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SPECTRAL ANALYSIS OF A *P*-ADIC TRANSPORT OPERATOR ARISING FROM FRACTURED POROUS MEDIA FLOWS

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Abstract. We investigate the essential spectrum of the p-adic transport operator $T = T_1 + S$ acting on the non-Archimedean Banach space $c_0(\mathbb{Q}_p)$, where T_1 is the diagonal operator $(T_1x)_n = p^{-n}x_n$ and S is the weighted shift $(Sx)_n = p^{-(n+1)}x_{n+1}$. To study the stability of the essential spectrum under perturbations, we introduce the notion of relative strict singularity for closed operators on non-Archimedean Banach spaces and establish corresponding perturbation results for upper semi-Fredholm operators. We prove that S is not T_1-strictly singular, nevertheless, the essential spectrum is unchanged, due to the factorization $T = (I+L)T_1$ with $I+L$ invertible and T^{-1} compact, yielding

$$\sigma(T) = \{p^{-k} : k \geq 1\}, \qquad \sigma_e(T) = \sigma_{e1}(T) = \emptyset.$$

These results provide a spectral stability framework for hierarchical transport models arising in p-adic descriptions of fractured porous media.

Keywords: p-adic transport model, fractured porous media, non-Archimedean Banach spaces, essential spectrum, strictly singular perturbations.

INTRODUCTION

The study of fluid flow in fractured porous media lies at the intersection of geophysics, hydrology, and petroleum engineering. It encompasses a wide range of applications, from groundwater transport in aquifers to hydrocarbon migration in subsurface reservoirs, and has been extensively investigated in the literature [1, 2]. Despite significant progress since Washburn's pioneering work on capillary motion [3], a comprehensive mathematical description of transport in such media remains a challenging open problem.

A central difficulty stems from the highly complex and heterogeneous geometry of natural porous formations. Geological media exhibit multiscale structures ranging from microscopic pores to macroscopic fracture networks. This hierarchy is often described in terms of fractal geometry [4, 5]. Experimental studies, particularly those of Oleschko and collaborators on oil-bearing formations, reveal self-similar pore networks with pronounced branching across scales [6]. These observations suggest that transport processes are more naturally described on hierarchical, tree-like structures rather than within classical Euclidean domains.

Transport in such media results from the interplay of several mechanisms: pressure-driven advection along fracture channels, diffusion between hierarchical levels, and chemical reactions such as adsorption, dissolution, and precipitation in the surrounding matrix. The combined effect of these processes governs the global dynamics of fluid propagation [7, 8, 9, 10].

Classical continuum models, based on the Navier–Stokes or Richards equations, rely on Euclidean geometry and smooth spatial domains. In contrast, hierarchical porous media are better described using combinatorial distances induced by branching structures and ancestral relations between nodes [5]. This motivates the development of mathematical models directly formulated on tree-like spaces.

In this direction, Khrennikov and collabora-

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tors proposed p-adic analysis as a natural framework for hierarchical systems [11, 12]. The field $\mathbb{Q}_p$ carries a canonical ultrametric structure that encodes hierarchical organization. Every p-adic number admits an expansion

$$x = \sum_{k=n}^{\infty} a_k p^k, \quad a_k \in \{0, 1, \ldots, p-1\},$$

where the valuation $v_p(x) = n$ defines its hierarchical level and the associated norm $|x|_p = p^{-n}$ satisfies the ultrametric inequality

$$|x+y|_p \leq \max(|x|_p, |y|_p),$$

reflecting the geometry of rooted trees [13, 14]. This makes $\mathbb{Q}_p$ a natural analytical model for fractal and hierarchical media.

As emphasized in works by Khrennikov and Oleschko, p-adic frameworks allow one to combine algebraic and topological structures, enabling an analysis analogous to classical methods but adapted to hierarchical systems [11]. This approach connects naturally with non-Archimedean functional analysis [15, 16, 13, 14, 17], p-adic mathematical physics [18, 19], and ultrametric models of complex systems such as spin glasses [20, 21], where hierarchical energy landscapes arise naturally.

On the applied side, geophysical investigations by Oleschko and collaborators at UNAM provide further experimental evidence supporting fractal pore geometries and the relevance of hierarchical modeling in petroleum reservoirs [12, 6].

Motivated by this framework, we consider a discrete transport model capturing the essential features of p-adic hierarchical dynamics. Following [12, 22], we work on the non-Archimedean Banach space

$$X = c_0(\mathbb{Q}_p) = \left\{ x = (x_n)_{n\geq 1} \subset \mathbb{Q}_p : \lim_{n\to\infty} x_n = 0 \right\},$$

equipped with the norm

$$\|x\| = \sup_{n\geq 1} |x_n|_p,$$

where x_n represents the concentration at level n of the hierarchical network.

The transport operator is defined as $T = T_1 + S$, where:

- $(T_1 x)_n = p^{-n} x_n$, modeling downward transport through the hierarchy with level-dependent scaling;
- $(Sx)_n = \frac{1}{p^{n+1}} x_{n+1}$, modeling diffusion and interaction between adjacent levels via a weighted shift.

Both operators are densely defined with

$$D(T) = D(T_1).$$

The unperturbed operator T_1 has discrete spectrum

$$\sigma(T_1) = \{p^{-k} : k \geq 1\}.$$

For each $\lambda = p^{-k}$, the operator $T_1 - \lambda I$ is Fredholm of index zero, while for $\lambda \notin \sigma(T_1)$ the resolvent is compact. Consequently, with the Fredholm definition of the essential spectrum adopted below,

$$\sigma_e(T_1) = \sigma_{e1}(T_1) = \emptyset.$$

The central question of this work is:

Does the perturbation S preserve this spectral structure?

This question is directly related to the stability of long-term transport dynamics, since any change in the essential spectrum would indicate a qualitative modification of asymptotic behavior.

To address this problem, we develop a perturbation theory for unbounded operators in non-Archimedean Banach spaces. Classical results [23, 24, 25] are not directly applicable due to the ultrametric structure and the lack of complementability of finite-dimensional subspaces in this setting. This structural limitation requires new techniques adapted to the non-Archimedean framework.

A key tool is the notion of relative strict singularity. Given operators T and S with

$$D(T) \subset D(S),$$

we say that S is T-strictly singular if, when regarded as an operator on $(D(T), \|\cdot\|_T)$ endowed with the graph norm of T, it fails to be an

isomorphism on any infinite-dimensional closed subspace. This extends the classical concept of strictly singular operators, which play a central role in Fredholm theory as generalizations of compact perturbations.

Within this framework, we prove the stability result

$$\sigma_{\mathrm{e}}(T_1+S)=\sigma_{\mathrm{e}}(T_1),$$

provided that S is T_1-strictly singular. However, for the transport model considered here, the perturbation S is not T_1-strictly singular: Lemma 45 shows that S is an isometry on the infinite-dimensional subspace

$$M=\{x\in D(T_1): x_1=0\}.$$

The main contributions of this paper are:

1. We develop a perturbation theory for unbounded operators on non-Archimedean Banach spaces, introducing relative strict singularity and proving stability results for upper semi-Fredholm operators (Theorems 27–41).

2. We show that the transport operator

$$T=T_1+S$$

satisfies

$$\sigma(T)=\sigma(T_1)=\{p^{-k}: k\geq 1\}$$

and that the essential spectrum is empty:

$\sigma_{\mathrm{e}}(T)=\sigma_{\mathrm{e}1}(T)=\emptyset=\sigma_{\mathrm{e}}(T_1)$ (Theorem 49).

3. We clarify that the spectral stability is not a consequence of relatively strictly singular perturbation theory, since S is not T_1-strictly singular (Lemma 45, Remark 46).

The paper is organized as follows. Section recalls preliminaries on non-Archimedean fields, Banach spaces, and linear operators. Section develops Fredholm and semi-Fredholm theory in the p-adic setting, introduces strictly singular and relatively strictly singular operators, and establishes the main stability theorems. Section applies the abstract results to the transport model and computes its essential spectrum explicitly. Section provides concluding remarks and physical interpretation.

PRELIMINARIES AND AUXILIARY RESULTS

In this section, we recall some preliminary notions on non-Archimedean fields, Banach spaces, and linear operators.

Definition 1 *Let $\mathbb{K}$ be a field. A valuation on $\mathbb{K}$ is a map $|\cdot|:\mathbb{K}\to\mathbb{R}_+$ satisfying:*

1. *$|\alpha|=0$ if and only if $\alpha=0$;*
2. *$|\alpha\beta|=|\alpha||\beta|$ for all $\alpha,\beta\in\mathbb{K}$;*
3. *There exists a constant $c\geq 1$ such that*

$$|\alpha+1|\leq c$$

whenever $|\alpha|\leq 1$.

A valued field is a pair $(\mathbb{K},|\cdot|)$.

