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# BIRMAN–SCHWINGER FRAMEWORK AND SPECTRAL COMPARISON OF BOSONIC AND FERMIONIC LATTICE TRIMERS

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**Abstract.** We compare two explicitly specified three-particle lattice models at total quasimomentum $\mathbf{K} = 0$: three identical bosons with attractive pair-contact interactions and a $2+1$ system of two identical fermions interacting attractively with a third particle. The common tool is the Birman–Schwinger principle, but the interaction channels and symmetry reductions are model-dependent; we therefore distinguish the shared method from the separate spectral conclusions. For the bosonic model, the strong-coupling analysis gives the ground-state asymptotics $z_1^s(\mu) = -3\mu + 6 - \frac{3}{2\mu} + O(\mu^{-2})$ and, for sufficiently large $\mu$, exactly two eigenvalues below $z_\mu(0) - \varepsilon$ for any fixed $\varepsilon \in (0, \delta_\infty)$, the second at the finite distance $\delta_\infty = 0.0354199892\ldots$. For the fermionic model, the threshold criterion is expressed through a Brillouin-zone integral and yields $\gamma_c = \pi/(\pi-2) \approx 2.75194$. In the strong-coupling regime $\lambda \to \infty$ and above this threshold, the relevant bound-state level is doubly degenerate by square-lattice symmetry. Near onset, the binding-energy scale is described by a linear-over-logarithmic law with the order of limits stated explicitly. We identify which conclusions are inherited from the companion analysis and which are used here to compare the two models. The asymptotic statements are compared with numerical finite-grid Faddeev computations and with the corresponding limiting channel equations.

**Keywords:** Birman–Schwinger principle; lattice Schrödinger operators; bound states; bosons; fermions; critical mass ratio.

## INTRODUCTION

The spectral theory of few-body Schrödinger operators on lattices lies at the crossroads of mathematical physics, quantum many-body theory, and experimental quantum simulation. Recent advances in ultracold atoms in optical lattices [1] and superconducting circuit simulators have made it possible to realize controllable quantum systems with tunable interactions and geometries, thus creating an urgent demand for exact, predictive spectral results.

The spectral theory of lattice Schrödinger operators has been developed by several groups; see e.g. [2, 3, 4, 5, 6] for general existence criteria, mixed-statistics systems, and the lattice Efimov effect with discrete-spectrum asymptotics. General criteria for zero-energy bound states on lattices were derived in [7]. Existence of three-particle bound states in one-dimensional optical lattices was established in [8]. A subspace-by-subspace classification of threshold states and their dependence on coupling constants for a family of finite-rank lattice Hamiltonians was given in [9], where a threshold resonance is distinguished from a genuine threshold eigenvalue within each invariant subspace. A detailed comparison of bosonic and fermionic trimers on the square lattice, including the identification of $\gamma_c$, is given in our companion paper [10]. In three dimensions, critical mass ratios for the repulsive $2+1$ fermionic trimer were determined for $K = 0$ in [11] and for general $K$ along the diagonal in [12].

The purpose of this note is a model-by-model comparison of two explicitly specified lattice systems at $K = 0$ within a shared Birman–Schwinger framework, not a new general Hunziker–van Winter–Zhislin (HVZ) theorem. We use three common steps: (i) reduction below the essential spectrum to a compact Birman–Schwinger

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(equivalently, Faddeev) operator, (ii) decomposition into invariant subspaces using discrete symmetries (parity and particle exchange), and (iii) strong-coupling approximation by a finite-rank principal part with a controlled remainder. These steps do not, by themselves, imply exact eigenvalue counts: such counts require a quantitative remainder bound and a separation from the threshold. Every model-specific result quoted below—the bosonic asymptotic $z_1^s(\mu)$, the two bosonic levels and the finite gap $\delta_\infty$, the critical integral and its closed form $\gamma_c = \pi/(\pi - 2)$, the fermionic asymptotic $z = -\lambda + e_0(\gamma) + O(\lambda^{-1})$, and the double degeneracy—is inherited from [10], where its model-specific derivation is given. The contribution of this note is the comparative organization and the explicit delineation of what the Birman–Schwinger principle does and does not imply without additional estimates. We do not claim that the framework is universal for all few-body systems, nor that it extends the established lattice HVZ theory; the latter is used in its standard form.

