L_D^2}{2d_*}\|\nabla \bar{u}_k\|_{L^\infty}^2 \|w\|_{L^2}^2.$$

The reaction term is at most $-\sigma_R\|w\|_{L^2}^2$. Summation and weighted-norm equivalence yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{P,L^2}^2 \leq C\|\nabla \bar{u}(t)\|_{L^\infty}^2 \|w\|_{P,L^2}^2.$$

Since $w(0) = 0$, Gronwall's inequality gives $w = 0$. □

Corollary 29 (Continuous dependence)**.** *For two strong solutions u and v of the same system in $\mathscr{D}_{\mathrm{diss}}(P,\sigma_R)$,*

$$\|u(t) - v(t)\|_{P,L^2}^2 \leq \|u(0) - v(0)\|_{P,L^2}^2 \times \exp\left(C\int_0^t \|\nabla v(s)\|_{L^\infty}^2 \,\mathrm{d}s\right).$$

If the diffusion tensors do not depend on the state, Theorem 25 applies.

The weak–strong argument also works with a one-sided Lipschitz reaction estimate $(R(x,a) - R(x,b))^T P(a-b) \leq L_R|a-b|^2$. The Gronwall exponent then contains an additional constant term. Such a bound follows from the stated coefficient regularity on either Q_M or the convex set Q_{M,δ_0}. Thus strict reaction dissipation improves the difference estimate but is not indispensable for weak–strong uniqueness on a fixed regular state region.

COMPACTNESS AND STABILITY UNDER COEFFICIENT PERTURBATIONS

Let $D_k^\varepsilon, R^\varepsilon$, $0 \le \varepsilon \le 1$, satisfy the conditions of $\mathscr{D}_0$ or $\mathscr{D}_0^{\delta_0}$ with common structural constants on the state region under consideration. In this section, a positive lower bound for the total concentration is an independent hypothesis. Define

$$Q_{M,3\delta_0/4} = \left\{ a \in Q_M : \sum_k a_k \ge \frac{3\delta_0}{4} \right\} \Subset \mathscr{O}_{\delta_0}$$

and

$$\omega_{\delta_0}(\varepsilon) = \max_k \left\| D_k^\varepsilon - D_k^0 \right\|_{L^\infty(\overline{\Omega}\times Q_{M,3\delta_0/4})} + \left\| R^\varepsilon - R^0 \right\|_{L^\infty(\overline{\Omega}\times Q_{M,3\delta_0/4})}.$$

The energy spaces and duality convention are those introduced in the weak-solution formulation.

Theorem 30 (Compactness at positive total concentration)**.** *Let $\varepsilon_n \downarrow 0$, and let u_n be energy-class weak solutions on $[0,T]$. Assume convergence of their initial data in $L^2(\Omega)^K$ and*

$$\|u_n\|_{L^\infty(0,T;L^2)} + \|u_n\|_{L^2(0,T;V)} + \|\partial_t u_n\|_{L^2(0,T;V')} \le C,$$
$$\sum_k u_{n,k} \ge \delta_0 > 0.$$

If $\omega_{\delta_0}(\varepsilon_n) \to 0$, then, after extraction of a subsequence,

$$u_n \to u \quad \text{strongly in } L^2(\Omega\times(0,T))^K \text{ and almost everywhere},$$

where u is a weak solution of the limit system with the limit initial trace.

Proof. The compact embedding $V \Subset L^2(\Omega)^K$ and the continuous embedding $L^2(\Omega)^K \hookrightarrow V'$ allow the use of Aubin–Lions–Simon compactness [26]. It gives strong L^2 convergence and, after subsequence extraction, almost-everywhere convergence. Also $\nabla u_n \rightharpoonup \nabla u$ in L^2. The rectangle bounds and the lower total-concentration estimate pass to the limit. Uniform convergence of the coefficients on the fixed compact state set and continuity of the limit coefficients imply

$$D_k^{\varepsilon_n}(x,u_n) \to D_k^0(x,u),$$
$$R^{\varepsilon_n}(x,u_n) \to R^0(x,u)$$

almost everywhere and strongly in every finite L^q. The coefficient sequences are uniformly bounded. Multiplying any fixed L^2 test field by the coefficient difference therefore gives strong L^2 convergence by dominated convergence. It follows that

$$D_k^{\varepsilon_n}(x,u_n)\nabla u_{n,k} \rightharpoonup D_k^0(x,u)\nabla u_k \quad \text{in } L^2.$$

Passage to the limit in (26) is valid.