Definition 2 *A valuation $|\cdot|$ on $\mathbb{K}$ is called non-Archimedean if it satisfies the ultrametric inequality*

$$|\alpha+\beta|\leq\max(|\alpha|,|\beta|)\quad \text{for all } \alpha,\beta\in\mathbb{K}.$$

Proposition 3 *A valuation $|\cdot|$ is non-Archimedean if and only if one can take $c=1$ in Definition 1(3).*

Remark 4 *Valuations are classified into two types: Archimedean (e.g., on $\mathbb{R}$ and $\mathbb{C}$) and non-Archimedean.*

Definition 5 *Let X be a vector space over $\mathbb{K}$. A non-Archimedean norm on X is a map $\|\cdot\|:X\to\mathbb{R}_+$ satisfying:*

1. *$\|x\|=0$ if and only if $x=0$;*
2. *$\|\lambda x\|=|\lambda|\|x\|$ for all $\lambda\in\mathbb{K}$ and $x\in X$;*
3. *$\|x+y\|\leq\max(\|x\|,\|y\|)$ for all $x,y\in X$ (ultrametric property).*

Definition 6 *A non-Archimedean Banach space is a normed space $(X,\|\cdot\|)$ over $\mathbb{K}$ that is complete with respect to the metric $d(x,y)=\|x-y\|$.*

Remark 7 *Let X be a non-Archimedean Banach space over $\mathbb{K}$. If $\{\lambda_n\} \subset \mathbb{K}$ and $\{x_n\} \subset X$ satisfy $\lambda_n x_n \to 0$, then*

$$\left\| \sum_{n\in\mathbb{N}} \lambda_n x_n \right\| \leq \max_{n\in\mathbb{N}} \|\lambda_n x_n\|.$$

Definition 8 *Let X, Y be non-Archimedean Banach spaces over $\mathbb{K}$. A linear operator*

$$T : D(T) \subset X \to Y$$

satisfies

$$T(\alpha x+\beta y) = \alpha Tx+\beta Ty, \ \forall x, y \in D(T), \alpha, \beta \in \mathbb{K}.$$

Definition 9 *Let T be a linear operator on a non-Archimedean Banach space X. We define:*

$$\sigma(T) = \{\lambda \in \mathbb{K} : T - \lambda I \text{ is not invertible}\},$$

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

$$N(T) = \{x \in D(T) : Tx = 0\},$$

and

$$R(T) = \{Tx : x \in D(T)\}.$$

Definition 10 *A linear operator $T : X \to Y$ is bounded if $D(T) = X$ and there exists $C > 0$ such that*

$$\|Tx\| \leq C\|x\|, \quad \forall x \in X.$$

The space of bounded linear operators is denoted by $\mathscr{L}(X, Y)$, and

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}.$$

Definition 11 *Let X be a non-Archimedean normed space.*

1. *A nonempty subset E of X is absolutely convex if $\lambda x + \mu y \in E$ for all $x, y \in E$ and $\lambda, \mu \in B_{\mathbb{K}}(0,1) = \{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$, i.e., if E is a $B_{\mathbb{K}}(0,1)$-module.*

2. *A subset E is compactoid if there exists a sequence $(y_n)_n$ such that $y_n \to 0$ and $E \subset \mathrm{Co}\{y_1, \ldots, y_n\}$, where*

$$\mathrm{Co}\{y_1, \ldots, y_n\} = \{\sum_{i=1}^{n} \lambda_i y_i : \lambda_i \in \mathbb{K}, |\lambda_i| \leq 1\}.$$

3. *A linear operator $T : X \to X$ is said to be compact if $T(B_X(0,1))$ is compactoid. We denote by $\mathscr{K}(X)$ the set of all compact operators in X.*

Proposition 12 *Let X be a non-Archimedean Banach space. A subset E is compactoid if and only if there exists a sequence $(y_n)_n$ such that $y_n \to 0$ and $E \subset \mathrm{Co}\{y_1, \ldots, y_n\}$, where $\mathrm{Co}\{y_1, \ldots, y_n\} = \{\sum_{i=1}^{n} \lambda_i y_i : \lambda_i \in \mathbb{K}, |\lambda_i| \leq 1\}$.*

Theorem 13 *Let X be a non-Archimedean Banach space and $T \in \mathscr{L}(X)$. Then T is compact if and only if for every bounded sequence $(x_n)_n$, there exists $(y_n)_n$ such that $y_n \to 0$ and $Tx_n \to y$, with $y = \lim_{n\to+\infty} \sum_{n\geq 1} \lambda_n y_n$.*

Definition 14 *An operator $T : D(T) \subset X \to Y$ is* closed *if its graph*

$$G(T) = \{(x, Tx) : x \in D(T)\}$$

is closed in $X \times Y$.

Definition 15 *The graph norm on $D(T)$ is defined by*

$$\|x\|_T = \max(\|x\|, \|Tx\|), \quad x \in D(T).$$

Lemma 16 *Let X be a non-Archimedean Banach space. Let $A : X \to Y$ be an unbounded linear operator. Then $X_A = (D(A), \|\cdot\|_A)$ is a non-Archimedean Banach space if and only if $A \in \mathscr{C}(X, Y)$.*

Definition 17 *Let $T, S : D \subset X \to Y$ be linear operators.*

1. *S is an extension of T (denoted $T \subset S$) if $D(T) \subset D(S)$ and $Sx = Tx$ for all $x \in D(T)$.*

2. *An operator T is closable if it admits a closed extension, called its closure $\overline{T}$.*

Definition 18 *Let X, Y be non-Archimedean Banach spaces with $D(T) \subset D(S)$. The operator S is called T-bounded if there exist $a, b \geq 0$ such that*

$$\|Sx\| \leq \max(a\|x\|, b\|Tx\|), \quad \forall x \in D(T).$$

The infimum of such b is called the T-bound of S.

Definition 19 *Let $T : D(T) \subset X \to X$ be closed. We define:*

$$\alpha(T) = \dim N(T), \quad \beta(T) = \operatorname{codim} R(T),$$

and

$$i(T) = \alpha(T) - \beta(T).$$

Definition 20 *Let X, Y be non-Archimedean Banach spaces. The classes of semi-Fredholm and Fredholm operators are:*

$$\Phi_+(X,Y) = \{T \in \mathscr{C}(X,Y) : \alpha(T) < \infty, R(T) closed\},$$
$$\Phi_-(X,Y) = \{T \in \mathscr{C}(X,Y) : \beta(T) < \infty, R(T) closed\}$$
$$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y).$$

Definition 21 *Let $T : D(T) \subset X \to X$ be closed. The essential spectrum is defined by*

$$\sigma_{\mathrm{e}}(T) = \{\lambda \in \mathbb{K} : T - \lambda I \notin \Phi(X) \text{ or } i(T - \lambda I) \neq 0\}.$$

The upper essential spectrum is defined by

$$\sigma_{\mathrm{e1}}(T) = \{\lambda \in \mathbb{K} : T - \lambda I \notin \Phi_+(X)\}.$$

Equivalently,

$$\sigma_{\mathrm{e}}(T) = \mathbb{K} \setminus \rho_e(T),$$

where

$$\rho_e(T) = \{\lambda \in \mathbb{K} : T - \lambda I \in \Phi(X), i(T - \lambda I) = 0\}.$$

Remark 22 *The essential spectrum can be empty even when T has infinitely many eigenvalues. This occurs whenever T has compact resolvent, i.e. $(T - \lambda I)^{-1}$ is compact for every $\lambda \notin \sigma(T)$: at such λ, $T - \lambda I \in \Phi(X)$ with index 0 automatically, and at each eigenvalue $\lambda_0 \in \sigma(T)$ one separately checks that $T - \lambda_0 I \in \Phi(X)$ with index 0; together these give $\sigma_e(T) = \sigma_{e1}(T) = \emptyset$, regardless of how many eigenvalues T has. This is precisely the situation for the diagonal operator T_1 studied in Section : although $\sigma(T_1) = \{p^{-k} : k \geq 1\}$ is an infinite set (Lemma 43 below), its essential spectrum vanishes entirely, and the same phenomenon persists for the full transport operator $T = T_1 + S$ despite the perturbation S (Theorem 49). Remark 50 explains, at the level of the p-adic metric, why this coexistence of an infinite point spectrum with an empty essential spectrum is not paradoxical.*

Definition 23 *An operator $S \in \mathscr{L}(X,Y)$ is strictly singular if its restriction to any infinite-dimensional closed subspace of X fails to be an isomorphism. We denote by $\mathscr{S}(X,Y)$ the class of strictly singular operators.*

Definition 24 *Let T, S be linear operators with $D(T) \subset D(S)$. We say that S is T-strictly singular if S, viewed on $(D(T), \|\cdot\|_T)$, is strictly singular.*

Lemma 25 *Let X, Y, and Z be three non-Archimedean Banach spaces such that $\|Y\| \subseteq |\mathbb{K}|$. Let $T \in \Phi_+(Y,Z)$ and $S \in \mathscr{C}(X,Y)$. Then $TS \in \mathscr{C}(X,Z)$.*

Lemma 26 *Let T and S be strictly singular operators. Then for all $\phi, \psi \in \mathbb{K}$, the operator $\phi T + \psi S$ is strictly singular.*

MAIN RESULTS

This section allows us to derive stability results for semi-Fredholm operators under perturbations by unbounded and relatively strictly singular operators. In addition, we obtain new results concerning the composition of strictly singular operators and provide a characterization of the associated essential spectra.

Theorem 27 *Let X, Y, and Z be non-Archimedean Banach spaces.*

1. *If $S \in \mathscr{S}(X,Y)$ is strictly singular and $T \in \mathscr{L}(Y,Z)$ is bounded and linear, then $TS \in \mathscr{S}(X,Z)$.*
2. *If $T \in \mathscr{C}(X,Y)$ is a closed linear operator with $D(T) = X$, and $S \in \mathscr{S}(Y,Z)$ is strictly singular, then*

$$ST \in \mathscr{S}(X,Z).$$

Proof. Recall that an operator $A \in \mathscr{L}(E,F)$ is strictly singular if there do not exist an infinite-dimensional subspace $M \subset E$ and a constant $c > 0$ such that

$$\|Ax\| \geq c\|x\|, \qquad x \in M.$$

(1) Suppose, by contradiction, that TS is not strictly singular. Then there exist an infinite-dimensional subspace $M \subset X$ and a constant $\alpha > 0$ such that

$$\|TSx\| \geq \alpha \|x\|, \qquad x \in M.$$

Since T is bounded,

$$\|TSx\| \leq \|T\| \, \|Sx\|, \qquad x \in M.$$

Hence

$$\|Sx\| \geq \frac{\alpha}{\|T\|} \|x\|, \qquad x \in M,$$

whenever $T \neq 0$. Thus S is bounded below on the infinite-dimensional subspace M, contradicting the strict singularity of S.

If $T = 0$, then $TS = 0$, which is trivially strictly singular. Therefore, in all cases,

$$TS \in \mathscr{S}(X, Z).$$

(2) Since T is closed and $D(T) = X$, the closed graph theorem implies that T is bounded.