## Methodological Framework

### *The models and conventions*

We use the normalized Haar measure $d\nu(p) = (2\pi)^{-2} dp_1 dp_2$ on $\mathbb{T}^2 = (-\pi, \pi]^2$ and write $\varepsilon(p) = 2 - \cos p_1 - \cos p_2$. At total quasimomentum $K = 0$, the bosonic free fiber is the multiplication operator by

$$E_B(p,q) = \varepsilon(p) + \varepsilon(q) + \varepsilon(p+q)$$

on the subspace consisting of $f$ with $f(p,q) = f(q,p) = f(p,-p-q)$ (symmetry under permutations of the three momenta $p, q, -p-q$).

The interacting fiber is

$$H_B(\mu;0) = E_B - \mu(V_{12} + V_{23} + V_{31}), \qquad \mu > 0,$$

where $V_{ij}$ is the contact-pair operator for pair $ij$; its fiber action is the corresponding partial integration in the relative momentum of that pair. The normalization of $V_{ij}$ is fixed by this displayed convention, so that the strong-coupling leading term is $-3\mu$.

For the $2+1$ model, $p, q$ are the momenta of the two identical fermions and $-p-q$ is the momentum of the third particle. The free fiber is multiplication by

$$E_F(p,q) = \varepsilon(p) + \varepsilon(q) + \gamma\varepsilon(p+q),$$

restricted to functions antisymmetric under $p \leftrightarrow q$. We take attractive contact interactions only between unlike particles and write

$$H_F(\gamma,\lambda;0) = E_F - \lambda(V_{13} + V_{23}), \qquad \lambda > 0.$$

$V_{13} + V_{23}$ commutes with the exchange $p \leftrightarrow q$, so its restriction to the antisymmetric subspace is well defined. Thus no interaction between the identical fermions is included. These conventions distinguish the two models; a single expression $H_0 - g\sum_{\langle ij\rangle} V_{ij}$ would not do so.

### *Birman–Schwinger reduction and counting convention*

For $z < \inf\sigma_{\mathrm{ess}}(H_\alpha)$, let $A_\alpha(z) = (H_{0,\alpha} - z)^{-1}$ be the free resolvent. Let $W_\alpha$ denote the channel map with the interaction normalization fixed in Section 2.1. The Birman–Schwinger operator

$$B_\alpha(z) = W_\alpha A_\alpha(z) W_\alpha^*$$

is bounded, self-adjoint and non-negative, but *not* compact: $B_\alpha = D_\alpha + T_\alpha$, where $D_\alpha$ is multiplication in the spectator momentum by $g\langle (E_a(k,Q) - z)^{-1}\rangle_k$ and $T_\alpha$ is the off-diagonal exchange part. For $z < z_{\mathrm{thr}} := \inf\sigma_{\mathrm{ess}}(H_\alpha)$, the function $\Delta_\alpha := 1 - D_\alpha$ is strictly positive and continuous, so $\Delta_\alpha \geq c(z) > 0$. The Faddeev operator $A_\alpha^{\mathrm{F}}(z) = \Delta_\alpha^{-1/2} T_\alpha(z) \Delta_\alpha^{-1/2}$ is compact (since $T_\alpha$ has a continuous kernel). By Sylvester's law of inertia applied to the congruence $B_\alpha(z) - 1 = \Delta_\alpha^{1/2}(A_\alpha^{\mathrm{F}}(z) - 1)\Delta_\alpha^{1/2}$, one has $N(B_\alpha(z) > 1) = N(A_\alpha^{\mathrm{F}}(z) > 1)$. The explicit channel kernels verifying this compactness are given in [13, Lemma 4.2] and in the Supplementary Material. References to the numbering of lemmas and theorems in [13] follow the current arXiv version (v2, 1 November 2026). The Birman–Schwinger principle [14] gives

$$N(H_\alpha < z) = N(B_\alpha(z) > 1),$$

with both counts including multiplicity. This principle does not by itself yield a count: a finite-rank approximation $A^{\mathrm{F,p}}$ must be accompanied by a norm estimate $\|A^{\mathrm{F}} - A^{\mathrm{F,p}}\| \le r(z)$ and a spectral separation from 1 larger than $r(z)$. We use this distinction below. The compactness of $A^{\mathrm{F}}_{\alpha}(z)$ and the explicit channel kernels are verified for the two models in [10].