The time-derivative bound passes weakly to the limit in $L^2(0,T;V')$. For each $\zeta \in V$,

$$|(u_n(t)-u_n(s),\zeta)_{L^2}| \le \|\partial_t u_n\|_{L^2(s,t;V')} |t-s|^{1/2} \|\zeta\|_V.$$

This equicontinuity, together with convergence on a dense set of times, identifies the weak L^2 trace at $t=0$ with the strong L^2 limit of the initial data. The limit belongs to $L^2(0,T;V)$ with time derivative in $L^2(0,T;V')$, hence it has a continuous L^2 representative. Its initial trace is therefore the required one. □

Theorem 31 (Quantitative stability)**.** *Let u^ε and u^0 be strong solutions such that*

$$\sum_k u_k^\varepsilon \ge \delta_0, \qquad \sum_k u_k^0 \ge \delta_0,$$
$$\nabla u^0 \in L^2(0,T;L^\infty).$$

Assume uniform ellipticity and uniform state-Lipschitz bounds for D_k^ε and R^ε on Q_{M,δ_0}, and $\omega_{\delta_0}(\varepsilon) \to 0$. Then

$$\sup_{0\le t\le T} \left\| u^\varepsilon(t) - u^0(t) \right\|_{L^2}^2 + \int_0^T \left\| \nabla(u^\varepsilon - u^0) \right\|_{L^2}^2 \, \mathrm{d}t \le C_T \left(\left\| u^{\varepsilon,\mathrm{in}} - u^{0,\mathrm{in}} \right\|_{L^2}^2 + |\Omega| \omega_{\delta_0}(\varepsilon)^2 \right). \tag{27}$$

The constant C_T is independent of ε. It depends on the common ellipticity and Lipschitz constants, K, T, and $\int_0^T \|\nabla u^0\|_{L^\infty}^2 \, \mathrm{d}t$.

Proof. Set $w = u^\varepsilon - u^0$ and test the difference equation by w. Decompose the kth flux difference as

$$\begin{aligned} & D_k^\varepsilon(x,u^\varepsilon)\nabla w_k \\ & \quad + [D_k^\varepsilon(x,u^\varepsilon) - D_k^\varepsilon(x,u^0)]\nabla u_k^0 \\ & \quad + [D_k^\varepsilon(x,u^0) - D_k^0(x,u^0)]\nabla u_k^0. \end{aligned}$$

The first term is coercive. Convexity of Q_{M,δ_0} and the common Lipschitz constant control the second term by Young's inequality. The third satisfies

$$\begin{aligned} & \left| \int_\Omega [D_k^\varepsilon(x,u^0) - D_k^0(x,u^0)]\nabla u_k^0 \cdot \nabla w_k \, \mathrm{d}x \right| \\ & \quad \le \frac{d_*}{4} \|\nabla w_k\|_{L^2}^2 + C|\Omega|\omega_{\delta_0}(\varepsilon)^2 \|\nabla u_k^0\|_{L^\infty}^2. \end{aligned}$$

The reaction difference is bounded pointwise by $L_R|w| + \omega_{\delta_0}(\varepsilon)$. Thus

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + d_*\|\nabla w\|_{L^2}^2 \\ & \quad \le C\left(1 + \|\nabla u^0\|_{L^\infty}^2\right)\|w\|_{L^2}^2 \\ & \qquad + C|\Omega|\left(1 + \|\nabla u^0\|_{L^\infty}^2\right)\omega_{\delta_0}(\varepsilon)^2. \end{aligned}$$

Integration and Gronwall's inequality prove (27), including the gradient term. □

Remark 32. *For a strong solution, $u^0 \in C([0,T];E_r^K)$, and $E_r \hookrightarrow C^{1,\mu}$. Hence the gradient condition in Theorem 31 is automatic on finite time intervals. The positive-time bounds of Theorem 9 provide constants independent of the initial trace norm on time intervals separated from zero.*

Corollary 33. *If the limit system in Theorem 30 has a strong solution with the limit initial state and satisfying the hypotheses of Theorem 28, then the entire sequence converges to that solution. The same conclusion holds for the local coefficient class whenever the one-sided Lipschitz version of the weak–strong argument applies on Q_{M,δ_0}.*

RESULTS AND DISCUSSION

Scope and limitations

The general existence and attractor results require the invariant rectangle and uniform ellipticity, but not strict reaction dissipativity. The latter is a separate hypothesis for the quantitative radius and singleton results. The trace-space compactness proved in Theorem 21 gives strong attraction directly; the two-space terminology organizes the energy estimates and does not represent an additional compactness principle.