Suppose, by contradiction, that ST is not strictly singular. Then there exist an infinite-dimensional subspace $M \subset X$ and a constant $\alpha > 0$ such that

$$\|STx\| \geq \alpha \|x\|, \qquad x \in M.$$

First, we claim that $T|_M$ is injective. Indeed, if $Tx = 0$ for some $x \in M$, then

$$0 = \|STx\| \geq \alpha \|x\|,$$

which implies $x = 0$. Thus $T|_M$ is injective.

Consequently, since M is infinite-dimensional,

$$T(M)$$

is an infinite-dimensional subspace of Y.

Now let $y \in T(M)$. Then $y = Tx$ for some $x \in M$. We have

$$\|Sy\| = \|STx\| \geq \alpha \|x\|.$$

Moreover, since T is bounded,

$$\|y\| = \|Tx\| \leq \|T\| \, \|x\|,$$

and therefore

$$\|x\| \geq \frac{1}{\|T\|} \|y\|.$$

Combining the above inequalities yields

$$\|Sy\| \geq \frac{\alpha}{\|T\|} \|y\|, \qquad y \in T(M).$$

Thus S is bounded below on the infinite-dimensional subspace $T(M)$. By the characterization recalled at the beginning of the proof, it suffices that the inequality hold on an infinite-dimensional subspace (not necessarily closed) to contradict strict singularity. Hence

$$ST \in \mathscr{S}(X, Z).$$

This completes the proof. □

Example 28 *Let $X = c_0(\mathbb{N}, \mathbb{Q}_p)$, the space of p-adic sequences tending to zero, equipped with the ultrametric norm*

$$\|(x_n)_{n \geq 1}\| = \sup_n |x_n|_p.$$

Define the operator $S : X \to X$ by

$$S(x_1, x_2, x_3, \ldots) = \left(p x_1, \, p^2 x_2, \, p^3 x_3, \ldots\right).$$

Since $|p^n|_p = p^{-n} \to 0$, the coefficients tend to 0 in $\mathbb{Q}_p$, so S is bounded and compact. Moreover, for any infinite-dimensional subspace $M \subset X$,

$$\inf_{x \in M, \|x\| = 1} \|Sx\| = 0,$$

so S is strictly singular.

More generally, let $(a_n)_{n \geq 1} \subset \mathbb{Q}_p$ be a sequence such that $a_n \to 0$. Define the diagonal operator

$$D : X \to X, \qquad D(x_1, x_2, \ldots) = (a_1 x_1, a_2 x_2, \ldots).$$

Then D is bounded. Moreover, since $a_n \to 0$, the operator D is compact. Indeed, for each $N \geq 1$, define

$$D_N(x_1, x_2, \ldots) = (a_1 x_1, \ldots, a_N x_N, 0, 0, \ldots).$$

The operator D_N has finite rank and

$$\|D - D_N\| = \sup_{n>N} |a_n|_p \longrightarrow 0.$$

Hence, D is the norm limit of finite-rank operators and therefore $D \in \mathscr{K}(X)$. In particular, D is strictly singular. For instance, if

$$a_n = p^n, \qquad |p|_p = p^{-1}, \text{so that } |a_n|_p = p^{-n} \to 0,$$

then D is a compact and strictly singular diagonal operator.

Remark 29 *On $X = c_0(\mathbb{N}, \mathbb{Q}_p)$, the shift operator $T : X \to X$,*

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots)$$

is not strictly singular. Indeed,

$$\|Tx\| = \|x\|, \qquad x \in X,$$

so that T is an isometry. More precisely, T is an isomorphism from X onto its range

$$T(X) = \{(y_n) \in X : y_{2k} = 0, k \geq 1\}.$$

Thus, T restricts to an isomorphism on an infinite-dimensional subspace, which rules out strict singularity. Consequently, this shift operator cannot be used as an example of a non-compact strictly singular operator. Any assertion concerning the existence of a non-compact strictly singular operator on this space requires an additional result establishing such an example.

Theorem 30 *Let X and Y be non-Archimedean Banach spaces, and let $T : D(T) \subset X \to Y$ be a closed linear operator (i.e., $T \in \mathscr{C}(X,Y)$). Let $S : D(S) \subset X \to Y$ be a linear operator such that $D(T) \subset D(S)$. Assume that S is T-bounded and T-strictly singular. Define the Banach space*

$$G_T := (D(T), \|\cdot\|_T),$$

where

$$\|x\|_T := \max\{\|x\|, \|Tx\|\}, \quad x \in D(T).$$

Then, regarded as an operator on G_T, the operator S is strictly singular.

Proof. First, note that G_T is a non-Archimedean Banach space. Indeed, the norm $\|\cdot\|_T$ satisfies the ultrametric inequality, and completeness follows from the fact that T is closed. If (x_n) is a Cauchy sequence in G_T, then (x_n) is Cauchy in X and (Tx_n) is Cauchy in Y, so there exists $x \in X$ such that $x_n \to x$ and $Tx_n \to y \in Y$. Since T is closed, $x \in D(T)$ and $Tx = y$, so $x \in G_T$.

Next, let

$$\widehat{S} := S|_{G_T} \in \mathscr{L}(G_T, Y)$$

be the restriction of S to G_T. The operator $\widehat{S}$ is bounded because S is T-bounded:

$$\begin{aligned} \|\widehat{S}x\| &= \|Sx\| \leq \max\{a\|x\|, b\|Tx\|\} \\ &\leq \max\{a,b\}\|x\|_T. \end{aligned}$$

Now, suppose by contradiction that $\widehat{S}$ is not strictly singular on G_T. Then there exists an infinite-dimensional subspace $M \subset D(T) \subset D(S)$ and a constant $\gamma > 0$ such that

$$\|\widehat{S}x\| \geq \gamma\|x\|_T = \gamma\max\{\|x\|, \|Tx\|\}, \quad \forall x \in M.$$

This directly contradicts the fact that S is strictly singular relative to T, because no T-strictly singular operator can be bounded below on an infinite-dimensional subspace. Therefore, any subspace $M \subset G_T$ where $\|\widehat{S}x\| \geq \gamma\|x\|_T$ must be finite-dimensional, and we conclude that $\widehat{S}$ is strictly singular on G_T. □

Example 31 *Let $X = c_0(\mathbb{N}, \mathbb{Q}_p)$ be the space of p-adic sequences converging to zero equipped with the ultrametric norm*

$$\|x\| = \sup_{n\in\mathbb{N}} |x_n|_p.$$

Define the left shift operator $T : D(T) \subset X \to X$ by

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots),$$

with domain

$$D(T) = \{(x_n) \in X : (x_{n+1}) \in X\}.$$

Then T is closed and induces the graph norm

$$\|x\|_T = \max\{\|x\|, \|Tx\|\}, \quad x \in D(T),$$

so that $G_T = (D(T), \|\cdot\|_T)$ is a non-Archimedean Banach space. Define $S : D(S) = D(T) \to X$ by

$$S(x_1, x_2, x_3, \ldots) = (0, x_1/2, 0, x_2/2, 0, x_3/2, \ldots).$$

Then for every $x \in D(T)$,

$$\|Sx\| \leq \frac{1}{2}\|x\| \leq \frac{1}{2}\|x\|_T,$$

so S is T-bounded. Moreover, the alternating zero structure of S prevents it from being bounded below on any infinite-dimensional subspace of $(D(T), \|\cdot\|_T)$. Hence S is strictly singular on G_T, i.e., T-strictly singular. This provides a concrete example illustrating Theorem 30.

Theorem 32 *Let X and Y be non-Archimedean Banach spaces, and let $T \in \Phi_+(X,Y)$ satisfy $\|D(T)\|_T \subset |\mathbb{K}|$. Let $S \in \mathscr{C}(X,Y)$ be a strictly singular operator. Then the perturbation $T+S$ also belongs to $\Phi_+(X,Y)$.*

Proof. Define the space

$$G_T := (D(T), \|x\|_T), \quad \|x\|_T := \max\{\|x\|, \|Tx\|\}.$$

As in Theorem 30, G_T is a non-Archimedean Banach space. Moreover, the domain of the sum satisfies

$$D(T+S) = D(T) \cap D(S) = D(T).$$

Since $T \in \mathscr{C}(X,Y)$, by Lemma 16, G_T is a non-Archimedean Banach space. We note that

$$T+S \in \mathscr{C}(X,Y)$$

such that

$$D(T+S) = D(T) \cap D(S) = D(T).$$

Indeed, consider the operator J defined by

$$J : D(T) \longrightarrow G_T, \quad x \longmapsto x.$$

The inverse J^{-1} is defined and bounded on G_T with $\|J^{-1}\| \leq 1$. Hence, $J \in \mathscr{C}(D(T), G_T)$. Consider the restrictions

$$\widehat{T} := T|_{G_T}, \quad \widehat{S} := S|_{G_T}.$$

Let $\|D(T)\|_T \subseteq |\mathbb{K}|$ and $T+S = (\widehat{T}+\widehat{S})J$, it follows from Lemma 25 that

$$T+S \in \mathscr{C}(X,Y).$$

Since S is strictly singular relative to T, Theorem 30 implies

$$\widehat{S} \in \mathscr{S}(G_T, Y),$$

and $\widehat{T} \in \Phi_+(G_T, Y)$ because T is semi-Fredholm.

Hence, by [26, Theorem 3.4], we deduce that the sum of a semi-Fredholm operator and a strictly singular operator remains semi-Fredholm. Thus:

$$\widehat{T}+\widehat{S} \in \Phi_+(G_T, Y).$$

Finally, since the norms $\|\cdot\|$ and $\|\cdot\|_T$ are equivalent on $D(T)$ and $S \in \mathscr{C}(X,Y)$, the operator $T+S$ is closed on X. Therefore, we can lift the result from G_T to X and conclude that

$$T+S \in \Phi_+(X,Y),$$

as claimed. □

Lemma 33 *Under the hypotheses of Theorem 35 below (in particular the growth condition (1)), for every $2 \leq j \leq n$ there exist constants $\alpha_j, \beta_j \geq 0$ such that*

$$\|T_j x\| \leq \max\{\alpha_j\|x\|, \beta_j\|T_1 x\|\}, \qquad \forall x \in D(T_1).$$

In particular, each T_j is T_1-bounded.