### *Invariant subspace decomposition*

The kernel of $\mathscr{B}_{\alpha}(z)$ inherits the parity and permutation symmetries of the free dispersion $E_B$ or $E_F$. For bosons ($\alpha = \mathrm{bos}$) this leads to a decomposition

$$L^2_s((\mathbb{T}^2)^2) = L^{s,e} \oplus L^{s,o},$$

where the even subspace $L^{s,e}$ contains the bound-state branches considered here. The absence of bound states in the odd subspace is a model-specific conclusion of the symmetry-reduced analysis in [10], not a consequence of parity alone. For fermions ($\alpha = \mathrm{ferm}$) the antisymmetry is built into the space from the beginning, but further decompositions based on the parity of the relative momenta can be performed, revealing critical subspaces associated with the two-particle channels.

### *Strong-coupling asymptotics and positivity*

For large coupling ($\mu \to \infty$ for bosons, $\lambda \to \infty$ for fermions) the Faddeev operator splits as $A^{\mathrm{F}}_{\alpha}(z) = A^{\mathrm{F,p}}_{\alpha}(z) + A^{\mathrm{F,r}}_{\alpha}(z)$, where $A^{\mathrm{F,p}}_{\alpha}(z)$ has finite rank and, for every fixed $\delta_0 > 0$, $\|A^{\mathrm{F,r}}_{\alpha}(z)\| = O(g^{-1}\ln(1/\delta_0))$ uniformly for $z \le z_{\mathrm{thr}} - \delta_0$; the logarithm is the two-dimensional threshold singularity of $\langle(\varepsilon+\delta_0)^{-1}\rangle$, and the estimate is not uniform as $\delta_0 \downarrow 0$. In the bosonic even sector, positivity of the relevant compact kernel and positivity improvement on the cone of nonnegative functions yield simplicity and strict positivity of the principal eigenfunction when the corresponding irreducibility hypothesis holds. Krein–Rutman does not by itself establish the total number of bound states; the latter also requires the finite-rank eigenvalue count and control of the remainder. A detailed asymptotic analysis of the Fredholm determinant yields explicit formulas for the eigenvalues; the model-specific derivation is in [10].

## Results and Discussion

### *Bosonic trimer (three identical bosons)*

For the three-boson system on $\mathbb{Z}^2$, the companion analysis [10] establishes the following model-specific strong-coupling conclusions, summarized here for comparison.

- For all sufficiently large $\mu$ the operator $H_{\mu}(\mathbf{0})$ has exactly two eigenvalues below $z_{\mu}(0) - \eta_{\mu}$, $\eta_{\mu} = 36/(\mu - 10)$ ($\mu > 16$). By the spectral transfer theorem [13, Theorem 4.6], the counting function is squeezed between those of the limit operator. Since the second limit eigenvalue satisfies $z_{\mu}(0) - z^s_2(\mu) = \delta_{\infty} + O(\mu^{-1})$ (third item), for every fixed $\varepsilon \in (0, \delta_{\infty})$ exactly two eigenvalues lie below $z_{\mu}(0) - \varepsilon$ for all sufficiently large $\mu$. An eigenvalue in the window of width $O(\mu^{-1})$ below $z_{\mu}(0)$ is not excluded by this estimate.

- The ground-state energy satisfies the asymptotics

  $$z^s_1(\mu) = -3\mu + 6 - \tfrac{3}{2\mu} + O(\mu^{-2}), \quad \mu \to \infty,$$

  and the corresponding eigenfunction is strictly positive in the momentum representation (Perron–Frobenius).