Theorem 14 offers explicit sufficient weights, not a necessary test or an improvement of the general theory of common diagonal stability. Remark 15 displays a diagonally stable matrix rejected by this test. Example 35, in contrast, shows how the accepted coefficient bounds yield a useful metric for a varying nonlinear Jacobian family. The gain is a verifiable PDE dissipation estimate with specified constants.

Large state derivatives of the diffusion tensors are allowed, but the regularity constants depend on their norms. Strict dissipativity of the reaction does not remove the mixed coefficient-difference term when two solutions with state-dependent diffusion are compared. Accordingly, general pairwise exponential contraction is asserted only in Theorem 25, where the diffusion is state-independent. The singleton result around a constant equilibrium has a different proof: comparison with that constant produces no such mixed term.

For normalized fractions, the positive lower bound on total concentration is an independent condition. With different diffusion tensors, the sum of the components need not satisfy a closed scalar equation; preservation of integrated mass alone does not give a pointwise lower bound. Example 36 verifies the condition for a common diffusion tensor and conservative exchange. Theorem 31 and (27) are finite-time statements: the constant C_T need not remain bounded as $T \to \infty$. Uniform-in-time perturbation estimates and continuity of attractors under coefficient perturbations are not established here. No numerical

scheme is analyzed; a structure-preserving discretization would require its own invariance, stability and convergence arguments.

Examples and independence of assumptions

Example 34 (Large diffusion derivatives with a simple reaction metric)**.** *Let $K = 2$, $M > 0$, $\mu_1, \mu_2, q > 0$, and $N_1, N_2 \geq 1$. Set*

$$D_k(x,a) = \left[1 + \frac{1}{2}\sin(N_k(a_1 + a_2))\right] I_d, \qquad k = 1,2,$$

and

$$R_1(a) = \mu_1(M/2 - a_1) + q(a_2 - a_1),$$
$$R_2(a) = \mu_2(M/2 - a_2) + q(a_1 - a_2).$$

The eigenvalues of D_k lie in $[1/2, 3/2]$, while the state derivatives grow linearly with N_k. The reaction points inward on all four faces of Q_M. Its Jacobian is constant and symmetric, and

$$z^T D_a R z = -\mu_1 z_1^2 - \mu_2 z_2^2 - q(z_1 - z_2)^2.$$

Thus $P = I$ already gives (3) with $\sigma_R = \min\{\mu_1, \mu_2\}$. The spectral test also holds:

$$\delta_k = \mu_k + q/2, \qquad \rho(H) = \frac{q}{2\sqrt{\delta_1 \delta_2}} < 1.$$

Theorem 24 applies at $a_* = (M/2, M/2)$. This example shows that large state derivatives of the diffusion tensors are admissible; a nontrivial common metric is not needed for its reaction.

More generally, $D_k = A_k(x) + \Theta_k(a)B_k(x)$ is uniformly elliptic if the factors have the prescribed $C^{2+\gamma}$ regularity,

$$A_k \geq 2d_* I, \qquad |\Theta_k| \|B_k\|_{\mathrm{op}} \leq d_*,$$

and A_k, B_k are symmetric. Large derivatives of Θ_k are compatible with this condition. The remaining assumptions of $\mathscr{D}_0$ must still hold.

Example 35 (A nonlinear family requiring a non-identity metric)**.** *Let $K = 2$, $M > 0$, and*

$$g_M(s) = \frac{M}{\pi}\sin\left(\frac{\pi s}{M}\right).$$

Use the diffusion tensors of Example 34 and define

$$R_1(a) = -(a_1 - M/2) + \frac{4}{5} g_M(a_2 - M/2),$$
$$R_2(a) = -10(a_2 - M/2) + 8 g_M(a_1 - M/2). \tag{28}$$

Since $|g_M(s)| \leq M/\pi$ for $|s| \leq M/2$, the inward margins at the lower faces are at least

$$M\left(\frac{1}{2} - \frac{4}{5\pi}\right) > 0, \qquad M\left(5 - \frac{8}{\pi}\right) > 0,$$

respectively; the upper-face reactions are at most the negatives of the same quantities. Thus the whole rectangle is invariant, including its edges and vertices.