Proof. We argue by induction on j. For $j = 2$, (1) with $k = 2$ gives the claim directly, with $\alpha_2 = a_1$, $\beta_2 = b_1$.

Assume the claim holds for $j-1$, i.e.

$$\|T_{j-1}x\| \leq \max\{\alpha_{j-1}\|x\|, \beta_{j-1}\|T_1 x\|\}$$

for all $x \in D(T_1)$. Since $D(T_1) \subset D(T_{j-1})$, the growth condition (1) at $k = j$ applies to every $x \in D(T_1)$ and yields

$$\begin{aligned}\|T_j x\| &\leq \max\{a_{j-1}\|x\|, b_{j-1}\|T_{j-1}x\|\} \\ &\leq \max\{a_{j-1}\|x\|, b_{j-1}\max\{\alpha_{j-1}\|x\|, \beta_{j-1}\|T_1 x\|\}\}.\end{aligned}$$

Using $c\max\{d,e\} = \max\{cd,ce\}$ for $c \geq 0$ and merging the two terms proportional to $\|x\|$, we obtain

$$\|T_j x\| \leq \max\{\alpha_j \|x\|, \beta_j \|T_1 x\|\},$$
$$\alpha_j := a_{j-1} + b_{j-1}\alpha_{j-1},$$

and

$$\beta_j := b_{j-1}\beta_{j-1},$$

which completes the induction. □

Lemma 34 *Under the hypotheses of Theorem 35 below, for every $2 \leq k \leq n$ the operator T_k is strictly singular relative to T_1: for every infinite-dimensional subspace $M \subset D(T_1)$,*

$$\inf_{x \in M\setminus\{0\}} \frac{\|T_k x\|}{\max\{\|x\|, \|T_1 x\|\}} = 0.$$

Proof. For $k = 2$ this is (2) itself, since

$$T_{k-1} = T_1$$

in that case. Let $3 \leq k \leq n$ and let $M \subset D(T_1)$ be infinite-dimensional. By Lemma 33 applied at $j = k-1$, there are constants $\alpha_{k-1}, \beta_{k-1} \geq 0$ with

$$\begin{aligned} \|T_{k-1}x\| &\leq \max\{\alpha_{k-1}\|x\|, \beta_{k-1}\|T_1 x\|\} \\ &\leq C_{k-1}\max\{\|x\|, \|T_1 x\|\}, \end{aligned}$$

$C_{k-1} := \max\{\alpha_{k-1}, \beta_{k-1}, 1\}$, for all $x \in D(T_1)$, so that

$$\max\{\|x\|, \|T_{k-1}x\|\} \leq C_{k-1}\max\{\|x\|, \|T_1 x\|\}.$$

Combined with (2), this gives, for every $x \in M \setminus \{0\}$,

$$\frac{\|T_k x\|}{\max\{\|x\|, \|T_1 x\|\}} \leq C_{k-1}\frac{\|T_k x\|}{\max\{\|x\|, \|T_{k-1}x\|\}}.$$

Taking the infimum over $x \in M \setminus \{0\}$ and using (2) yields the stated equality. □

Theorem 35 *Let X and Y be non-Archimedean Banach spaces. Let $T_1 \in \Phi(X,Y)$ and let $T_2, \ldots, T_n$, $n \geq 2$, be (possibly unbounded) linear operators from X into Y. Assume that*

$$D(T_1) \subset D(T_2) \subset \cdots \subset D(T_n),$$

and that for each $2 \leq k \leq n$ and all $x \in D(T_{k-1})$

$$\|T_k x\| \leq \max(a_{k-1}\|x\|, b_{k-1}\|T_{k-1}x\|), \qquad (1)$$

where $a_{k-1}, b_{k-1} \geq 0$ are constants.

Moreover, assume that each T_k is strictly singular relative to T_{k-1} in the sense that for every $2 \leq k \leq n$ and every infinite-dimensional subspace $M \subset D(T_{k-1})$

$$\inf_{x \in M\setminus\{0\}} \frac{\|T_k x\|}{\max\{\|x\|, \|T_{k-1}x\|\}} = 0. \qquad (2)$$

Then the sum operator

$$T_1 + T_2 + \cdots + T_n \in \Phi_+(X,Y).$$

Proof. Let $D(T_1)$ denote the domain of T_1. We equip $D(T_1)$ with the graph norm

$$\|x\|_{T_1} := \max\{\|x\|, \|T_1 x\|\}.$$

Since T_1 is closed, the space

$$G_{T_1} := (D(T_1), \|\cdot\|_{T_1})$$

is a non-Archimedean Banach space. For each $k = 1, \ldots, n$, let $\widetilde{T}_k$ denote the restriction of T_k to G_{T_1}. In particular, $\widetilde{T}_1$ is bounded on G_{T_1} and

$$\|\widetilde{T}_1 x\| \leq \|x\|_{T_1}, \quad x \in G_{T_1},$$

and hence $\|\widetilde{T}_1\| \leq 1$.

We first observe that the relative strict singularity assumptions remain valid after passing to the graph norm. Indeed, for every $k = 2, \ldots, n$ and every $x \in D(T_1)$, since $D(T_1) \subset D(T_{k-1})$ and $\widetilde{T}_{k-1}x = T_{k-1}x$, we have

$$\begin{aligned} \max\{\|x\|, \|T_{k-1}x\|\} &= \max\{\|x\|, \|\widetilde{T}_{k-1}x\|\} \\ &\leq \max\{\|x\|_{T_1}, \|\widetilde{T}_{k-1}x\|\}. \end{aligned}$$

Consequently,

$$\frac{\|\widetilde{T}_k x\|}{\max\{\|x\|_{T_1}, \|\widetilde{T}_{k-1}x\|\}} = \frac{\|T_k x\|}{\max\{\|x\|_{T_1}, \|T_{k-1}x\|\}}.$$

Therefore, for every infinite-dimensional subspace M of G_{T_1}

$$\inf_{x \in M\setminus\{0\}} \frac{\|T_k x\|}{\max\{\|x\|_{T_1}, \|T_{k-1}x\|\}} = 0.$$

Thus, $\widetilde{T}_k$ is strictly singular relative to $\widetilde{T}_{k-1}$ on G_{T_1}.

By Lemma 33 and Lemma 34, each

$$T_k \ (2 \le k \le n)$$

is T_1-bounded and strictly singular relative to T_1. Theorem 30, applied with T_1 in place of T, therefore shows that each restriction

$$\widetilde{T}_k \ \ (2 \le k \le n)$$

is strictly singular on G_{T_1}.

We now show by induction on $m = 2, \dots, n$ that

$$\widetilde{T}_2 + \cdots + \widetilde{T}_m$$

is strictly singular on G_{T_1}. The case $m = 2$ is exactly the statement just obtained for $k = 2$. Assume

$$\widetilde{T}_2 + \cdots + \widetilde{T}_{m-1} \in \mathscr{S}(G_{T_1}, Y)$$

for some $m \ge 3$. Since $\widetilde{T}_m \in \mathscr{S}(G_{T_1}, Y)$ as well, Lemma 26 (with $\phi = \psi = 1$, applied to the two strictly singular operators $\widetilde{T}_2 + \cdots + \widetilde{T}_{m-1}$ and $\widetilde{T}_m$ on G_{T_1}) gives

$$\widetilde{T}_2 + \cdots + \widetilde{T}_m = (\widetilde{T}_2 + \cdots + \widetilde{T}_{m-1}) + \widetilde{T}_m \in \mathscr{S}(G_{T_1}, Y),$$

which completes the induction. Consequently,

$$\widetilde{S} := \widetilde{T}_2 + \cdots + \widetilde{T}_n \in \mathscr{S}(G_{T_1}, Y).$$

Since $T_1 \in \Phi_+(X, Y)$, the operator

$$\widetilde{T}_1 : G_{T_1} \longrightarrow Y$$

also belongs to $\Phi_+(G_{T_1}, Y)$. Indeed, passing from $D(T_1)$ endowed with the original norm to the graph norm does not change the underlying vector space, and the kernel and range of the operator remain unchanged.

Applying Theorem 32 to the Φ_+-operator $\widetilde{T}_1$ and its strictly singular perturbation $\widetilde{S}$, we obtain

$$\widetilde{T}_1 + \widetilde{S} = \widetilde{T}_1 + \widetilde{T}_2 + \cdots + \widetilde{T}_n \in \Phi_+(G_{T_1}, Y).$$

The operator on the left-hand side is precisely the restriction of

$$T := T_1 + \cdots + T_n$$

to $D(T_1)$, considered as an operator from the graph space G_{T_1} into Y.

Finally, the semi-Fredholm property is preserved when the same closed operator is considered on $D(T_1)$ with the original norm. Since $T = T_1 + \cdots + T_n$ is closed on $D(T_1)$ by hypothesis, it follows that

$$T_1 + T_2 + \cdots + T_n \in \Phi_+(X, Y).$$

This completes the proof. □

Theorem 36 *Under the assumptions of Theorem 35, if in addition*

$$r(T_1)\|x\| < \|T_1 x\|, \quad \textit{for all } x \in D(T_1),$$

then the upper essential spectrum satisfies

$$\sigma_{e1}(T_1 + T_2 + \cdots + T_n) = \sigma_{e1}(T_1).$$

Proof. Assume that

$$r(T_1)\|x\| < \|T_1 x\|, \quad \text{for all } x \in D(T_1).$$

We first prove the inclusion

$$\sigma_{e1}(T_1 + \cdots + T_n) \subseteq \sigma_{e1}(T_1).$$

Let $\lambda \notin \sigma_{e1}(T_1)$, so that $T_1 - \lambda I \in \Phi_+(X)$. We distinguish two cases.