- The second level is $z^s_2(\mu) = -\mu + C + O(\mu^{-1})$ with $C = 4 - \delta_{\infty} = 3.9645800108\ldots$, where $\delta_{\infty} = 0.0354199892\ldots$ is the unique root of $b_0(\delta) = 1/(1+\delta)$ with $b_0(\delta) = \langle(\varepsilon+\delta)^{-1}\rangle$. Since $z_{\mu}(0) = -\mu + 4 - 4/\mu + O(\mu^{-3})$, the second level stays at the finite distance $z_{\mu}(0) - z^s_2(\mu) = \delta_{\infty} + O(\mu^{-1})$ below the threshold; it is a shallow atom–dimer bound state, not a level approaching the threshold. This is a two-dimensional effect: the $s$-channel eigenvalue $m_s(\delta) = 2 + \delta - 1/b_0(\delta)$ is strictly decreasing, since $m'_s(\delta) = 1 + b'_0(\delta)/b_0(\delta)^2 < 0$

by the strict Cauchy–Schwarz inequality applied to the non-constant function $(\varepsilon+\delta)^{-1}$ on $\mathbb{T}^2$, and $m_s(0^+) = 2 > 1$ because $b_0$ diverges logarithmically; on $\mathbb{Z}^3$ the integral $b_0(0)$ is finite and this mechanism is absent.

These results are summarized in Fig. 1(a),(b).

***Fermionic*** $2+1$ ***trimer***

For the fermionic $2+1$ system on $\mathbb{Z}^2$ the picture is fundamentally different. The antisymmetry of the wave function under exchange of the two identical fermions introduces a strong Pauli suppression that can only be overcome when the mass ratio $\gamma = m_f/m_3$ exceeds a critical value $\gamma_c$.

The analysis follows the same Birman–Schwinger strategy as in the bosonic case, but the antisymmetry of the Hilbert space leads to a more intricate Fredholm determinant $\Delta_{\gamma,\lambda}(p,z)$, whose zeroes determine the two-particle threshold $z_{\rm thr}(\gamma,\lambda)$. In three dimensions, critical mass ratios for the repulsive $2+1$ fermionic trimer have been determined for $K=0$ in [11] and for general $K$ along the diagonal in [12]; these three-dimensional results concern a different spectral regime (repulsive contact, eigenvalues to the right of the essential spectrum) and are cited here for context rather than as direct evidence for the two-dimensional assertion below.

Our main result is:

- There exists a critical mass ratio $\gamma_c > 0$ such that:
  1. In the strong-coupling threshold criterion, if $\gamma < \gamma_c$, the principal Birman–Schwinger threshold eigenvalue remains below 1. The stronger statement that no bound state exists for every finite $\lambda > 0$ is asserted in [10] and is not inferred here from the strong-coupling expansion alone.
  2. If $\gamma > \gamma_c$, then for all sufficiently large $\lambda$ the model has the doubly degenerate bound-state level identified in [10], with asymptotics $z(\gamma,\lambda) = -\lambda + e_0(\gamma) + O(\lambda^{-1})$ as $\lambda \to \infty$, where $e_0(\gamma) = 2(1+\gamma) - \delta_F(\gamma)$, and $\delta_F(\gamma) = \lim_{\lambda\to\infty}(z_{\rm thr} - z)$ is the unique positive root of $\gamma s_1(\delta) = 1$, $s_1(\delta) = \langle \sin^2 q_1/(\varepsilon+\delta)\rangle$; this root exists iff $\gamma > \gamma_c = 1/s_1(0)$, since $s_1$ is strictly decreasing.
- The critical value is given by an integral over the Brillouin zone,

$$\gamma_c^{-1} = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \frac{(\sin q_1 + \sin q_2)^2}{4 - 2(\cos q_1 + \cos q_2)}\, dq_1 dq_2, \tag{1}$$

  which yields $\gamma_c = \pi/(\pi-2) \approx 2.7519383938$ in closed form (Appendix).
- Define the threshold detachment in the ordered limit $\lambda \to \infty$ first and then $\gamma \downarrow \gamma_c$. The asymptotic law derived in [10] is

$$\delta_F(\gamma) \sim \frac{2\pi}{\gamma_c^2}\,\frac{\gamma-\gamma_c}{\ln(1/(\gamma-\gamma_c))}, \qquad \gamma \downarrow \gamma_c,$$

  ($\sim$ means that the ratio of the two sides tends to 1), i.e. linear up to a logarithm. This is a spectral threshold law, not by itself a thermodynamic classification as a second-order phase transition.