Put $c_j(a) = \cos(\pi(a_j - M/2)/M) \in [0,1]$ on Q_M. The Jacobian family is

$$D_a R(a) = \begin{pmatrix} -1 & \frac{4}{5}c_2(a) \\ 8c_1(a) & -10 \end{pmatrix}. \tag{29}$$

It is nonconstant and generally nonsymmetric. The exact entrywise bounds are

$$m_1 = 1, \quad m_2 = 10, \quad b_{12} = \frac{4}{5}, \quad b_{21} = 8,$$

so $\delta_1 = 3/5$, $\delta_2 = 6$. For $s = (1,1)^T$, Theorem 14 gives

$$H = \begin{pmatrix} 0 & 20/3 \\ 1/15 & 0 \end{pmatrix}, \qquad \rho(H) = \frac{2}{3},$$
$$(I - H)^{-1} s = \begin{pmatrix} 69/5 \\ 48/25 \end{pmatrix}, \tag{30}$$
$$P = \mathrm{diag}(69/5, 48/25), \qquad \sigma_R = \frac{3}{5}.$$

This single P works for every $a \in Q_M$. At $a_* = (M/2, M/2)$, however, the symmetric part of $D_a R$ has off-diagonal entry $22/5$ and determinant $-234/25 < 0$. Indeed, for $z = (4,1)^T$,

$$z^T D_a R(a_*) z = 46/5 > 0.$$

Hence the identity metric cannot satisfy (3), even locally at a_*.

Since $R(a_*)=0$, the attractor is $\{a_*\}$ and (24) becomes

$$\|\alpha(t)-a_*\|_{L^2}^2 \le \frac{115}{16}e^{-2t/23}\|\alpha^{\mathrm{in}}-a_*\|_{L^2}^2. \quad (31)$$

The stated rate follows from the sufficient comparison bound and is not claimed optimal. It remains valid for arbitrarily large fixed N_k, although the positive-time regularity constants depend on N_k.

Example 36 (Conservative exchange and positive total concentration)**.** *For $K=2$, take*

$$R^{\mathrm{tr}}(a)=q\begin{pmatrix}a_2-a_1\\ a_1-a_2\end{pmatrix}, \qquad q>0.$$

This symmetric exchange reaction points inward on every face of Q_M, and $R_1^{\mathrm{tr}}+R_2^{\mathrm{tr}}=0$. Its Jacobian has the nonzero kernel vector $(1,1)^T$. For distinct points a,b on the diagonal of Q_M, $R^{\mathrm{tr}}(a)-R^{\mathrm{tr}}(b)=0$. Therefore no positive diagonal P and $\sigma_R>0$ can satisfy (3). Together with, for example, $D_k=I_d$, this gives a system in $\mathscr{D}_0$ with an invariant rectangle but without strict diagonal dissipativity.

This exchange reaction also gives a concrete local-class example. Let $0<\delta_0\le 2M$ and take the same tensor for both components:

$$D_1(x,a)=D_2(x,a)=\left[1+\frac{1}{2}\sin\left(\frac{Na_1}{a_1+a_2}\right)\right]I_d, \quad (32)$$

where $N\ge 1$ and $a_1+a_2>0$. These tensors are smooth away from zero total concentration, uniformly elliptic with bounds $1/2$ and $3/2$, and regular on K_{δ_0}. For a local strong solution write $m=\alpha_1+\alpha_2$ and $D=D_1=D_2$. Adding the equations gives

$$m_t=\nabla\cdot(D(x,\alpha)\nabla m), \qquad D(x,\alpha)\nabla m\cdot n=0.$$

If $m(0)\ge\delta_0$, testing with the negative truncation yields, for $w=(\delta_0-m)_+$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2+\int_\Omega \nabla w^T D(x,\alpha)\nabla w\,\mathrm{d}x=0. \quad (33)$$

Thus $w=0$. Apply this identity on the local existence interval before the total concentration could leave $\{m>\delta_0/2\}$. Continuity and the bound $m\ge\delta_0$ prevent such an exit; the coordinate-face conditions likewise preserve Q_M. Therefore $\mathscr{X}_r^{\delta_0}$ is positively invariant, and Theorem 11 gives a global strong solution for every compatible initial state in this space. The lower concentration bound is proved here because the two tensors coincide; the conclusion is not extended to unequal tensors.

Example 37 (Monotonicity without invariance)**.** *The scalar reaction $R(a)=-a-r$, $r>0$, is strictly monotone, but at $a=0$ it points out of the nonnegative half-line. Thus strict reaction monotonicity does not imply the inward-pointing condition.*

Conclusions

For the specified diagonal quasilinear class, invariant-state bounds and scalar boundary regularity yield global strong solutions, uniform positive-time smoothing, and an attractor compact and attracting in the strong trace norm. The entrywise reaction criterion supplies a common diagonal metric and an explicit dissipation margin for the quantitative energy and singleton results. The nonlinear example shows a varying Jacobian family where this metric succeeds and the identity fails. The perturbation theory is finite-time and, for normalized fractions, conditional on a positive total concentration; the common-diffusion example verifies that condition. Extensions to genuinely nondiagonal or degenerate diffusion, uniform-in-time coefficient perturbations, and structure-preserving discretizations require additional analysis.

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