Case A: $|\lambda| \le r(T_1)$. Then for every $x \in D(T_1)$,

$$\|(T_1 - \lambda I)x\| = \max(\|T_1 x\|, |\lambda|\|x\|) = \|T_1 x\|.$$

Hence, the graph norm induced by $T_1 - \lambda I$ coincides with that of T_1. Since $T_2, \dots, T_n$ are strictly singular relative to T_1, they are also strictly singular relative to $T_1 - \lambda I$. Applying Theorem 35 to the operator $T_1 - \lambda I$ (which belongs to $\Phi_+(X)$) and the relatively strictly singular perturbations $T_2, \dots, T_n$, we obtain

$$T_1 + T_2 + \cdots + T_n - \lambda I \in \Phi_+(X).$$

Thus $\lambda \notin \sigma_{e1}(T_1 + \cdots + T_n)$.

Case B: $|\lambda| > r(T_1)$. Then $\lambda \notin \sigma(T_1)$ (since $r(T_1)$ is the spectral radius), so $T_1 - \lambda I$ is bijective with bounded inverse. Moreover, the graph

norm induced by $T_1-\lambda I$ is equivalent to that of T_1. Because

$$\begin{aligned}\|(T_1-\lambda I)x\| &= \max(\|T_1x\|,|\lambda|\|x\|)\\ &\leq |\lambda|\max(\|x\|,\|T_1x\|)\end{aligned}$$

and conversely. Hence, $T_2,\dots,T_n$ remain strictly singular relative to $T_1-\lambda I$. Applying Theorem 35 again, we get

$$T_1+T_2+\cdots+T_n-\lambda I\in\Phi_+(X),$$

so $\lambda\notin\sigma_{e1}(T_1+\cdots+T_n)$. This proves

$$\sigma_{e1}(T_1+\cdots+T_n)\subseteq\sigma_{e1}(T_1).$$

Conversely, we prove

$$\sigma_{e1}(T_1)\subseteq\sigma_{e1}(T_1+\cdots+T_n).$$

Assume that

$$\lambda\notin\sigma_{e1}(T_1+\cdots+T_n).$$

Then

$$T_1+T_2+\cdots+T_n-\lambda I\in\Phi_+(X).$$

As above, we distinguish two cases.

Case A: $|\lambda|\leq r(T_1)$. Since $T_2,\dots,T_n$ are strictly singular relative to $T_1-\lambda I$, the operators $-T_2,\dots,-T_n$ are also strictly singular relative to $T_1-\lambda I$. Hence, applying Theorem 35 in reverse form, we obtain

$$\begin{aligned}T_1-\lambda I &= (T_1+T_2+\cdots+T_n-\lambda I)\\ &- (T_2+\cdots+T_n)\in\Phi_+(X).\end{aligned}$$

Case B: $|\lambda|>r(T_1)$. Then $\lambda\notin\sigma(T_1)$, so $T_1-\lambda I\in\Phi_+(X)$ automatically.

Therefore, in both cases, $\lambda\notin\sigma_{e1}(T_1)$.

We conclude that

$$\sigma_{e1}(T_1+T_2+\cdots+T_n)=\sigma_{e1}(T_1).$$

□

Proposition 37 *Let X and Y be non-Archimedean Banach spaces, let*

$$T:D(T)\subset X\to Y$$

be a closed linear operator, and let $S\in\mathscr{L}(X,Y)$ be T-bounded with T-bound $\delta<1$. Let

$$i_T:G_T\to X$$

be the canonical projection defined on the graph

$$G_T=\{(x,Tx)\mid x\in D(T)\}\subset X\times Y,$$

endowed with the graph norm

$$\|(x,Tx)\|_{G_T}=\max\{\|x\|,\|Tx\|\}.$$

Then the operator

$$(T+S)\circ i_T:G_T\longrightarrow Y$$

is closed.

Proof. S is T-bounded if there exist two constants $a,b\geq 0$ satisfying

$$\|Sx\|\leq\max\{a\|x\|,b\|Tx\|\},\quad\text{for all } x\in D(T). \tag{3}$$

We have

$$\|Tx\|=\|(T-S+S)x\|\leq\max\{\|(T+S)x\|,\|Sx\|\}.$$

Then, since $b<1$,

$$\|Tx\|\leq\max\{\|(T+S)x\|,a\|x\|\},\quad\text{for all } x\in D(T).$$

Now, suppose $x_n\in D(T+S)=D(T)$ is a sequence such that $x_n\to x$ and $(T+S)x_n\to y$. Then, there exists $n_0\in\mathbb{N}$ such that for all $n,m\geq n_0$, we have

$$\|T(x_n-x_m)\|\leq\max\{\|(T+S)(x_n-x_m)\|,a\|x_n-x_m\|\}.$$

It follows that the sequence (Tx_n) is Cauchy in the Banach space Y, hence convergent: $Tx_n\to z$. Since T is a closed operator, we conclude that $x\in D(T)$ and $Tx=z$. Moreover, we have

$$Sx_n=(T+S)(x_n)-Tx_n\to y-z.$$

In addition,

$$\|S(x_n - x_m)\| \leq \max\{b\|T(x_n - x_m)\|, a\|x_n - x_m\|\} \to 0,$$

which implies

$$Sx = y - z.$$

Consequently,

$$(T+S)x = y,$$

and therefore $x \in D(T+S)$. Indeed, consider the operator i_T defined by

$$i_T : G_T = \{D(T), \|\cdot\|_T\} \longrightarrow X, \quad x \longmapsto x,$$

the identity map i_T is continuous from G_T to X, with $\|i_T\| \leq 1$. Hence, $i_T \in \mathscr{L}(G_T, X)$, i.e., i_T is a bounded linear operator. Now consider the composition

$$(T+S)_{i_T} = (T+S) \circ i_T.$$

Since $S+T$ is defined on $D(T) \subset G_T$, and i_T is continuous, it follows that $(T+S)_{i_T}$ is closed. □

Theorem 38 *Let X and Y be non-Archimedean Banach spaces, and let $T : D(T) \subset X \to Y$ be a closed linear operator. Let i_T denote the canonical identification of the graph space*

$$G_T = (D(T), \|\cdot\|_T), \ \ \|x\|_T = \max\{\|x\|, \|Tx\|\}.$$

Define the operator $T_{i_T} : G_T \to Y$ by

$$T_{i_T}(x) = Tx, \quad x \in D(T).$$

Then the following equivalences hold:

1. $T \in \Phi_+(X,Y) \iff T_{i_T} \in \Phi_+(G_T, Y)$.
2. $T \in \Phi_-(X,Y) \iff T_{i_T} \in \Phi_-(G_T, Y)$.

Proof. We prove both equivalences simultaneously. Recall that

$$T \in \Phi_+(X,Y)$$

Equivalently,

$$\dim\ker(T) < \infty \text{ and } \operatorname{ran}(T) \text{ is closed in } Y.$$

and

$$T \in \Phi_-(X,Y).$$

Equivalently,

$$\operatorname{Ran}(T) \text{ is closed and } \operatorname{codim}\operatorname{Ran}(T) < \infty.$$

Let $G_T = (D(T), \|\cdot\|_T)$ be the graph space of T, and consider the canonical inclusion

$$i_T : G_T \to X, \qquad i_T(x) = x.$$

The map i_T is linear, bounded, and injective, and its range is $D(T)$. Since T is closed, the operator

$$T_{i_T} := T \circ i_T : G_T \to Y$$

is well-defined and closed. Because i_T is injective and $T_{i_T}(x) = T(x)$ for all $x \in D(T)$, we have

$$\operatorname{Ker}(T_{i_T}) = \operatorname{Ker}(T),$$

hence

$$\dim\operatorname{Ker}(T_{i_T}) = \dim\operatorname{Ker}(T).$$

Moreover, since $i_T(G_T) = D(T)$, it follows that

$$\operatorname{Ran}(T_{i_T}) = \operatorname{Ran}(T),$$

and therefore

$$\operatorname{Ran}(T_{i_T}) \text{ is closed } \iff \operatorname{Ran}(T) \text{ is closed},$$

as well as

$$\operatorname{codim}\operatorname{Ran}(T_{i_T}) = \operatorname{codim}\operatorname{Ran}(T).$$

Thus T and T_{i_T} share the same kernel dimension and the same range properties. Consequently,

$$T \in \Phi_+(X,Y) \iff T_{i_T} \in \Phi_+(G_T, Y),$$

and

$$T \in \Phi_-(X,Y) \iff T_{i_T} \in \Phi_-(G_T, Y).$$

□

Theorem 39 *Let X and Y be non-Archimedean Banach spaces, and let $T : D(T) \subset X \to Y$ be a closed linear operator such that $T \in \Phi_+(X,Y)$. Let $G_T = (D(T), \|\cdot\|_T)$ be the graph space of T, where*

$$\|x\|_T = \max\{\|x\|, \|Tx\|\},$$

and let

$$i_T : G_T \to X$$

be the canonical embedding defined by $i_T(x) = x$. Assume that $S : D(T) \to Y$ is a linear operator such that $S \circ i_T \in \mathscr{L}(G_T, Y)$ is strictly singular. Then the perturbed operator $T + S$ belongs to $\Phi_+(X,Y)$.

Proof. Let $T \in \Phi_+(X,Y)$ and

$$S_{i_T} := S \circ i_T = S|_{G_T} \in \mathscr{S}(G_T, Y).$$

Therefore, S is a strictly singular operator relative to T. Applying Theorem 32, we have $T + S \in \Phi_+(X,Y)$. □

The reverse inclusion in Theorem 41 below relies on the following representation principle, which extends to the non-Archimedean setting the classical fact (established in the Fredholm/semi-Fredholm perturbation theory of [27, 16]) that a Fredholm-type defect can always be repaired by a strictly singular perturbation once a bounded approximation property is available.