The criterion $\gamma > \gamma_c$ above is a strong-coupling statement at $K = 0$. The full Brillouin-zone dependence of the critical mass ratio $\gamma_c(\mathbf{K})$ on $\mathbb{T}^2$, the second fermionic threshold $\gamma_c^{(2)}(\mathbf{K})$, the closed forms at the symmetry points, and the

onset laws for both the $s$-wave and the pure $p$-wave channels are analysed in [13].

The phase diagram and level detachment are illustrated in the Fig. 1(c),(d). The observed double degeneracy is consistent with the symmetry of the effective atom–dimer Hamiltonian $h_F$ under the point group $D_4$ of the square lattice. $D_4$ has four one-dimensional irreps $(A_1, A_2, B_1, B_2)$ and one two-dimensional irrep $E$. The double degeneracy requires that the level transform in $E$ (at $K = 0$; the pair $\sin p_1, \sin p_2$ spans $E$ and is odd under inversion). This identification is a model-specific spectral statement cited from [13, Theorem 4.6]. This is the lattice analogue of the universal trimers of Kartavtsev and Malykh [15] and Petrov [16]. Symmetry alone does not establish existence or uniqueness of the level.

### *Comparison and unifying perspective*

Table 1 summarizes the key differences between bosonic and fermionic trimers, highlighting the shared methodological core and the distinct physical outcomes dictated by quantum statistics.

Table 2 places our work in the context of related results.

## Implications and Outlook

The spectral results provide mathematical benchmarks that may be relevant to the following applications; these applications are motivation, not consequences demonstrated by the present analysis.

- **Quantum simulation.** The critical mass ratio $\gamma_c \approx 2.75194$ provides a design rule for experiments: to observe a fermionic trimer, the two identical fermions must be the heavier (or less mobile) species, e.g. $^{40}$K with $^{6}$Li ($m_f/m_3 \approx 6.6$), $^{161}$Dy with $^{40}$K ($\approx 4.0$) or $^{53}$Cr with $^{6}$Li ($\approx 8.8$); on a lattice the relevant $\gamma$ is the ratio $J_3/J_f$ of tunnelling amplitudes, which can be tuned by species-selective lattice depths.
- **Computational materials science.** The invariant subspace decomposition and the strong-coupling asymptotics provide a computational benchmark for exact finite-volume Faddeev solvers of few-body lattice models.
- **Mathematical physics.** The work uses the established spectral theory of lattice Schrödinger operators and compares two model-specific threshold mechanisms; no new HVZ theorem is claimed here.

Future directions include mixed statistics (Bose–Fermi mixtures), higher dimensions, and additional interaction channels.

## Conclusions

We have compared two three-particle lattice models using the Birman–Schwinger principle, invariant-subspace decomposition, and strong-coupling asymptotics. The common method does not make the two models identical and is not claimed to be universal for all few-body lattice systems. Applying this framework to three identical bosons and to the fermionic $2+1$ trimer reveals a striking dichotomy: for all sufficiently large $\mu$ the bosonic operator has two bound states, while fermions exhibit a sharp mass-ratio threshold $\gamma_c$ below which no binding occurs in the strong-coupling threshold criterion; a statement for arbitrary finite $\lambda$ would require the monotonicity argument of [10]. The onset at $\gamma_c$ is continuous and obeys the stated linear-over-logarithmic threshold law in the specified order of limits; we do not infer a thermodynamic "second-order transition" from this spectral law alone. These results provide benchmarks for quantum simulators.

The methodological framework presented here complements recent advances in the spectral theory of lattice operators, including general criteria for zero-energy bound states [7], existence results for three-particle systems in one dimension [8], and the analysis of mixed-statistics systems [3]. A unified perspective on bosonic and fermionic trimers on the square lattice is provided in our companion paper [10].