Hypothesis 40 *Let X be a non-Archimedean Banach space with the bounded approximation property. For every $A \in \mathscr{L}(X)$ and every $\lambda \in \mathbb{K}$ such that $A - \lambda I \in \Phi_+(X)$, there exists a strictly singular operator $S \in \mathscr{S}(X)$ such that $A + S - \lambda I$ is invertible.*

We use Hypothesis 40 only in the proof of the inclusion $(\supseteq)$ below; the inclusion $(\subseteq)$ is established independently and unconditionally.

Theorem 41 *Let X be a non-Archimedean Banach space satisfying the bounded approximation property, and let $A \in \mathscr{L}(X)$. Assume Hypothesis 40 holds for X. Then:*

$$\sigma_{e1}(A) = \bigcap_{S \in \mathscr{S}(X)} \sigma(A + S).$$

Proof. We prove the two inclusions separately.

$(\subseteq)$ Suppose $\lambda \notin \bigcap_{S \in \mathscr{S}(X)} \sigma(A + S)$. Then there exists $S \in \mathscr{S}(X)$ such that $\lambda \in \rho(A + S)$, i.e., $A + S - \lambda I$ is invertible. Hence,

$$\begin{aligned} A - \lambda I &= A - \lambda I = (A + S - \lambda I) - S \\ &= (A + S - \lambda I)(I - (A + S - \lambda I)^{-1} S). \end{aligned}$$

Since $(A + S - \lambda I)^{-1}$ is bounded and S is strictly singular, the operator

$$K := (A + S - \lambda I)^{-1} S$$

is strictly singular. Thus $I - K$ is a strictly singular perturbation of the identity. Because $A + S - \lambda I$ is invertible, it belongs to $\Phi_+(X)$. By the stability result (Theorem 32), strictly singular perturbations preserve the upper semi-Fredholm property. Therefore,

$$A - \lambda I \in \Phi_+(X),$$

which implies $\lambda \notin \sigma_{e1}(A)$. This proves

$$\sigma_{e1}(A) \subseteq \bigcap_{S \in \mathscr{S}(X)} \sigma(A + S).$$

$(\supseteq)$ Assume $\lambda \notin \sigma_{e1}(A)$, so that $A - \lambda I \in \Phi_+(X)$. By Hypothesis 40 (the representation principle), there exists a strictly singular operator $S \in \mathscr{S}(X)$ such that $A + S - \lambda I$ is invertible. Hence,

$$\lambda \notin \sigma(A + S),$$

and therefore

$$\lambda \notin \bigcap_{S \in \mathscr{S}(X)} \sigma(A + S).$$

Thus $\bigcap_{S \in \mathscr{S}(X)} \sigma(A + S) \subseteq \sigma_{e1}(A)$. Combining both inclusions yields

$$\sigma_{e1}(A) = \bigcap_{S \in \mathscr{S}(X)} \sigma(A + S).$$

□

APPLICATION TO A p-ADIC TRANSPORT MODEL

In this section, we apply the abstract results developed above to a transport model in disordered porous media recently formulated within the framework of p-adic analysis. This model, introduced by Khrennikov [11], describes fluid propagation (e.g., water or oil) through complex pore networks in fractured rock formations. Let

$$X = c_0(\mathbb{Q}_p)$$

denote the non-Archimedean Banach space of p-adic sequences tending to 0, equipped with the norm

$$\|x\| = \sup_{n\geq 1} |x_n|_p.$$

Define an operator T_1 on the domain

$$D(T_1) = \{x \in X : p^n |x_n|_p \to 0\}$$

by

$$(T_1 x)_n = p^{-n} x_n.$$

The growth condition $p^n|x_n|_p \to 0$, as opposed to the stronger requirement $\sup_n p^n|x_n|_p < \infty$, is precisely what ensures that $T_1 x$ again belongs to $c_0(\mathbb{Q}_p)$. On the same domain, introduce the shift-type operator

$$D(S) = D(T_1), \qquad (Sx)_n = p^{-(n+1)} x_{n+1},$$

and then set

$$T = T_1 + S, \qquad D(T) = D(T_1).$$

Since $D(T_1)$ contains all finitely supported sequences, it is dense in X. Moreover, on $D(T_1)$ one has the relation

$$S = LT_1,$$

where L is the left shift.

Lemma 42 *The operator $T_1 : D(T_1) \to X$ is bijective and closed. Moreover,*

$$\|x\|_{T_1} = \|T_1 x\| \qquad (x \in D(T_1)),$$

and its inverse $T_1^{-1} : X \to X$, given by

$$T_1^{-1} y = (p^n y_n)_{n\geq 1},$$

is compact. In particular, the map $x \mapsto T_1 x$ defines an isometry from the graph G_{T_1} onto $c_0(\mathbb{Q}_p)$.

Proof. If $y \in c_0(\mathbb{Q}_p)$ and $x_n = p^n y_n$, then

$$|x_n|_p = p^{-n}|y_n|_p \to 0 \quad \text{and} \quad p^n|x_n|_p = |y_n|_p \to 0,$$

so $x \in D(T_1)$ and $T_1 x = y$. Since

$$\|T_1 x\| = \sup_n p^n |x_n|_p \geq p\|x\|,$$

we get

$$\|x\|_{T_1} = \|T_1 x\|.$$

For compactness, set $z_n = p^n e_n$. Then $\|z_n\| = p^{-n} \to 0$ and

$$T_1^{-1}\big(B_X(0,1)\big) = \Big\{\textstyle\sum_n \lambda_n z_n : |\lambda_n|_p \leq 1\Big\},$$

which is compactoid (Proposition 12), equivalently, its closure $\prod_n p^n\mathbb{Z}_p$ is compact. The operator T_1^{-1} is thus compact.

Finally, we prove that T_1 is closed. Let $x^{(m)} \in D(T_1)$ be such that $x^{(m)} \to x$ in X and $T_1 x^{(m)} \to y$ in X. Since T_1^{-1} is bounded,

$$x^{(m)} = T_1^{-1}(T_1 x^{(m)}) \to T_1^{-1} y.$$

Hence $x = T_1^{-1} y \in D(T_1)$ and $T_1 x = y$. Thus T_1 is closed. □

Lemma 43 *The spectrum of T_1 is given by*

$$\sigma(T_1) = \{p^{-k} : k \geq 1\}.$$

If $\lambda = p^{-k}$ for some $k \geq 1$, then

$$\mathrm{Ker}(T_1 - \lambda I) = \mathbb{Q}_p e_k, \;\; R(T_1 - \lambda I) = \{y \in X : y_k = 0\}.$$

Consequently, $T_1 - \lambda I$ is a Fredholm operator of index 0, i.e. $T_1 - \lambda I \in \Phi(X)$ with

$$\mathrm{ind}(T_1 - \lambda I) = 0.$$

On the other hand, whenever $\lambda \notin \sigma(T_1)$, the resolvent $(T_1 - \lambda I)^{-1}$ is compact.

Proof. Since $T_1 - \lambda I = \mathrm{diag}(p^{-n} - \lambda)_{n\geq 1}$ and $|p^{-n}|_p = p^n \to \infty$, the ultrametric inequality gives, for all n large enough that $p^n > |\lambda|_p$,

$$|p^{-n} - \lambda|_p = \max\{|p^{-n}|_p, |\lambda|_p\} = p^n \to \infty.$$

Hence the diagonal inverse $(y_n/(p^{-n}-\lambda))$ is bounded, maps X into $D(T_1)$, and is compact whenever no diagonal entry vanishes, since its entries $1/(p^{-n}-\lambda)$ have p-adic modulus $p^{-n}\to 0$ for large n; otherwise $p^{-k}=\lambda$ for some k, and the kernel and range are as stated. □

Lemma 44 *For every $x\in D(T_1)$,*

$$\|Sx\| = \sup_{m\geq 2} p^m |x_m|_p \leq \|T_1 x\|,$$

with equality when $x=e_m$ and $m\geq 2$. Hence S is T_1-bounded with T_1-bound exactly 1. However, S is unbounded on X, since $\|Se_m\|=p^m$.

Proof. Since $(Sx)_n = p^{-(n+1)}x_{n+1}$,

$$\begin{aligned}\|Sx\| &= \sup_{n\geq 1} p^{n+1}|x_{n+1}|_p \\ &= \sup_{m\geq 2} p^m |x_m|_p \\ &\leq \sup_{m\geq 1} p^m |x_m|_p = \|T_1 x\|,\end{aligned}$$

as the first supremum runs over the smaller index set $\{m\geq 2\}\subseteq\{m\geq 1\}$.

For $x=e_m$ with $m\geq 2$, only the m-th coordinate contributes, so

$$\|T_1 e_m\| = p^m = \|Se_m\|.$$

Hence $\|S\|_{T_1}\leq 1$ by the inequality above, while equality on the T_1-unit vectors $e_m/\|T_1 e_m\|$ gives $\|S\|_{T_1}\geq 1$. Therefore $\|S\|_{T_1}=1$ exactly. Finally, for $m\geq 1$,

$$\|Se_m\| = \sup_{n\geq 1} p^{n+1}|(e_m)_{n+1}|_p = p^m \to \infty$$

while $\|e_m\|=1$, so S is unbounded on X. □

Lemma 45 *S is not T_1-strictly singular, and consequently not T_1-compact. Indeed, on the infinite-dimensional closed subspace*

$$M=\{x\in D(T_1) : x_1=0\}$$

of G_{T_1}, one has $\|Sx\|=\|x\|_{T_1}$.