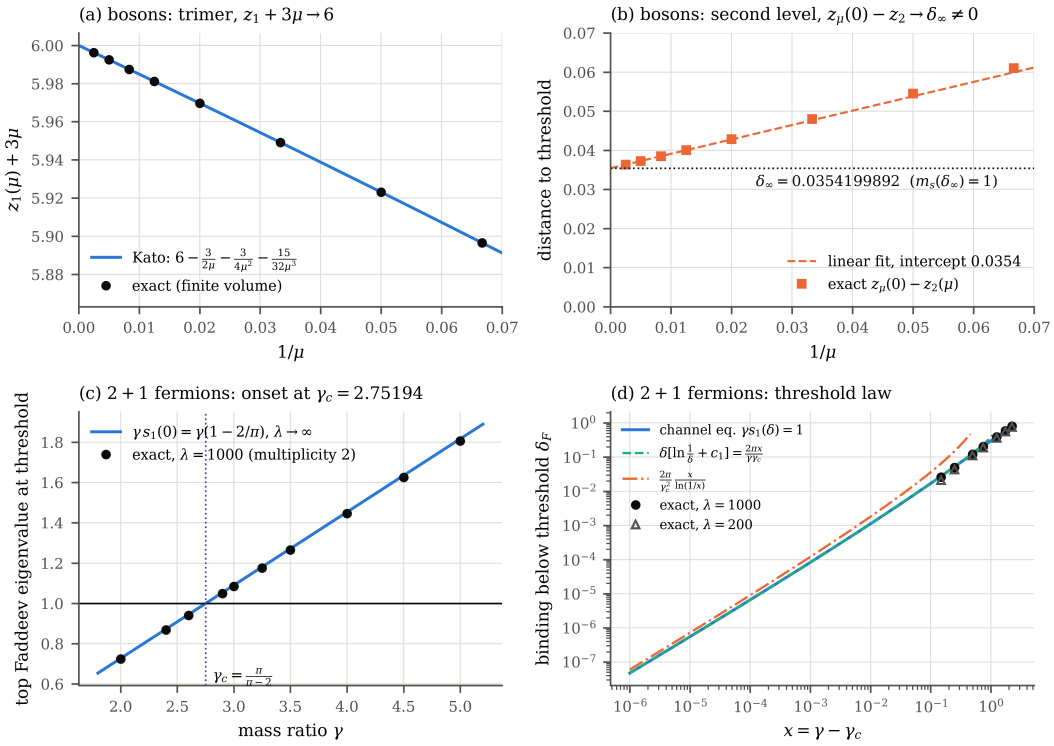



**Figure 1 -** Bosonic and fermionic trimers on $\mathbb{Z}^2$ at $K = 0$. Markers show numerical Faddeev computations using a $32 \times 32$ momentum-grid discretization; curves show the corresponding asymptotic expansions or limiting channel equations. (a) Bosons: $z_1(\mu) + 3\mu$ versus $1/\mu$ and the leading Kato terms $6 - 3/(2\mu) - 3/(4\mu^2) - 15/(32\mu^3)$. (b) Bosons: the distance $z_\mu(0) - z_2(\mu)$ and its limiting value $\delta_\infty = 0.0354199892\ldots$, determined by $b_0(\delta_\infty) = 1/(1 + \delta_\infty)$. (c) $2+1$ fermions: the largest threshold Faddeev eigenvalue for $\lambda = 1000$, compared with its strong-coupling limit $\gamma s_1(0) = \gamma(1 - 2/\pi)$; the limiting crossing occurs at $\gamma_c = \pi/(\pi - 2)$. (d) Fermionic binding energy $\delta_F$ versus $x = \gamma - \gamma_c$: finite-coupling computations for $\lambda = 200, 1000$, the root of the limiting channel equation, the implicit linear-over-logarithmic approximation, and its leading asymptotic form are distinguished. The constant $c_1$ in the figure legend corresponds to $\kappa_0 = 1 + 4\ln 2 - \pi$ in the text.

**Table 1 –** Spectral characteristics of bosonic and fermionic trimers on a square lattice at $K = 0$. The entries for each model are inherited from [10]; the table is a comparative summary, not an independent derivation.

| Aspect | Bosonic (3 identical bosons) | Fermionic $(2+1)$ |
|---|---|---|
| Hilbert space | symmetric $L_s^2$ | antisymmetric $L_a^2$ |
| Key parameter | coupling $\mu$ | mass ratio $\gamma$, coupling $\lambda$ |
| Bound states below $\sigma_{\text{ess}}$ | exactly two below $z_\mu(0) - \eta_\mu$ (large $\mu$) | for $\gamma > \gamma_c$ and sufficiently large $\lambda$, one doubly degenerate level |
| Ground-state asymptotics | $z_1^s = -3\mu + 6 - \frac{3}{2\mu} + O(\mu^{-2})$ | $z = -\lambda + 2(1+\gamma) - \delta_F(\gamma) + O(\lambda^{-1})$ |
| Threshold behaviour | second level at distance $\delta_\infty = 0.0354$ below threshold | $\delta_F[\ln(1/\delta_F) + \kappa_0] = \frac{2\pi(\gamma - \gamma_c)}{\gamma\gamma_c}(1 + o(1))$, $\kappa_0 = 1 + 4\ln 2 - \pi = 0.6309960686\ldots$ |
| Main mathematical tool | Birman–Schwinger + Krein–Rutman | Birman–Schwinger + Faddeev reduction |

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## EVALUATION OF THE CRITICAL INTEGRAL

$$I = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{(\sin q_1 + \sin q_2)^2}{4 - 2(\cos q_1 + \cos q_2)}\, dq_1 dq_2.$$

**Table 2 –** Comparison of spectral properties of few-body lattice models. The present note is a comparative synthesis; it does not claim a new existence or counting theorem.

| Work | System | Dim. | Key result | Main tool |
|---|---|---|---|---|
| This comparative note | bosonic and $2+1$ models | 2D | model-by-model synthesis; no new HVZ theorem | Birman–Schwinger + symmetry reduction |
| Abdullaev et al. (companion [10]) | bosons and $2+1$ fermions | 2D | model-specific threshold and strong-coupling results | Birman–Schwinger reduction |
| Khalkhuzhaev et al. [12] | $2+1$ fermions (repulsive) | 3D | critical $\gamma_s(K)$, $\gamma_{as}(K)$ | BS + invariant subspaces |
| Khalkhuzhaev et al. [11] | $2+1$ fermions (repulsive, $K=0$) | 3D | critical mass ratios; simple eigenvalue | BS + Faddeev |

We evaluate $I$, since $|\sin q_1 \sin q_2| \le \frac{\pi^2}{4}\varepsilon(q)$, the cross term $2\sin q_1 \sin q_2/(2\varepsilon)$ is bounded in a neighborhood of the origin (where $\varepsilon(q)$ vanishes), and therefore the singularity is removable. This term is odd in $q_1$, so it integrates to zero; with the $q_1 \leftrightarrow q_2$ symmetry,

$$I = \frac{2}{(2\pi)^2}\int_{[-\pi,\pi]^2} \frac{\sin^2 q_1}{4-2(\cos q_1+\cos q_2)}\,dq_1 dq_2.$$

Since the integrand is nonnegative, we may integrate in $q_2$ first by Tonelli's theorem (for $q_1 \neq 0$ one has $a = 2-\cos q_1 > 1$ and $\int_{-\pi}^{\pi} \frac{dq_2}{2(a-\cos q_2)} = \frac{\pi}{\sqrt{a^2-1}}$); using $a^2-1 = (1-\cos q_1)(3-\cos q_1)$, one obtains

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin^2 q_1}{\sqrt{(1-\cos q_1)(3-\cos q_1)}}\,dq_1.$$

The substitutions $u = q_1/2$ and $v = \cos u$ give $I = \frac{4}{\pi}\int_0^1 \frac{v^2\,dv}{\sqrt{2-v^2}}$, and then $v = \sqrt{2}\sin\theta$ gives

$$I = \frac{4}{\pi}\int_0^{\pi/4} 2\sin^2\theta\,d\theta = \frac{4}{\pi}\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi-2}{\pi},$$

hence $\gamma_c = I^{-1} = \pi/(\pi-2) \approx 2.7519383938\ldots$. The model-specific reduction of the Birman–Schwinger threshold to this integral, with all channel kernels and Jacobians, is given in [10].

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