Proof. The subspace $M=\{x\in D(T_1): x_1=0\}$ is closed in G_{T_1}, being the kernel of the continuous coordinate functional $x\mapsto x_1$, and is infinite-dimensional since it contains all e_m with $m\geq 2$. For $x\in M$, $x_1=0$, so

$$\|Sx\| = \sup_{m\geq 2} p^m|x_m|_p = \sup_{m\geq 1} p^m|x_m|_p = \|T_1 x\| = \|x\|_{T_1}.$$

Thus $S|_M$ is an isometry, hence an isomorphism onto its range, so S is not T_1-strictly singular. Since T_1-compact operators are T_1-strictly singular, S is not T_1-compact either. □

Remark 46 *Lemma 45 shows that the perturbation S lies outside the framework of Section : its relative bound is 1, not <1, and it is an isometry on $M\subset G_{T_1}$. Consequently, the abstract perturbation theory developed in Section (Theorems 32–41) does not apply directly to S. The spectral stability in Section is established by a different method.*

Lemma 47 *The operator $I+L$ is invertible on $c_0(\mathbb{Q}_p)$, and*

$$\|I+L\|\leq 1, \qquad \|(I+L)^{-1}\|\leq 1.$$

Its inverse is given explicitly by

$$\big((I+L)^{-1}z\big)_n = \sum_{k\geq n}(-1)^{n+k}z_k \qquad (z\in c_0(\mathbb{Q}_p)).$$

Proof. Let $z\in c_0(\mathbb{Q}_p)$. Since $z_k\to 0$, the series defining

$$y_n = \sum_{k\geq n}(-1)^{n+k}z_k$$

converges in $\mathbb{Q}_p$ by the ultrametric convergence criterion. Moreover,

$$|y_n|_p \leq \sup_{k\geq n}|z_k|_p \longrightarrow 0 \qquad (n\to\infty),$$

so $y=(y_n)_{n\geq 1}\in c_0(\mathbb{Q}_p)$, with $\|y\|\leq\|z\|$. A direct computation gives

$$y_n + y_{n+1} = z_n \qquad (n\geq 1),$$

i.e. $(I+L)y=z$, so $I+L$ is surjective and

$$(I+L)^{-1}z=y$$

with $\|(I+L)^{-1}\|\leq 1$.

For injectivity, suppose $(I+L)y=0$, i.e.

$$y_n+y_{n+1}=0$$

for all n. Then

$$|y_n|_p=|y_{n+1}|_p,$$

so $|y_n|_p=|y_1|_p$ is constant in n. Since $y\in c_0(\mathbb{Q}_p)$, this forces $|y_1|_p=0$, hence

$$y=0.$$

Therefore $I+L$ is injective, and being surjective it is invertible. Finally, $\|I+L\|\leq 1$ follows from $\|Iy\|\leq\|y\|$ and $\|Ly\|\leq\|y\|$ together with the ultrametric inequality. □

Theorem 48 *The operator T factors as*

$$T=(I+L)T_1,\qquad D(T)=D(T_1).$$

It is closed and bijective from $D(T_1)$ onto X, and its inverse

$$T^{-1}=T_1^{-1}(I+L)^{-1}$$

is compact.

Proof. For $x\in D(T_1)$, the relation $S=LT_1$ gives

$$Tx=T_1x+Sx=T_1x+LT_1x=(I+L)T_1x,$$

so $T=(I+L)T_1$ on $D(T_1)$. By Lemma 47, the map

$$\Phi:X\times X\to X\times X,\qquad (x,y)\mapsto(x,(I+L)y),$$

is a homeomorphism. It carries the graph G_{T_1} onto

$$\Phi(G_{T_1})=\{(x,(I+L)T_1x):x\in D(T_1)\}=G_T,$$

so G_T is closed, being the homeomorphic image of the closed set G_{T_1}.

Bijectivity is now immediate: $T_1:D(T_1)\to X$ is bijective by Lemma 42, and $I+L$ is invertible on X by Lemma 47; hence their composition $T=(I+L)T_1$ is bijective, with inverse

$$T^{-1}=T_1^{-1}(I+L)^{-1}.$$

Finally, T^{-1} is compact because it is the composition of the compact operator T_1^{-1} (which maps X into $D(T_1)$) with the bounded operator $(I+L)^{-1}$, and the ideal of compact operators is closed under composition with bounded operators on either side. □

Theorem 49 *(a) The spectra of T and T_1 agree:*

$$\sigma(T)=\{p^{-k}:k\geq 1\}=\sigma(T_1).$$

(b) For each eigenvalue $\lambda=p^{-k}$, the kernel of $T-\lambda I$ is the one-dimensional subspace $\mathbb{Q}_p v^{(k)}$, spanned by the vector with coordinates

$$v_1^{(k)}=1,\qquad v_{n+1}^{(k)}=(p^{n+1-k}-p)v_n^{(k)}\ (n<k),$$

and

$$v_n^{(k)}=0\ (n>k).$$

The range $R(T-\lambda I)$ is closed of codimension 1, so $T-\lambda I\in\Phi(X)$ has index 0.

(c) For $\lambda\notin\sigma(T)$, the resolvent $(T-\lambda I)^{-1}$ is compact. More generally, $T-\lambda I\in\Phi(X)$ with index 0 for every $\lambda\in\mathbb{Q}_p$. Therefore $\sigma_e(T)=\sigma_{e1}(T)=\emptyset=\sigma_e(T_1)$ and $\rho_e(T)=\mathbb{Q}_p$.

Proof. Put $x=T_1^{-1}y=(p^n y_n)_n$, $y\in c_0(\mathbb{Q}_p)$. Then $(T-\lambda I)x=w$ is equivalent to

$$y_{n+1}=w_n-a_ny_n,\qquad a_n=1-\lambda p^n.$$

Choose n_0 with $|\lambda|_p p^{-n}<1$ for $n\geq n_0$; then $|a_n|_p=1$ for $n\geq n_0$.

Case 1: $\lambda\notin\{p^{-k}\}$. Here $a_n\neq 0$ for all n. Set $P_1=1$, $P_{n+1}=-a_nP_n$, then

$$P_n\neq 0\text{ and }|P_n|_p=c>0\text{ for }n\geq n_0.$$

- If $w=0$, then $y_n=P_ny_1$ and $|y_n|_p=c|y_1|_p$ for $n\geq n_0$, forcing $y_1=0$.

- Define

$$v_n = -\sum_{j\geq n} \frac{w_j}{P_{j+1}}, \qquad y_n = P_n v_n.$$

The series converge because

$$|w_j/P_{j+1}|_p = |w_j|_p/c \to 0.$$

One checks

$$v_{n+1} = v_n + w_n/P_{n+1},$$

hence

$$y_{n+1} = w_n - a_n y_n, \text{ and } |y_n|_p \leq \sup_{j\geq n} |w_j|_p \to 0.$$

Thus

$$y \in c_0(\mathbb{Q}_p), \|y\| \leq C\|w\|, \text{ and } x = T_1^{-1} y$$

satisfies $\|x\| \leq \|y\|$. So $T - \lambda I$ is bijective with bounded inverse, which is compact since it factors through T_1^{-1}. Hence $\lambda \in \rho(T)$.

Case 2: $\lambda = p^{-k}$**.** Now $a_k = 0$ and $a_n \neq 0$ for $n \neq k$.

- If $w = 0$, then $y_{k+1} = 0$, hence $y_n = 0$ for $n > k$, while y_1 is free and $y_2, \ldots, y_k$ are determined by it. Thus $\mathrm{Ker}(T - \lambda I)$ is one-dimensional; translating back by $x_n = p^n y_n$ gives $v^{(k)}$ equivalently,

$$x_{n+1} = (\lambda p^{n+1} - p) x_n,$$

which vanishes exactly at $n = k$).

- For general w, the equation at $n = k$ forces $y_{k+1} = w_k$ and leaves y_k free; $y_2, \ldots, y_k$ are solved from y_1 freely, and for $n > k$ the recursion is determined by $y_{k+1} = w_k$. Set

$$P'_{k+1} = 1, \;\; P'_{n+1} = -a_n P'_n \;\; (n > k)$$

and write $y_n = P'_n v_n$ for $n \geq k+1$, then

$$v_{k+1} = w_k \text{ and } v_{n+1} = v_n + w_n/P'_{n+1}.$$

Since $|P'_n|_p$ is eventually constant and nonzero, $y_n \to 0$ iff $v_n \to 0$ iff

$$\phi(w) = w_k + \sum_{j\geq k+1} \frac{w_j}{P'_{j+1}} = 0.$$

The functional ϕ is continuous and non-zero (e.g. $\phi(e_k) = 1$), so

$$R(T - \lambda I) = \mathrm{Ker}\, \phi$$

has codimension 1 and is closed. Therefore

$$T - \lambda I \in \Phi(X)$$

with

$$\alpha(T - \lambda I) = 1, \beta(T - \lambda I) = 1,$$

and index 0.

□

Remark 50 *It might seem paradoxical that T has infinitely many eigenvalues p^{-k}, $k \geq 1$, and yet, by Theorem 49(c), an empty essential spectrum. In the classical Archimedean setting, an infinite sequence of eigenvalues of a compact-resolvent operator typically accumulates at a point of the essential spectrum (there is no isolated infinity of eigenvalues without an accumulation point). Here the situation is structurally different: the ultrametric topology of $\mathbb{Q}_p$ prevents the sequence $(p^{-k})_{k\geq 1}$ from accumulating anywhere in $\mathbb{Q}_p$. Indeed, fix $\lambda \in \mathbb{Q}_p$ with $\lambda \neq p^{-n_0}$ for every n_0. For all n large enough that $p^n > |\lambda|_p$, the ultrametric inequality gives*

$$\begin{aligned} |p^{-n} - \lambda|_p &= \max\{|p^{-n}|_p, |\lambda|_p\} \\ &= \max\{p^n, |\lambda|_p\} = p^n \to \infty, \end{aligned}$$

so $p^{-n} \not\to \lambda$: the eigenvalues p^{-n} move ever further away, in the p-adic metric, from every fixed point of $\mathbb{Q}_p$, rather than accumulating near one. For instance, with $p = 3$ and $\lambda = 2$ (so $|\lambda|_3 = 1$), one has

$$|3^{-n} - 2|_3 = \max\{3^n, 1\} = 3^n \longrightarrow \infty,$$

so 3^{-n} never approaches $\lambda = 2$, however large n becomes. This is the structural reason behind the apparent paradox: each eigenvalue p^{-k} is an isolated point that stays p-adically far from every other point of $\mathbb{Q}_p$, so the sequence carries no limit point capable of generating an obstruction to the Fredholm property. An infinite discrete spectrum is therefore entirely compatible with an empty essential spectrum in this non-Archimedean setting.

Remark 51 *Since $T^{-1} = T_1^{-1}(I+L)^{-1}$, and T_1^{-1} is diagonal while $(I+L)^{-1}$ is upper triangular with diagonal entries 1, the inverse T^{-1} is itself upper triangular with diagonal entries*

$$p^n \cdot 1 = p^n \qquad (n \geq 1).$$

Hence the nonzero spectrum of T^{-1} is

$$\{p^n : n \geq 1\},$$

and therefore

$$\sigma(T) = \{p^{-n} : n \geq 1\},$$

in agreement with Theorem 49.

On the other hand, although $\sigma_{e1}(T) = \sigma_{e1}(T_1)$, this coincidence cannot be attributed to relatively strictly singular perturbation theory, because both sets are empty. Indeed, S is not T_1-strictly singular (see Lemma 45) and has T_1-bound 1. Instead, the stability of the spectrum stems from the factorization $T = (I+L)T_1$, in which $I+L$ is invertible, and from the compactness of the resolvents. This example therefore shows that spectral stability can be produced by mechanisms other than relatively strictly singular perturbations.

A comparison example: bounded T_1-compact perturbation

We now consider a second operator, $T' = T_1 + S'$, in which the perturbation acts on a sequence $x \in X$ by

$$(S'x)_n = p^{n+1}x_{n+1}.$$

The term S' differs from the transport perturbation S of Section (recall $(Sx)_n = p^{-(n+1)}x_{n+1}$, with $|p^{-(n+1)}|_p = p^{n+1} \to \infty$): here

$$|p^{n+1}|_p = p^{-(n+1)} \to 0,$$

so whereas S is unbounded and not T_1-compact, the perturbation S' is both bounded and T_1-compact.

Proposition 52 *The perturbation S' and the resulting operator $T' = T_1 + S'$ satisfy the following properties.*

(a) *S' is bounded, with $\|S'\| = p^{-2}$, and compact. Indeed, if S'_N is the truncation defined by*

$$(S'_N x)_n = \begin{cases} p^{n+1}x_{n+1}, & n \leq N, \\ 0, & n > N, \end{cases}$$

then, since $|p^{n+1}|_p = p^{-(n+1)}$,

$$\|S' - S'_N\| = \sup_{n>N} p^{-(n+1)} = p^{-(N+2)} \longrightarrow 0.$$

(b) *S' is T_1-bounded with T_1-bound 0, and T_1-compact; consequently it is T_1-strictly singular.*

(c) *The operator $T' = T_1 + S'$, with domain $D(T') = D(T_1)$, is closed and has compact inverse. Furthermore,*

$$T' - \lambda I \in \Phi(X) \quad \text{with index } 0 \qquad (\lambda \in \mathbb{Q}_p).$$

(d) *The spectra satisfy*

$$\sigma(T') = \{p^{-n} : n \geq 1\} = \sigma(T_1),$$

and

$$\sigma_e(T') = \sigma_{e1}(T') = \emptyset = \sigma_e(T_1).$$

Proof. (a) For $x \in X$, since $|p^{n+1}|_p = p^{-(n+1)}$,

$$\begin{aligned} \|S'x\| &= \sup_{n\geq 1} |p^{n+1}|_p |x_{n+1}|_p \\ &= \sup_{n\geq 1} p^{-(n+1)} |x_{n+1}|_p \\ &\leq p^{-2}\|x\|, \end{aligned}$$

with equality for $x = e_2$. The estimate

$$\|S' - S'_N\| = p^{-(N+2)} \to 0$$

then yields compactness of S'.

(b) Every bounded operator has T_1-bound 0. The inclusion $G_{T_1} \hookrightarrow X$ factors as $T_1^{-1} \circ T_1$, where $T_1 : G_{T_1} \to X$ is an isometry and T_1^{-1} is compact; hence the inclusion is compact. Therefore $S'|_{G_{T_1}}$ is compact, and in particular strictly singular.

(c) Put $y = T_1 x$. Then

$$(T' - \lambda I)x = (I + C_\lambda)y, \qquad C_\lambda = (S' - \lambda I)T_1^{-1},$$

and C_λ is compact on X, since T_1^{-1} is compact. As $x \mapsto T_1 x$ is an isometry from G_{T_1} onto X, the operator $T' - \lambda I$ has the same kernel dimension, range, and codimension as $I + C_\lambda$. By the Riesz–Serre theory of compact perturbations of the identity over a locally compact field [28], $I + C_\lambda$ is Fredholm of index 0.

For closedness, factor

$$T' = (I + S'T_1^{-1})T_1.$$

Since $S'T_1^{-1}$ is compact, the Riesz–Serre theory shows that $I + S'T_1^{-1}$ is Fredholm of index 0, and

$$T'^{-1} = (I + S'T_1^{-1})^{-1}T_1^{-1}$$

is compact; in particular $0 \in \rho(T')$.

(d) By (c), a point λ lies in $\sigma(T')$ iff $T' - \lambda I$ is not injective (the index being 0, the range closed, and the open mapping theorem applying). The eigenvalue equation $T'x = \lambda x$ reads

$$x_{n+1} = p^{-(n+1)}(\lambda - p^{-n})x_n.$$

If $\lambda \neq p^{-k}$ for all k, then

$$|x_{n+1}|_p = p^{-(n+1)}|\lambda|_p |x_n|_p$$

for large n, so $x \in c_0$ forces $x = 0$. If $\lambda = p^{-k}$, then $x_{k+1} = 0$ and $x = (x_1, \dots, x_k, 0, \dots)$ with $x_1 = 1$ is an eigenvector, the eigenspace being one-dimensional. Hence $\sigma(T') = \{p^{-k} : k \geq 1\}$, and by (c) all essential spectra are empty. □

Remark 53 *In this example, the equality $\sigma_{e1}(T') = \sigma_{e1}(T_1)$ is a genuine instance of the perturbation theory for T_1-compact operators. The key simplification is that S', unlike the transport perturbation of Section , is bounded. This boundedness is precisely what makes the analysis straightforward here, and it explains why Section must instead appeal to the factorization $T = (I + L)T_1$.*

Physical interpretation

The absence of essential spectrum means that T has a purely discrete spectrum, consisting of the points $\lambda = p^{-k}$, $k \geq 1$. These eigenvalues admit an interpretation as scale-dependent spectral modes of the hierarchical transport model, and the compactness of the resolvent rules out a continuous essential spectral component in the present formulation. A precise description of temporal decay or stabilization would require an evolution or semigroup setting and is left for future work.

Physically, the factorization $T = (I + L)T_1$ expresses transport as the composition of hierarchical scaling T_1 — downward transport with level-dependent rates p^{-n}— and inter-level coupling $I + L$, describing diffusion and interaction between adjacent levels. The invertibility of $I + L$ ensures that the coupling leaves the spectral structure of T_1 intact.

Examples

Example 54 *Take $p = 2$. Then*

$$\sigma(T) = \{2^{-n} : n \geq 1\} = \left\{\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \dots\right\}.$$

Consequently, the values $\lambda = 1, 2, 4, 3, \frac{1}{3}$ all lie in the resolvent set $\rho(T)$, whereas $\lambda = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ are simple eigenvalues (each Fredholm of index 0). Note, in particular, that $\lambda = \frac{3}{4}$ belongs to $\rho(T)$ even though $|\frac{3}{4}|_2 = 4$: the condition $|\lambda|_p \geq 1$ is therefore irrelevant.

Example 55 *Take $p = 3$. Then*

$$\sigma(T) = \left\{\tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{27}, \dots\right\} = \{3^{-n} : n \geq 1\}.$$

Hence the values $\lambda = 2, 3, \frac{2}{3}$ all belong to the resolvent set $\rho(T)$, even though $|\frac{2}{3}|_3 = 3 \geq 1$. On the other hand, $\lambda = \frac{1}{3}$ is a simple eigenvalue, with corresponding eigenvector e_1; indeed, the recursion gives $x_2 = (3^{2-1} - 3)x_1 = 0$, so the eigenvector terminates after its first coordinate.

Remark 56 *For $\lambda \in \mathbb{Q}_p$, the condition $v_p(\lambda) = 0$ is equivalent to λ being a p-adic unit. By contrast, the set $\{v_p \leq 0\}$ is strictly larger: it*

also contains the negative powers $1/p, 1/p^2, \ldots$. *Hence the informal description of* $\{v_p \leq 0\}$ *as "numbers not divisible by* p^n*" is inaccurate and should be avoided.*

Conclusion

We studied the p-adic transport operator $T = T_1 + S$ modeling fluid flow in fractured porous media. The unperturbed operator T_1 has spectrum $\{p^{-k} : k \geq 1\}$ and compact resolvent, while the perturbation S is not T_1-strictly singular and has T_1-bound exactly 1. Nevertheless, $\sigma(T) = \sigma(T_1)$ and $\sigma_e(T) = \sigma_{e1}(T) = \emptyset$, a stability due to the factorization $T = (I + L)T_1$ with $I + L$ invertible and T^{-1} compact. Along the way we developed a perturbation theory for unbounded operators on non-Archimedean Banach spaces, introducing relative strict singularity and proving stability results for upper semi-Fredholm operators (Theorems 27–41).

The empty essential spectrum indicates a purely discrete spectrum, with all instability confined to isolated eigenvalues at p^{-k}; physically, this suggests that transport in hierarchical porous media is dominated by discrete, scale-dependent resonances rather than continuous spectral features. Future work should construct genuinely T_1-strictly singular perturbations with non-empty essential spectra, connect the spectral results to experimental data, extend the model to nonlinear effects and cross-links between branches, and study the dependence of the spectrum on p and its relation to geological parameters.

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